

COURSE 7

Korovkin theorems. Modulus of continuity

Remark 1 The second Korovkin theorem is regarding the space $C_{2\pi}$, of 2π -periodical real and continuous functions. It is considered the norm $\|f\| := \sup_{x \in E} |f(x)|$, with E , for example, chosen as $[-\pi, \pi]$. Notice that for $\forall f \in C_{2\pi}$, $f(-\pi) = f(\pi)$. The test functions are e_0 , $\sin$, $\cos$.

Theorem 2 (The second Korovkin Theorem) Let $(L_m)_{m \in \mathbb{N}}$ be a sequence of linear and positive operators $L_m : C_{2\pi} \rightarrow C_{2\pi}$. If $(L_m f_i)_{m \in \mathbb{N}}$ converges uniformly to f_i , $i \in \{1, 2, 3\}$, where $f_1 = e_0$, $f_2 = \sin$, $f_3 = \cos$ then for any $f \in C_{2\pi}$, $(L_m f)_{m \in \mathbb{N}}$ converges uniformly to f .

Korovkin closure

Let $E \neq \emptyset$ be a compact topological space and $\mathcal{T}$ a subset of $C(E)$. If the constant function **1** is in $\mathcal{T}$, then the set $\mathcal{T}$ is called a *test set*.

A sequence $(L_n)_{n \geq 1}$ of linear and positive operators, $L_n : C(E) \rightarrow C(E)$, is called a *$\mathcal{T}$ -admissible sequence* if the sequence $(L_n g)_{n \geq 1}$ converges uniformly to g in E for all $g \in \mathcal{T}$, i.e., $\lim_{n \rightarrow \infty} \|L_n g - g\| = 0$, $(\forall) g \in \mathcal{T}$.

A function $f \in C(E)$ is called *Korovkin function with respect to $\mathcal{T}$* if the sequence $(L_n f)_{n \geq 1}$ converges uniformly to f on E for all the sequences of linear and positive operator $(L_n)_{n \geq 1}$ that are $\mathcal{T}$ -admissible.

The Korovkin closure of the test set $\mathcal{T}$, denoted by $Kor(\mathcal{T})$, is the set of all Korovkin functions $f \in C(E)$, with respect to $\mathcal{T}$.

So, Popoviciu-Bohman-Korovkin theorems state that

$$\text{Kor}(\{e_0, e_1, e_2\}) = C([a, b]), \quad \text{Kor}(\{e_0, \sin, \cos\}) = C_{2\pi}.$$

Let X, Y be normed linear spaces. We define

$$\mathcal{L}^*(X, Y) := \{A : X \rightarrow Y \mid A \text{ linear and continuous}\}.$$

Remark 3 We have $\mathcal{L}^*(X, Y) \leq \mathcal{L}(X, Y) \leq Y^X$.

Definition 4 Let X, Y be normed vectorial spaces. The operator $A : X \rightarrow Y$ is called **Lipschitz-operator** if there exists $L \geq 0$ such that

$$\|Af\|_Y \leq L\|f\|_X, \quad (\forall) f \in X. \quad (1)$$

Theorem 5 Let X, Y be normed spaces and $A \in L(X, Y)$. A is continuous iff A is a Lipschitz-operator.

Lemma 6 Let $A \in L^*(X, Y)$. The set

$$\mathcal{L}_A := \{L \geq 0 \mid \|Af\|_Y \leq L\|f\|_X, f \in X\}$$

admits a minimum.

Proof. Let $f \in X$, $f \neq 0$. If $L \in \mathcal{L}_A$ arbitrary fixed, then $\|Af\|_Y \leq L\|f\|_X$ whence $\|f\|_X^{-1}\|Af\|_Y \leq L$. So $\|f\|_X^{-1}\|Af\|_Y \leq \inf \mathcal{L}_A \Leftrightarrow \|Af\|_Y \leq \inf \mathcal{L}_A \cdot \|f\|_X$, so $\inf \mathcal{L}_A \in \mathcal{L}_A$. If $\inf \mathcal{L}_A$ is in the set it follows that $\inf \mathcal{L}_A = \min \mathcal{L}_A$. ■

Definition 7 Let $A \in \mathcal{L}^*(X, Y)$. The smallest number L that satisfies relation (1) is called the norm of the operator A , denoted by $\|A\|$.

Therefore, there is the relation

$$\|Af\|_Y \leq \|A\|\|f\|_X.$$

MODULUS OF CONTINUITY

Definition 8 Let $f \in C_B(I)$, I being an interval. The application $\omega_f : [0, \infty) \rightarrow \mathbb{R}$,

$$\omega_f(\delta) := \sup\{|f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta\} \quad (2)$$

is called **the modulus of continuity** of f .

