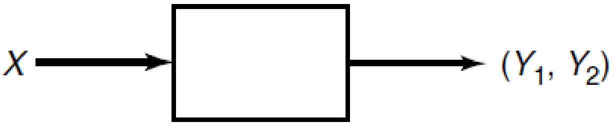
Gaussian Channel - Seminar 8

December 2, 2013

Problem 1 (Channel with two independent looks at Y) *Let Y_1 and Y_2 be conditionally independent and conditionally identically distributed given X.*

(a) Show that $I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2)$.

(b) Conclude that the capacity of the channel



is less than twice the capacity of the channel

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Solution.

(a)

$$\begin{aligned} I(X;Y_1,Y_2) &= H(Y_1,Y_2) - H(Y_1,Y_2|X) & (1)\\ &= H(Y_1) + H(Y_2) - I(Y_1;Y_2) - H(Y_1|X) - H(Y_2|X) & (2)\\ &= I(X;Y_1) + I(X;Y_2) - I(X_1;Y_2) & (3)\\ &= 2I(X;Y_1) - I(Y_2,Y_1). & (4) \end{aligned}$$

(2) since Y_1 and Y_2 are conditionally independent given X (4) follows since Y_1 and Y_2 are conditionally identically distributed.

(b) The capacity of the sigle look channel $X \to Y_1$ is

$$C_1 = \max_{p(x)} I(X; Y_1).$$

The capacity of the channel $X \to (Y_1, Y_2)$ is

$$\begin{aligned} C_2 &= \max_{p(x)} I(X; Y_1, Y_2) \\ &= \max_{p(x)} 2I(X; Y_1) - I(Y_1; Y_2) \\ &\leq \max_{p(x)} 2I(X; Y_1) = 2C_1. \end{aligned}$$

Hence two independent looks cannot be more than twice as good as one look.

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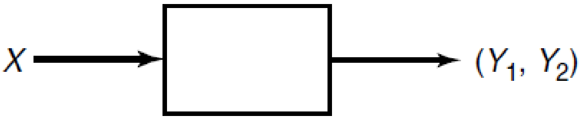

Problem 2 (Two-look Gaussian channel) *Consider the ordinary Gaussian channel with two correlated looks at X, that is, $Y = (Y_1, Y_2)$, where*

$$\begin{aligned} Y_1 &= X + Z_1 \\ Y_2 &= X + Z_2 \end{aligned}$$

with a power constraint P on X, and $(Z_1, Z_2) \sim N_2(0, K)$, where

$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}.$$

Find the capacity C for

(a) $\rho = 1$

(b) $\rho = 0$

(c) $\rho = -1$.

Solution. It is clear that the input distribution that maximizes the capacity is $X \sim N(0, P)$. Evaluate the mutual information for this distribution

$$\begin{aligned} C_2 &= \max I(X; Y_1, Y_2) \\ &= h(Y_1, Y_2) - h(Y_1, Y_2|X) \\ &= h(Y_1, Y_2) - h(Z_1, Z_2|X) \\ &= h(Y_1, Y_2) - h(Z_1, Z_2). \end{aligned} \tag{5}$$

Now, since $(Z_1, Z_2) \sim N\left(0, \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}\right)$, we have

$$h(Z_1, Z_2) = \frac{1}{2} \log (2\pi e)^2 |K_Z| = \frac{1}{2} \log (2\pi e)^2 N^2(1 - \rho^2).$$

Since $Y_1 = X + Z_1$ and $Y_2 = X + Z_2$, we have

$$(Y_1, Y_2) \sim N\left(0, \begin{bmatrix} P+N & P+N\rho \\ P+N\rho & P+N \end{bmatrix}\right),$$

and

$$h(Y_1, Y_2) = \frac{1}{2}\log(2\pi e)^2 |K_Y| = \frac{1}{2}\log(2\pi e)^2 \left(N^2(1-\rho^2) + 2PN(1-\rho)\right).$$

(5)$\Longrightarrow$

$$\begin{aligned} C_2 &= h(Y_1, Y_2) - h(Z_1, Z_2) \\ &= \frac{1}{2}\log\left(1 + \frac{2P}{N(1+\rho)}\right). \end{aligned}$$

(a) $\rho = 1$, $C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$, which is the capacity of a single look channel. This is not surprising, since in this case $Y_1 = Y_2$.

(b) $\rho = 0$. In this case

$$C = \frac{1}{2}\log\left(1 + \frac{2P}{N}\right),$$

which correspond to using twice the power in a single look. The capacity is the same as the capacity of the channel $X \to (Y_1 + Y_2)$.

(c) $\rho = -1$. In this case $C = \infty$, which is not surprising, since if we add Y_1 and Y_2, we can recover X exactly.

