Gaussian Channel

The Most Important Continuous Channel

Radu Trîmbițaș

UBB

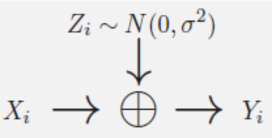
November 2012

Outline I

Continuous Channels

- So far, we have considered discrete channels which are modeled by conditional probability distributions $p(y|x)$.
- That is, for a given $x \in \mathcal{X}$, $p(y|x)$ models the form of distortion that x undergoes when it is being sent from source to receiver.
- Real channels are continuous as are real signals. What really happens to a continuous random variable X is that we have $Y = \nu(X)$ where ν is a random function that may or may not be dependent on X.
- This is quite hard to analyze so we may consider only additive noise $Y = X + \nu$ where ν is a random variable.
- We further simplify by saying that ν and X are idependent and moreover that ν is Gaussian, leading to the Gaussian channel.

Gaussian Channel



- Above is our model, where $Y_i = X_i + Z_i$, with $Z_i \sim N(0; \sigma^2)$ and Zi, Xi independent.
- If $\sigma^2 = 0$, what is the capacity of this channel?
- If $\sigma^2 = 0$, capacity is infinite since one can perfectly send an arbitrarily precise real number (consider arithmetic coding, it sends a number all within $[0, 1)$).
- If $\sigma^2 > 0$, what is the capacity?
- If $\sigma^2 > 0$, capacity is still infinite, since we can make input power as large as we want, effectively removing a finite strict subinterval within $[0, 1)$.
- If input power is constrained as well (which is also more practical and realistic), then the problem becomes interesting.

Power constraint

- Average power constraint: for any codeword $(x_1, \ldots x_n)$ of length n, we require that
$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P. \tag{1}$$
where P is the average power$\approx E(X^2)$
- This communication channel models many practical channels, including radio and satellite links.
- The additive noise in such channels may be due to a variety of causes.
- However, by the central limit theorem, the cumulative effect of a large number of small random effects will be approximately normal, so the Gaussian assumption is valid in a large number of situations.

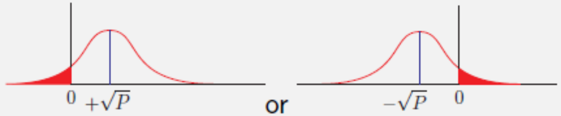
Example I

- Send 1 bit over channel at a time (obviously sub-optimal use of the channel).
- $X \in \{-\sqrt{P}, \sqrt{P}\}$ means that $E(X^2) = P$, so this satisfies the constraint.
- For a uniform source distribution, decode as $+\sqrt{P}$ if $Y > 0$ and $-\sqrt{P}$ if $Y < 0$.
- Error:

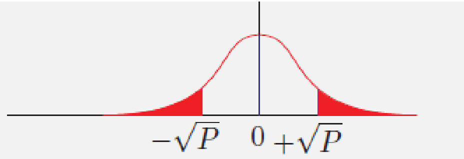
$$\begin{aligned} P_e &= \frac{1}{2}P(Y < 0|X = \sqrt{P}) + \frac{1}{2}P(Y > 0|X = -\sqrt{P}) \\ &= \frac{1}{2}P(Z < -\sqrt{P}|X = \sqrt{P}) + \frac{1}{2}P(Z > \sqrt{P}|X = -\sqrt{P}) \\ &= P(Z > \sqrt{P}). \end{aligned}$$

Example II

- The two separate error types



- Lead to total error

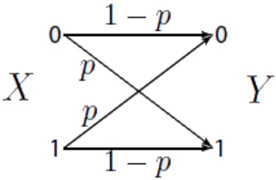


- We have that

$$P(Z > \sqrt{P}) = 1 - \Phi\left(\frac{\sqrt{P}}{\sigma^2}\right).$$

Example III

- In fact, we have essentially just turned a Gaussian channel into a discrete BSC:



  where $p = P_e$ for the Gaussian.
- This will be the common idea. We convert continuous channels into discrete ones with appropriate encodings.
- This is essentially a process of vector quantization (where under different quantization schemes, we study the tradeoffs that exist when coding). Tradeoffs take the form of rate vs. distortion under the power constraints.