Remark 9 (i) For ω_f is also used notation $\omega(f; \cdot)$.

(ii) Definition (2) can be written as

$$\omega_f(\delta) := \sup\{|f(x+h) - f(x)| : x, x+h \in I, 0 \leq h \leq \delta\}.$$

(iii) Let $I = [a, b]$ and $L := b - a$. For $(\forall) \delta \geq L$, $\omega_f(\delta)$ has the same value, $\omega_f(\delta) = \text{diam}(\text{Im } f)$.

Theorem 10 (Properties of the modulus of continuity) Let $f \in C_B(I)$. ω_f has the following properties:

1° $\omega_f \geq 0$, 2° $\omega_f(0) = 0$, 3° ω_f is ascending

4° ω_f is subadditive 5° ω_f is uniform continuous

6° $(\forall) \delta \geq 0, (\forall) n \in \mathbb{N}, \omega_f(n\delta) \leq n\omega_f(\delta),$

7° $(\forall) \delta \geq 0, (\forall) \lambda > 0, \omega_f(\lambda\delta) \leq (\lambda + 1)\omega_f(\delta),$

8° $(\forall) \delta > 0, |f(y) - f(x)| \leq (1 + \delta^{-2}(y - x)^2)\omega_f(\delta).$

9° $(\forall) f_1, f_2 \in C_B(I), \omega(f_1 f_2; \delta) \leq \|f_1\|\omega(f_2; \delta) + \|f_2\|\omega(f_1; \delta).$

10° If

$\text{Lip}_M \alpha := \{f : I \rightarrow \mathbb{R} \mid (\forall) x_1, x_2 \in I, |f(x_1) - f(x_2)| \leq M|x_1 - x_2|^\alpha\}$

is the class of functions that satisfy Lipschitz condition on I , of exponent α ($0 < \alpha \leq 1$) and coefficient $M > 0$, then

$$f \in \text{Lip}_M \alpha \Leftrightarrow \omega_f(\delta) \leq M\delta^\alpha.$$

Proof. 1° It is obvious by definition (2).

2° We have

$$\omega_f(0) = \sup_{\substack{x', x'' \in I \\ |x' - x''| \leq 0}} |f(x') - f(x'')| = \sup_{x' = x'' \in I} |f(x') - f(x'')| = 0.$$

3° Let $0 \leq \delta_1 < \delta_2$ and

$$D_k = \{(x', x'') \mid x', x'' \in I, |x' - x''| \leq \delta_k\}, \quad k = 1, 2.$$

We have $D_1 \subset D_2$. Let $D := D_2 \setminus D_1$. As $\sup_{D_2} = \sup_{D_1 \cup D} \geq \sup_{D_1}$ we get $\omega_f(\delta_2) \geq \omega_f(\delta_1)$, so ω_f is ascending.

4° Let $\delta_1 \geq 0, \delta_2 \geq 0$ be arbitrary given and $x', x'' \in I$ such that $|x' - x''| \leq \delta_1 + \delta_2$. Let x be between x' and x'' such that $|x - x'| = \delta_1$ and $|x'' - x| \leq \delta_2$, we may write

$$|f(x') - f(x'')| \leq |f(x') - f(x)| + |f(x) - f(x'')| \leq \omega_f(\delta_1) + \omega_f(\delta_2).$$

Applying $\sup_{|x' - x''| \leq \delta_1 + \delta_2}$, we get $\omega_f(\delta_1 + \delta_2) \leq \omega_f(\delta_1) + \omega_f(\delta_2)$.

5° Let $\varepsilon > 0$ be given. We show that $(\exists)\delta > 0$ such that $(\forall) u \geq 0$, $(\forall) v \geq 0$, with $|u - v| < \delta$ it follows $|\omega_f(u) - \omega_f(v)| < \varepsilon$.

As $f \in C_B(I)$ we have $\lim_{t \rightarrow 0} \omega_f(t) = 0 = \omega_f(0)$, i.e., ω_f is continuous at zero. For given ε we have

$$(\exists) \delta > 0, (\forall) t \in [0, \delta], |\omega_f(t)| < \varepsilon. \quad (3)$$

On the other hand, by monotony and subadditivity we have

$$0 \leq \omega_f(t_1 + t_2) - \omega_f(t_1) \leq \omega_f(t_2), \quad (\forall) t_1 \geq 0, t_2 \geq 0. \quad (4)$$

We choose $u = t_1 + t_2$, $v = t_1$ so $t_2 = u - v \geq 0$ and by (4) we get

$$|\omega_f(u) - \omega_f(v)| \leq \omega_f(|u - v|), \quad u \geq 0, v \geq 0. \quad (5)$$

We choose u, v such that $|u - v| < \delta$, with δ fixed by (3). By (5) and (3), knowing that ω_f is ascending, it follows $|\omega_f(u) - \omega_f(v)| \leq \omega_f(|u - v|) \leq \omega_f(\delta) < \varepsilon$.

