Lecture 13

6. Type II Errors, Power of a Test and the Neyman-Pearson Lemma

We are returning now to hypothesis testing. Recall that for a target parameter θ, we are testing

$$H_0 : \quad \theta = \theta_0, \text{ versus one of}$$
$$H_1 : \quad \begin{cases} \theta < \theta_0 \\ \theta > \theta_0 \\ \theta \neq \theta_0, \end{cases} \tag{6.1}$$

The "goodness" of a test is measured by the two probabilities of risk

$$\begin{aligned} \alpha &= P(\text{type I error}) &= P(\text{reject } H_0 \mid H_0) \\ \beta &= P(\text{type II error}) &= P(\text{not reject } H_0 \mid H_1). \end{aligned}$$

The smaller both of them are, the more reliable the test is. For some problems, a type I error is more dangerous, while for others, a significant type II error is unacceptable. In general, α is preset, at most 0.05 and the test is designed so that β is also small enough to be acceptable.

6.1 Type II Errors and Power of a Test

So far, type II errors were not discussed. That is because the computation of β can be more difficult. The condition that H_1 is true *does not* specify an actual value for the unknown parameter and thus, does not identify a distribution, for which the probability can be computed.

The simple condition that a parameter θ is less than, greater than or not equal to a value is not enough to help us compute the probability.
However, if the alternate H_1 is also a simple hypothesis

$$H_1 : \theta = \theta_1,$$

then β can be computed.

Thus, β, unlike α, depends on the value specified in the alternative hypothesis,

$$\beta = \beta(\theta_1).$$

Example 6.1.

Let us consider again the problem in Example 4.2. in Lecture 11 (or Example 4.4. in Lecture 10): *The number of monthly sales at a firm is known to have a mean of* 20 *and a standard deviation of* 4 *and all salary, tax and bonus figures are based on these values. However, in times of economical recession, a sales manager fears that his employees do not average* 20 *sales per month, but less, which could seriously hurt the company. For a number of* 36 *randomly selected salespeople, it was found that in one month they averaged* 19 *sales. At the* 5% *significance level, does the data confirm or contradict the manager's suspicion?*

Now let us find β for the test

$$\begin{aligned} H_0 &: \quad \mu = \mu_0 = 20 \\ H_1 &: \quad \mu = \mu_1 = 18 < 20, \end{aligned}$$

i.e. find $\beta(\mu_1)$.

Solution.

We tested a left-tailed alternative for the mean

$$H_0: \quad \mu = 20$$
$$H_1: \quad \mu < 20.$$

The population standard deviation was given, $\sigma = 4$ and for a sample of size $n = 36$, the sample mean was $\overline{X} = 19$. For the test statistic

$$TS = Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \in N(0,1),$$

the observed value was

$$Z_0 = \frac{\overline{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{19 - 20}{\frac{4}{6}} = -1.5.$$

At the significance level $\alpha = 0.05$, we have determined the rejection region

$$RR = \left\{ Z_0 = \frac{\overline{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \leq z_{0.05} \right\} = \left\{ \frac{\overline{X} - 20}{\frac{4}{6}} \leq -1.645 \right\}$$
$$= \left\{ \overline{X} \leq -1.645 \cdot \frac{4}{6} + 20 \right\} = \left\{ \overline{X} \leq 18.9 \right\}.$$

Then, in a similar fashion, we compute

$$\beta(\mu_1) = P(\text{not reject } H_0 \mid H_1) = P\left(\overline{X} > 18.9 \mid \mu = \mu_1 \right).$$

If the true value of μ is μ_1, then the statistic

$$Z_1 = \frac{\overline{X} - \mu_1}{\frac{\sigma}{\sqrt{n}}} = \frac{\overline{X} - 18}{\frac{4}{6}}$$

has a Standard Normal $N(0, 1)$ distribution.

Hence,

$$\begin{aligned}
\beta(\mu_1) &= P\left(\overline{X} > 18.9 \mid \mu = \mu_1\right) \\
&= P\left(\frac{\overline{X} - 18}{\frac{4}{6}} > \frac{18.9 - 18}{\frac{4}{6}} \mid \mu = 18\right) \\
&= P(Z_1 > 1.35 \mid Z_1 \in N(0,1)) \\
&= 1 - P(Z_1 \leq 1.35 \mid Z_1 \in N(0,1)) \\
&= 1 - \Phi(1.35) \ = \ 0.0885,
\end{aligned}$$

where

$$\Phi(x) = F_Z(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}}\, dx$$

is Laplace's function, the cdf of a $N(0,1)$ variable.

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Remark 6.2.