Capacity of Gaussian Channel I

We need a capacity notion, but here under a power constraint.

Definition 1

The *information capacity* of the Gaussian channel with power constraint P is

$$C = \max_{f(x):EX^2 \leq P} I(X;Y). \tag{2}$$

- $I(X;Y)$ has a nice form in this case, as

$$\begin{aligned} I(X;Y) &= h(Y) - h(Y|X) = h(Y) - h(X+Z|X) \\ &= h(Y) - h(Z|X) \\ &= h(Y) - h(Z) \end{aligned}$$

- But since Z is Gaussian, $h(Z) = \frac{1}{2} log(2\pi e \sigma^2)$ where σ^2 is the noise power, $E\left(Z^2\right) = \sigma^2 = N$, with $E\left(Z\right) = 0$.

Capacity of Gaussian Channel II

- We also saw earlier, since Gaussians have maximum entropy for a given 2nd moment, that if $EX = 0$, $var(X) = K$, then

$$h(X) \leq \frac{1}{2} \log \left[2\pi e^2 |K|\right]$$

- Also,

$$\begin{aligned} E\left(Y^2\right) &= E(X+Z)^2 = E\left(X^2\right) + 2E\left(X\right)E\left(Y\right) + E\left(Y^2\right) \\ &= \underbrace{P}_{signal\ \ power} + \underbrace{\sigma^2}_{noise\ \ power} \end{aligned}$$

Capacity of Gaussian Channel III

- Thus, we can upper bound the mutual information as follows:

$$I(X;Y) = h(Y) - h(Z) \leq \frac{1}{2}\log\left(2\pi e\left(P+\sigma^2\right)\right) - \frac{1}{2}\log\left(2\pi e\sigma^2\right)$$
$$= \frac{1}{2}\log\left(1+\frac{P}{\sigma^2}\right) = \frac{1}{2}\log\left(1+\frac{P}{N}\right) = \frac{1}{2}\log\left(1+SNR\right),$$

  where SNR is the signal to noise ratio.
- We can achieve the bound on $h(Y)$ by ensuring Y is Gaussian, and this is the case if X is Gaussian (sums of Gaussians are Gaussian).
- The capacity of the Gaussian channel is

$$C = \frac{1}{2}\log\left(1+\frac{P}{\sigma^2}\right) = \frac{1}{2}\log\left(1+SNR\right)$$

- Makes sense: the maximum transmission rate obtained when $X \sim N(0;P)$.

Capacity of Gaussian Channel IV

- Rate depends on SNR - if signal level is allowed to be much larger than noise, then rate should increase (log when information measured in bits).
- In fact, from this we get the standard 6.02 dB SNR/bit for audio. I.e., $16 = 1/2 \log(1 + SNR)$, or $2^{32} = 1 + SNR$ or $SNR = 2^{32} - 1$.
- $10 \log_{10}(SNR) = 10 \times 32 log_{10}(2) = 96.33$ dB.
- And $96.33/16 = 6.02$ dB/bit.
- Every additional bit (on an audio CD) adds 6.02 dB of SNR.