6° We prove using induction. For $n = 1$ it is obvious. We suppose $\omega_f(n\delta) \leq n\omega_f(\delta)$ for $n > 1$ fixed. By the subadditivity, it follows

$$\omega_f((n + 1)\delta) = \omega_f(n\delta + \delta) \leq \omega_f(n\delta) + \omega_f(\delta)$$

$$\leq n\omega_f(\delta) + \omega_f(\delta) = (n+1)\omega_f(\delta).$$

7° We have $\lambda < [\lambda] + 1 \leq \lambda + 1$, using properties 3°, 6° (for $n = [\lambda] + 1$) and 1°, we may write

$$\omega_f(\lambda\delta) \leq \omega_f([\lambda] + 1)\delta \leq ([\lambda] + 1)\omega_f(\delta) \leq (\lambda + 1)\omega_f(\delta).$$

8° Let $\delta > 0$. $|f(x) - f(y)| \leq \sup_{\substack{u, v \in I \\ |u-v| \leq |x-y|}} |f(u) - f(v)| = \omega_f(|x - y|)$. If

$|x - y| < \delta$, by monotony of function ω_f we have

$$\omega_f(|x - y|) \leq \omega_f(\delta) \leq (1 + \delta^{-2}(x - y)^2)\omega_f(\delta).$$

If $|x - y| \geq \delta$ then we apply 7° for $\lambda := \delta^{-1}|x - y|$ and having $\lambda \leq \lambda^2$ we get

$$\begin{aligned} \omega_f(|x - y|) &= \omega_f(\delta^{-1}|x - y|\delta) \leq (1 + \delta^{-1}|x - y|)\omega_f(\delta) \\ &\leq (1 + \delta^{-2}(x - y)^2)\omega_f(\delta). \end{aligned}$$

9° Using the identity

$$\begin{aligned} &(f_1 f_2)(x + h) - (f_1 f_2)(x) \\ &= f_1(x + h)(f_2(x + h) - f_2(x)) + f_2(x)(f_1(x + h) - f_1(x)), \end{aligned}$$

with $x, x + h \in I$, for $0 \leq h \leq \delta$ we get

$$\begin{aligned} |(f_1 f_2)(x + h) - (f_1 f_2)(x)| &\leq |f_1(x + h)| |f_2(x + h) - f_2(x)| \\ &\quad + |f_2(x)| |f_1(x + h) - f_1(x)| \\ &\leq \|f_1\| \omega(f_2; \delta) + \|f_2\| \omega(f_1; \delta), \end{aligned}$$

using Remark 9. Applying $\sup_{0 \leq h \leq \delta}$ we get

$$\omega(f_1 f_2; \delta) \leq \|f_1\| \omega(f_2; \delta) + \|f_2\| \omega(f_1; \delta).$$

10° We have

$$\begin{aligned} \text{Lip}_M \alpha &:= \{f : I \rightarrow \mathbb{R} \mid (\forall) x_1, x_2 \in I, |f(x_1) - f(x_2)| \leq M |x_1 - x_2|^\alpha\}, \\ M &> 0, \alpha > 0. \end{aligned}$$

If f satisfies Lipschitz condition of exponent α and coefficient M on I , f is uniformly continuous on I .

For $\alpha > 1$, $f \in \text{Lip}_M \alpha$ implies that f is constant on I (it is proved that f' is zero on I). (That's why the study of this class is interesting for $\alpha \leq 1$.)

Let $f \in \text{Lip}_M \alpha$. We choose arbitrary x_1, x_2 in I such that $|x_1 - x_2| \leq \delta$. We have $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|^\alpha \leq M\delta^\alpha$ and applying $\sup_{\substack{u, v \in I \\ |u-v| \leq \delta}}$ we get $\omega_f(\delta) \leq M\delta^\alpha$.

Reciprocally, let $\omega_f(\delta) \leq M\delta^\alpha$. Let $x_1 \neq x_2$ arbitrary chosen in I and $\delta = |x_1 - x_2| > 0$. We have

$$\begin{aligned} |f(x_1) - f(x_2)| &\leq \sup_{\substack{u, v \in I \\ |u-v| \leq \delta}} |f(u) - f(v)| \\ &= \omega_f(\delta) \leq M\delta^\alpha = M|x_1 - x_2|^\alpha \end{aligned}$$

so $f \in \text{Lip}_M \alpha$. ■

Definition 11 An application $\Omega : [0, \infty) \rightarrow \mathbb{R}$ with properties $\Omega(0) = 0$, Ω is ascending, subadditive and continuous is called **modulus of continuity**.