Let us take a closer look at the computation of α and β in the previous example. We used the fact that the variable

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

has a $N(0, 1)$ distribution. So, when the true value of μ is $\mu_0 = 20$, then

$$Z_0 = Z(\mu = \mu_0) \in N(0, 1)$$

and when the value is $\mu_1 = 18$, then

$$Z_1 = Z(\mu = \mu_1) \in N(0, 1).$$

However, in the end, we expressed the error probabilities α and β, by looking at the distribution of $\overline{X}$ *by itself*, not its reduced version.

Remark (Cont).

In other words, we used the fact that, when the true value of μ is $\mu_0 = 20$, then

$$\overline{X} \in N(\mu_0, \sigma/\sqrt{n}) \text{ and } \alpha = P(\overline{X} \leq 18.9),$$

while when the true value is $\mu_1 = 18$, then

$$\overline{X} \in N(\mu_1, \sigma/\sqrt{n}) \text{ and } \beta = P(\overline{X} > 18.9).$$

This can be seen graphically in Figure 1.

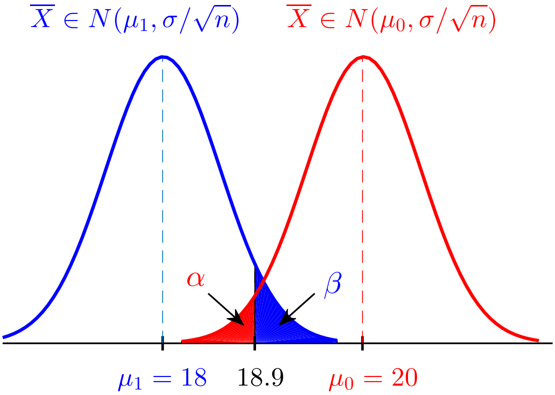

Figure 1: Type I and type II errors

In order to have a better control over β, we introduce the following notion.

Definition 6.3.

The **power of a test** on a parameter θ, unknown, is the probability of rejecting the null hypothesis

$$\pi(\theta^*) = P(\text{reject } H_0 \mid \theta = \theta^*) = P(TS \in RR \mid \theta = \theta^*), \quad (6.2)$$

when the true value of the parameter is $\theta = \theta^*$.

Notice that the power of a test is, usually, a function of the parameter θ, because the alternative hypothesis includes a set of parameter values.

Indeed, if the null hypothesis is true, i.e. $\theta = \theta_0$, then

$$\pi(\theta_0) = P(TS \in RR \mid \theta = \theta_0) = P(\text{reject } H_0 \mid H_0) = \alpha. \quad (6.3)$$

For any *other* value (in the alternative hypothesis H_1) $\theta = \theta_1 \neq \theta_0$,

$$\begin{aligned} \pi(\theta_1) &= P(\text{reject } H_0 \mid \theta = \theta_1) = P(\text{reject } H_0 \mid H_1) \\ &= 1 - P(\text{not reject } H_0 \mid H_1) = 1 - \beta(\theta_1). \quad (6.4) \end{aligned}$$

So, basically, the power of a test is the probability of rejecting a *false* null hypothesis. The larger the power is, the smaller β is, which is what we want in a test.

Then we can state a hypothesis testing problem the following way:
For a parametric test where both hypotheses are simple

$$\begin{aligned} H_0: &\quad \theta = \theta_0 \\ H_1: &\quad \theta = \theta_1, \end{aligned}$$

we preset $\alpha = \pi(\theta_0)$ and we determine a rejection region *RR* for which the power

$$\pi(\theta_1) \; = \; 1 - \beta(\theta_1)$$

is the largest possible. Such a test is called a **most powerful test**.

6.2 The Neyman-Pearson Lemma (NPL)

Most powerful tests cannot always be found. The following result gives a procedure for finding such a test, when both hypotheses tested are simple.

Lemma 6.4 (Neyman-Pearson (NPL)).

Let X be a characteristic with pdf $f(x;\theta)$, with $\theta \in A \subset \mathbb{R}$, unknown. Suppose we test on θ the simple hypotheses

$$\begin{aligned} H_0: &\quad \theta = \theta_0 \\ H_1: &\quad \theta = \theta_1, \end{aligned}$$

based on a random sample $X_1, \ldots, X_n$. Let $L(\theta) = L(X_1, \ldots, X_n; \theta)$ denote the likelihood function of this sample. Then for a fixed $\alpha \in (0, 1)$, a most powerful test is the test with rejection region given by

$$RR = \left\{ \frac{L(\theta_1)}{L(\theta_0)} \geq k_\alpha \right\}, \tag{6.5}$$

where the constant $k_\alpha > 0$ depends only on α and the sample variables.

Example 6.5.

Suppose X_1 represents a single observation from a probability density given by

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1}, & \text{if } x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Find the NPL most powerful test that at the 5% significance level tests

$$\begin{aligned} H_0: &\quad \theta = 1 \quad (= \theta_0) \\ H_1: &\quad \theta = 30 \quad (= \theta_1). \end{aligned}$$

Also, find β for that test.