Capacity of the channel definitions I

Definition 2

An (M, n) code for the Gaussian channel, with power constraint P, consist of

1. An index set $\{1, 2, \ldots, M\}$
2. An encoding function $X : \{1, 2, \ldots, M\} \to \mathcal{X}^n$ giving codewords $X^n(1), X^n(2), \ldots, X^n(M)$ with

$$\sum_{i=1}^{n} x_i^2(w) \leq nP, \qquad w = 1, 2, \ldots, n.$$

3. A decoding function $g : \mathcal{Y}^n \to \{1, 2, \ldots, M\}$.

Capacity of the channel definitions II

Definition 3

The *rate* is

$$R = \frac{\log M}{n} \text{ bits per channel use.}$$

Definition 4

The average probability of error is

$$P_e^{(n)} = \frac{1}{2^{nR}} \sum \lambda_i.$$

Definition 5

A rate R is *achievable* if $\exists$ a sequence of $(2^{nR}, n)$ codes satisfying the power constraint P s.t. $\lambda^{(n)} \to 0$ as $n \to \infty$. The capacity of the channel is the supremum over all achievable rates.

The Coding Theorem for Gaussian Channels

Theorem 6

The capacity of a Gaussian channel with input power constraint P and noise variance $\sigma^2 = N$ is

$$C = \frac{1}{2} \log \left(1 + \frac{P}{N}\right) \text{ bits per transmission.} \tag{3}$$

Proof sketch...

- Typical X set $A_\varepsilon^{(n)}$ has volume $\leq 2^{n(h(X)+\varepsilon)}$
- Conditional typical Y volume $\leq 2^{n(h(Y|X)+\varepsilon)} = 2^{n(h(Z)+\varepsilon)}$
- Unconditional typical set volume of $Y \leq 2^{n(h(Y)+\varepsilon)}$, but $h(Y) \leq \frac{1}{2} \log \left[2\pi e \left(P + \sigma^2\right)\right]$ and $h(Z) = \frac{1}{2} \log \left(2\pi e \sigma^2\right)$.

...

The Coding Theorem for Gaussian Channels

. . . proof sketch.

- How many X-conditional volumes can we pack into total available volume?

$$\leq \frac{2^{nh(Y)}}{2^{nh(Z)}} = \frac{2^{n\frac{1}{2}\log\left[2\pi e\left(P+\sigma^2\right)\right]}}{2^{n\frac{1}{2}\log[2\pi e\sigma^2]}} = 2^{\frac{n}{2}\log\left(\frac{P+\sigma^2}{\sigma^2}\right)} = \left(\frac{P+\sigma^2}{\sigma^2}\right)^{\frac{n}{2}}$$

- The above is measured in counts for n channel usages. To convert it into bits per channel use, we take log and divide by n to get

$$R = \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right)$$

- Assuming no overlap of volumes which is best we can do.

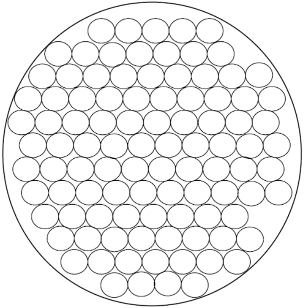
$\square$

The Coding Theorem for Gaussian Channels

- Moreover, if everything is jointly Gaussian, and i.i.d., then the typical volumes will be spheres, and we can relate the volume of a sphere to the typical set to get the radius (in n-Dim)

$$r_{\sigma^2} = \Gamma^{1/2}\left(\frac{n}{2}+1\right)\left(2e\sigma^2\right)^{\frac{1}{2}}.$$

- figure



Proof differences from discrete proof

proof Δ from discrete proof.

- We need to show that if $R < C$, $\exists$ a code with $P_e^n \to 0$ when $n \to \infty$.
- We do random codeword generation (like in discrete case) but in this case from Gaussians with $E\left(X^2\right) = P - \varepsilon$ so that

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \to P - \varepsilon, \qquad \text{as } n \to \infty$$

- Also have an additional source of possible error

$$E_0 = \left\{ \frac{1}{n} \sum x_i^2(1) > P \right\}$$

...

proof diff. from discrete proof

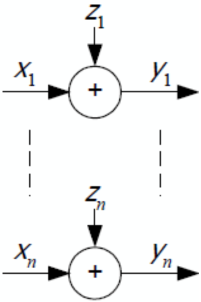
proof Δ from discrete proof.