Remark 12 This definition is justified by the fact that any function, having these properties, is its own modulus of continuity:

$$\Omega(\delta) = \omega_\Omega(\delta), \quad (\forall) \delta \geq 0.$$

Proof. By Remark 9 we have $\omega_{\Omega}(\delta) = \sup_{\substack{x \geq 0 \\ 0 \leq h \leq \delta}} |\Omega(x+h) - \Omega(x)|$.

By monotony and subadditivity we have

$$0 \leq \Omega(x+h) - \Omega(x) \leq \Omega(h) \leq \Omega(\delta)$$

so $\omega_{\Omega}(\delta) \leq \Omega(\delta)$.

On the other hand,

$$\begin{aligned} \Omega(\delta) &= \Omega(0 + \delta) - \Omega(0) \leq \\ &\leq \sup\{|\Omega(x+h) - \Omega(x)| : x \geq 0, 0 \leq h \leq \delta\} = \omega_{\Omega}(\delta). \end{aligned}$$

■

Having some extra hypothesis about f , the computation of ω_f becomes easier .

Theorem 13 Let $f \in C([a, b])$ be a convex function on $[a, b]$ and $\delta \in [0, b - a]$.

(i) If f is ascending on $[a, b]$ then

$$\omega_f(\delta) = f(b) - f(b - \delta).$$

(ii) If f is descending on $[a, b]$ then

$$\omega_f(\delta) = f(a) - f(a + \delta).$$

Proof. (i) f being ascending we have

$$\omega_f(\delta) = \sup\{f(x + \delta) - f(x) : x, x + \delta \in [a, b]\} \quad (6)$$

If $\delta = 0$ then $\omega_f(\delta) = 0 = f(b) - f(b - \delta)$.

If $\delta = b - a$ then $\omega_f(\delta) = f(b) - f(a) = f(b) - f(b - \delta)$.

If $\delta \in (0, b - a)$ we will use the following result:

Lemma [Gh.Siretchi [Prop. 7.1.4., pp. 230]] Let $I \subset \mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ be a convex function. Then f has finite two-sided derivatives in any $x_0 \in \text{Int}(I)$ and we have

$f'_s(x_1) \leq f'_d(x_1) \leq f'_s(x_2) \leq f'_d(x_2)$, $(\forall) x_1, x_2 \in \text{Int}(I)$, with $x_1 < x_2$,
so the left-hand and the right-hand derivatives, f'_s, f'_d , are ascending .

Here we have $I = [a, b]$ and define $H : [a, b - \delta] \rightarrow \mathbb{R}$, $H(x) = f(x + \delta) - f(x)$. By Lemma we get

$$H'_d(x) = f'_d(x + \delta) - f'_d(x) \geq 0, \quad (\forall) x \in (a, b - \delta).$$

H being ascending on $(a, b - \delta)$ and continuous on $[a, b - \delta]$ will be ascending on $[a, b - \delta]$. So,

$$f(x + \delta) - f(x) = H(x) \leq H(b - \delta) = f(b) - f(b - \delta), \quad (\forall) x \in [a, b - \delta].$$

and taking into account (6) we get

$$\omega_f(\delta) = f(b) - f(b - \delta).$$

(ii) We define $F : [a, b] \rightarrow \mathbb{R}$, $F(x) = f(b + a - x)$; f being convex and descending on $[a, b]$ imply that F is convex and ascending on $[a, b]$. But $\omega_F(\delta) = \omega_f(\delta)$, $(\forall) \delta \in [0, b - a]$.

By (i) we get

$$\omega_f(\delta) = \omega_F(\delta) = F(b) - F(b - \delta) = f(a) - f(a + \delta).$$

■

Lemma 14 *Let $f \in C([a, b])$ concave on $[a, b]$ and $\delta \in [0, b - a]$.*

(i) If f is descending on $[a, b]$ then

$$\omega_f(\delta) = f(b - \delta) - f(b).$$

(ii) If f is ascending on $[a, b]$ then

$$\omega_f(\delta) = f(a + \delta) - f(a).$$

Proof. As f is concave on $[a, b]$ it follows that $(-f)$ is convex on $[a, b]$. We apply Th. 13 to $(-f)$, knowing that $\omega_{-f} = \omega_f$. ■

Example 15 Compute $\omega(\sin; \delta)$ and $\omega(\cos; \delta)$, for $0 \leq \delta \leq \frac{\pi}{2}$.