Note that the capacity of the above channel in all cases is the same as the capacity of the cannel $X \to (Y_1 + Y_2)$. ■

Problem 3 (Output power constraint) *Consider an additive white Gaussian noise channel with an expected output power constraint P. Thus, $Y = X + Z$, $Z \sim N(0, \sigma^2)$, Z is independent of X, and $EY^2 \le P$. Find the channel capacity.*

Solution. Since Z is independent of X, we have

$$EY^2 = EX^2 + EZ^2 = EX^2 + \sigma^2,$$

and by the output power constraint, we have

$$EX^2 \le P - \sigma^2.$$

In the following, we assume $P > \sigma^2$, since otherwise the problem is uninteresting, since the output power constraint would be violated by the noise alone. Now, for a maximum expected output power P, the entropy of Y is maximized when $Y \sim N(0, P)$, which is achieved when $X \sim N(0, P - \sigma^2)$. Therefore, the channel is equivalent to one with an input power constraint $EX^2 \leq P - \sigma^2$, and it follows that the capacity is

$$C = \frac{1}{2}\log_2\left(1 + \frac{P-\sigma^2}{\sigma^2}\right) = \frac{1}{2}\log_2\left(\frac{P}{\sigma^2}\right).$$

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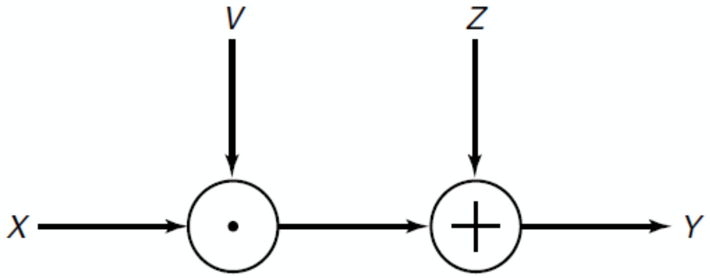
Problem 4 (Fading channel) *Consider an additive noise fading channel*



$$Y = XV + Z,$$

where Z is additive noise, V is a random variable representing fading, and Z and V are independent of each other and of X. Argue that knowledge of the fading factor V improves capacity by showing that

$$I(X;Y|V) \geq I(X;Y).$$

Solution. We have

$$\begin{aligned} I(X;Y|V) &= H(X|V) - H(X|Y,V) \\ &= H(X) - H(X|Y,V) \\ &\geq H(X) - H(X|Y) \\ &= I(X;Y), \end{aligned}$$

where the second equality follows since X and V are independent, and the inequality follows since conditioning reduces entropy. Therefore, as is intuitively reasonable, knowledge of the fading factor improves capacity. ■

Problem 5 (Exponential noise channels) $Y_i = X_i + Z_i$, *where Z_i is i.i.d. exponentially distributed noise with mean μ. Assume that we have a mean constraint on the signal (i.e., $EX_i \leq \lambda$). Show that the capacity of such a channel is $C = \log\left(1 + \frac{\lambda}{\mu}\right)$.*

Solution. We assume $X_i \geq 0$ for all i, since otherwise the capacity is unbounded when the only constraint is on the mean of the input (i.e., we could have unbounded power). Now, since

$$EX_i \leq \lambda,$$

it follows that

$$EY_i \leq \mu + \lambda.$$

Furthermore, from the non-negativity constraints on X_i and Z_i, we also have $Y_i \geq 0$ for all i. Since the noise is independent of the input, we have

$$\begin{aligned} I(X_i; Y_i) &= h(Y_i) - h(Y_i|X_i) \\ &= h(Y_i) - h(X_i + Z_i|X_i) \\ &= h(Y_i) - h(Z_i). \end{aligned}$$

Now, since Y_i is a non-negative random variable with $EY_i \geq \mu + \lambda$, it follows from Example 12.2.4 (Cover & Thomas) that $h(Y_i)$ is maximized when Y_i is exponentially distributed. Furthermore, from Problem 8.1, we know that the differential entropy of an exponential distribution is a monotonically increasing function of the mean. Therefore,

$$h(Y_i) \leq \log_2(e(\mu + \lambda)),$$

and it follows that

$$C \leq \log_2(e(\mu + \lambda)) - \log_2(e\mu)) = \log_2\left(1 + \frac{\lambda}{\mu}\right).$$

To confirm that $C = \log_2\left(1 + \frac{\lambda}{\mu}\right)$, we need to show that X_i can be chosen such that $X_i + Z_i$ has an exponential distribution with mean $\mu + \lambda$. We will do so by explicitly deriving the required distribution. The characteristic function of an exponential random varible Z with mean is

$$\varphi_Z(t) = \frac{\frac{1}{\rho}}{\frac{1}{\rho} - it}.$$

Since

$$\varphi_Y = \varphi_X \varphi_Z$$

we have that the required φ_X is given by

$$\begin{aligned} \varphi_X(t) &= \frac{1 - i\mu t}{1 - i(\mu + \lambda)t} \\ &= \frac{1}{1 - i(\mu + \lambda)t} - \left(\frac{\mu}{\mu + \lambda}\right)\left(\frac{1}{1 - i(\mu + \lambda)t}\right) + \frac{\mu}{\mu + \lambda} \end{aligned}$$

Finally, taking the inverse transform, we have

$$f_X(x) = \frac{\lambda}{\mu + \lambda} \frac{e^{-\frac{x}{\mu + \lambda}}}{\mu + \lambda} + \frac{\mu}{\mu + \lambda}\delta(x), \text{ for } x \geq 0.$$

Observe that this solution specifies that we should transmit zero with probability $\mu/(\mu + \lambda)$, and otherwise transmit an exponentially distributed nonzero value with as large a mean value as possible. (In a sense we are "saving up" for the possibility of a "louder" transmission whenever we send zero.) ■