- We add E_0 to the other errors we previously had in discrete case (note again we use same trick to show that considering only message 1 is sufficient due to symmetry).
- By the weak law of large numbers, $E_0 \to 0$ also as $n \to \infty$ as do the other sources of errors.

$\square$

Parallel Gaussian Channels

- Suppose we have k independent Gaussian channels with a common power constraint.



$$
\begin{aligned}
&Y_j = X_j + Z_j,\ j = 1, 2, \ldots, k,\\
&Z_j \sim N(0, N_j),\\
&Z_i,\ Z_j \text{ independent for } i \neq j\\
&Z \sim N\left(0, \begin{bmatrix} N_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & N_k \end{bmatrix}\right)
\end{aligned}
$$

- power constraint

$$
E\left(\sum_{i=1}^{n} X_j^2\right) \leq P.
$$

- Goal: to distribute the total power among the chanel in order to maximize the total capacity.

Parallel Channels I

- Find maximum capacity

$$C = \max_{f(x_{1:k}):\sum E(X_i^2) \le P} I\left(X_{1:k}, Y_{1:k}\right) \tag{4}$$

- We have

$$I\left(X_{1:k}, Y_{1:k}\right) = h(Y_{1:k}) - h(Y_{1:k}|X_{1:k}) = h(Y_{1:k}) - h(Z_{1:k}|X_{1:k}) \tag{5}$$

$$= h(Y_{1:k}) - h(Z_{1:k}) = h(Y_{1:k}) - \sum_i h(Z_i) \tag{6}$$

$$\le \sum_j \left[h(Y_j) - h(Z_j)\right] \le \sum_j \frac{1}{2}\log\left(1 + \frac{P_i}{N_i}\right) \tag{7}$$

- with $P_i = EX_i^2$ and $\sum_i P_i = P$.

Parallel Channels II

- The way to solve this is to solve the optimization problem

$$
\begin{aligned}
& \max_{(P_{1:k})} \sum_i \frac{1}{2} \log\left(1 + \frac{P_i}{N_i}\right) && (8) \\
\text{subject to} \quad & \sum_i P_i = P && (9)
\end{aligned}
$$

- Or in Lagrangian form

$$
J(P_{1:k}) = \sum_i \frac{1}{2} \log\left(1 + \frac{P_i}{N_i}\right) + \lambda \left(\sum_i P_i - P\right)
$$

Lagrangian optimization

$$\begin{aligned} &\text{find} && \min_x f_0(x) \\ &\text{subject to} && f_i(x) \leq 0,\ \ i = 1, \ldots, m \\ & && h_i(x) = 0,\ \ i = 1, \ldots, p \end{aligned}$$

Lagrangian form

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

and define dual objective

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu).$$

Strong duality

- Notation: $\lambda_{opt} = \lambda^*$, $\nu_{opt} = \nu^*$
- Strong duality means $f(x_{opt}) = g(\lambda_{opt}, \nu_{opt})$
- Karush-Kuhn-Tucker conditions for optimality,

$$\begin{aligned} f_i(x_{opt}) &\leq 0, && i = 1, \ldots, m \\ h_i(x_{opt}) &= 0, && i = 1, \ldots, p \\ \lambda_i^* &\geq 0, && i = 1, \ldots, m \\ \lambda_i^* f_i(x_{opt}) &= 0, && i = 1, \ldots, m \end{aligned}$$

and

$$\nabla_x L|_{x=x_{opt}} = 0$$

Back to our problem

- We get Lagrangian

$$L(x,\lambda,\nu) = -\sum_{i=1}^{k} \frac{1}{2}\log\left(1+\frac{P_i}{N_i}\right) - \sum_{i=1}^{k}\lambda_i P_i + \nu\left(\sum_{i=1}^{k} P_i - P\right).$$

