Error detecting and correcting codes I,II - Seminar 9 and 10

January 7, 2014

Problem 1 *Which of the following received words contain detectable errors when using the (3, 2) parity check code?*

$$110, 010, 001, 111, 101, 000.$$

Solution. 010, 001, 111. ■

Problem 2 *Decode the following words using the (3, 1) repeating code to correct errors:*

$$111, 011, 101, 010, 000, 001.$$

Which of the words contain detectable errors?

Problem 3 *An ancient method of detecting errors when performing the arithmetical operations of addition, multiplication, and subtraction is the method known as* ***casting out nines****. For each number occurring in a calculation, a check digit is found by adding together the digits in the number and casting out any multiples of nine. The original calculation is then performed on these check digits instead of on the original numbers. The answer obtained, after casting out nines, should equal the check digit of the original answer. If not, an error has occurred. For example, check the following:*

$$9642 \times (425 - 163) = 2526204.$$

Add the digits of each number; $9+6+4+2 = 21 = 2 \times 9 + 3$, $4+2+5 = 9+2$, $1+6+3 = 9+1$. *Cast out the nines and perform the calculation on these check digits:*

$$3 \times (2 - 1) = 3.$$

Now 3 is the check digit for the answer because $2+5+2+6+2+0+4 = 2 \times 9 + 3$*; hence this calculation checks. Why does this method work?*

Problem 4 *Find the redundancy of the English (and Romanian) language. Copy a paragraph from a book leaving out every nth letter, and ask a friend to try to read the paragraph. (Try* $n = 2, 3, 4, 5, 6$*. If a passage with every fifth letter missing can usually be read, the redundancy is at least 15 or 20%.)*

Problem 5 *Is* $1 + x^3 + x^4 + x^6 + x^7$ *or* $x + x^2 + x^3 + x^6$ *a code word in the* $(8, 4)$ *polynomial code generated by* $p(x) = 1 + x^2 + x^3 + x^4$*?*

Solution.

$$rem(1 + x^3 + x^4 + x^6 + x^7, 1 + x^2 + x^3 + x^4) \bmod 2 = x^3 + x + 1 \qquad No$$
$$rem(x + x^2 + x^3 + x^6, 1 + x^2 + x^3 + x^4) \bmod 2 = 0. \qquad Yes$$

■

Problem 6 *Write down all the code words in the* $(6, 3)$*-code generated by* $p(x) = 1 + x^2 + x^3$*.*

Solution. The codewords are: 000000, 110001, 111010, 001011, 101100, 011101, 010110, 100111. The message length is $k = 3$ and the codeword length is $n = 6$. There are $2^k = 8$ messages, 000, 001, 010, 011, 100, 101, 110, 111. For 001, $m(x) = x^2$, $x^3 m(x) = x^5$; the check digits are given by the coefficients of the remainder of $x^3 m(x)$ divided by $p(x)$. See the Maple worksheet `sem9p6.mws`. ■

Problem 7 *Design a code for messages of length 20, by adding as few check digits as possible, that will detect single, double, and triple errors. Also give a shift register encoding circuit for your code.*

Problem 8 *Show that the Hamming distance between vectors has the following properties:*

(1) $d(u, v) = d(v, u)$*.*

(2) $d(u, v) + d(v, w) \geq d(u, w)$*.*

(3) $d(u, v) \geq 0$ *with equality if and only if* $u = v$*.*

(This shows that d is a metric on the vector space.)

Problem 9 *Decode the following, using the (6,3)-code given in Table 10, Slide 65: 000101, 011001, 110000.*

Solution. 101, 001, 100. ■

Problem 10 *Find the minimum Hamming distance between the code words of the code with generator matrix* G*, where*

$$G^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Discuss the error-detecting and error-correcting capabilities of this code, and write down the parity check matrix.

Solution. Minimum distance = 3. It detects two errors and correct one error.

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

■

Problem 11 *Find the generator and parity check matrices for the* $(4, 1)$*-code generated by* $1 + x + x^2 + x^3$*.*

Solution.

$$G^T = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

■

Problem 12 *Find the generator and parity check matrices for the (7,3)-code generated by* $(1 + x)(1 + x + x^3)$*.*

Problem 13 *Find the generator and parity check matrices for the (9,4)-code generated by* $1 + x^2 + x^5$*.*

Solution. ■