- KKT conditions are:

$$\forall i\ P_i \geq 0,\ \sum_i P_i^* = P,\ \forall i\ \lambda_i^* \geq 0, \lambda_i^* P_i^* = 0$$

and also $\forall i$

$$-\frac{1}{(1+P_i/N_i)}\frac{1}{N_i} - \lambda_i^* + \nu^* = 0.$$

KKT Conditions

- From the Lagrangian gradient conditions we can further get

$$-\frac{1}{N_i + P_i} - \lambda_i^* + \nu^* = 0 \Longrightarrow -\frac{1}{N_i + P_i} + \nu^* = \lambda_i^* \geq 0$$

- We then eliminate λ_i^* to get KKT conditions in form

$$P_i^* \geq 0, \qquad \sum_i P_i^* = P$$

$$\left(\nu^* - \frac{1}{N_i + P_i}\right) P_i^* = 0, \qquad \nu^* \geq \frac{1}{N_i + P_i^*}$$

Final solution

- So, P_i^* must have form

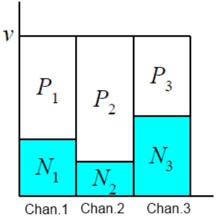
$$P_i^* = \left(\frac{1}{\nu^*} - N_i\right)_+$$

where $a_+ = \max\{0, a\}$.

- With the last constraint, we have that

$$\sum_i \left(\frac{1}{\nu^*} - N_i\right)_+ = P$$

- This leads to the famous water filling idea for parallel channels.



Capacity of Parallel Channels

- Hence, the final capacity is

$$\begin{aligned} C_n &= \frac{1}{2}\sum_{i=1}^{k} \log\left(1+\frac{P_i}{N_i}\right) \\ &= \frac{1}{2}\sum_{i=1}^{k} \log\left(1+\frac{\left(\frac{1}{\nu^*}-N_i\right)_+}{N_i}\right) \text{ bits per parallel channel use} \end{aligned}$$

- In units of bits per transmission (bits per single channel transmission, take the average):

$$C_n = \frac{1}{2n}\sum_{i=1}^{k} \log\left(1+\frac{(1/\nu^*-N_i)_+}{N_i}\right)$$

Colored Noise I

- Consider the case when the noise is dependent
- Suppose $Y = X + Z$ where $E\left(ZZ^T\right) = K_Z$ and $E\left(XX^T\right) = K_X$
- We want to find K_X to maximize capacity subject to power constraint:

$$\sum E\left(X_i^2\right) \leq nP \Longleftrightarrow tr(K_X) \leq nP$$

- Find noise eigenvectors: $K_Z = Q\Lambda Q^T$ with $QQ^T = I$.
- Now

$$Q^T Y = Q^T X + Q^T Z = Q^T X + W$$

where

$$E\left(WW^T\right) = E\left(Q^T ZZ^T Q\right) = E\left(Q^T K_Z Q\right) = \Lambda$$

is diagonal.

Colored Noise II

- W_i are now independent (so previous result on Parallel Gaussian Channels applies)
- Power constraint is unchanged $tr\left(Q^T K_X Q\right) = tr\left(K_X Q Q^T\right) = tr(K_X)$
- Use water-filling and indep. messages $Q^T K_X Q + \Lambda = \nu I$
- Choose $Q^T K_X Q = \nu I - \Lambda$ where $\nu = P + n^{-1} tr\left(\Lambda\right) \Longrightarrow K_X = Q\left(\nu I - \lambda\right) Q^T$.

Colored Noise I

- If Z is from a stationary process then $diag(\Lambda) \to$ power spectrum $N(f)$, as $k \to \infty$
- To achieve capacity use waterfilling on noise power spectrum

$$P = \int_{-W}^{W} (\nu - N(f))_+ \, df$$

$$C = \frac{1}{2} \int_{-W}^{W} \log\left(1 + \frac{(\nu - N(f))_+}{N(f)}\right) df$$

- Waterfilling on spectral domain