Solution.

Since our sample has size 1, we have

$$\frac{L(\theta_1)}{L(\theta_0)} = \frac{f(X_1;\theta_1)}{f(X_1;\theta_0)} = \frac{30X_1^{29}}{1} = 30X_1^{29}.$$

So the rejection region given by the NPL is

$$RR = \{30X_1^{29} \geq k_\alpha\} = \{X_1 \geq K_\alpha\},$$

where $K_\alpha = \left(\frac{1}{30}k_\alpha\right)^{1/29}$.

We find the value of K_α from

$$\begin{aligned}\alpha &= P(X_1 \in RR \mid H_0) = P(X_1 \geq K_\alpha \mid \theta = 1) \\ &= \int_{K_\alpha}^{1} dx = 1 - K_\alpha,\end{aligned}$$

i.e. $K_\alpha = 1 - \alpha = 0.95$.

So, of all tests for testing H_0 versus H_1, based on a sample of size 1, the observation X_1, at the significance level $\alpha = 0.05$, the most powerful test has rejection region

$$RR = \{X_1 \geq 0.95\}.$$

For this test,

$$\begin{aligned}\beta(\theta_1) &= P(X_1 < K_\alpha \mid \theta = 30) = \int_0^{K_\alpha} 30x^{29}\, dx \\ &= x^{30}\Big|_0^{K_\alpha} = (K_\alpha)^{30} = (1-\alpha)^{30} = 0.166\end{aligned}$$

and the power is

$$\pi(\theta_1) = 1 - \beta(\theta_1) = 0.834.$$

Note that the error probability β that we obtained is unacceptably large, but considering that the estimation was based on a sample of size one, we cannot expect too much accuracy.

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Remark 6.6.

Notice that the rejection region and, hence, the most powerful test we found in Example 6.5, depend on the value stated in H_1. For a different value of θ_1, we would have found a *different* rejection region. That is usually the case.

However, sometimes, a test obtained with the NPL actually maximizes the power for every value in H_1, i.e. even if H_1 is not a simple hypothesis. Such a test is called a **uniformly most powerful test**.

Example 6.7.

Let $X_1, \ldots, X_n$ be a random sample drawn from a Normal $N(\mu, \sigma)$ distribution, with $\mu \in \mathbb{R}$ unknown and $\sigma > 0$ known. At the significance level $\alpha \in (0, 1)$, find a most powerful right-tailed test for testing

$$\begin{aligned} H_0 &: \quad \mu = \mu_0 \\ H_1 &: \quad \mu > \mu_0. \end{aligned}$$

Solution.

First we use the NPL to find a most powerful test for a simple alternative, i.e.

$$\begin{aligned} H_0 &: \quad \mu = \mu_0 \\ H_1 &: \quad \mu = \mu_1 > \mu_0. \end{aligned}$$

We have the Normal pdf

$$f(x; \mu) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(- \frac{(x - \mu)^2}{2\sigma^2} \right), \ \forall x \in \mathbb{R}.$$

The likelihood function is

$$L(\mu) \;=\; \prod_{i=1}^{n} f(X_i;\mu) \;=\; \Big(\frac{1}{\sigma\sqrt{2\pi}}\Big)^n \exp\Big(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2\Big).$$

Then, by the NPL, we find

$$\frac{L(\mu_1)}{L(\mu_0)} \;=\; \exp\Big(\frac{1}{2\sigma^2}\Big[\sum_{i=1}^{n}(X_i-\mu_0)^2 - \sum_{i=1}^{n}(X_i-\mu_1)^2\Big]\Big) \;\geq\; k_\alpha,$$

or, taking the logarithm ln (which is an increasing function) on both sides,

$$\frac{1}{2\sigma^2}\Big[\sum_{i=1}^{n}(X_i-\mu_0)^2 - \sum_{i=1}^{n}(X_i-\mu_1)^2\Big] \;\geq\; \ln k_\alpha,$$

$$\sum_{i=1}^{n}X_i^2 - 2\mu_0\sum_{i=1}^{n}X_i + n\mu_0^2 - \Big(\sum_{i=1}^{n}X_i^2 - 2\mu_1\sum_{i=1}^{n}X_i + n\mu_1^2\Big) \;\geq\; 2\sigma^2\ln k_\alpha.$$

After cancellations and using $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, we have

$$2n\overline{X}(\mu_1 - \mu_0) \;\geq\; 2\sigma^2 \ln k_\alpha + n(\mu_1^2 - \mu_0^2).$$

Since $\mu_1 > \mu_0$, we get

$$\overline{X} \;\geq\; \frac{\sigma^2 \ln k_\alpha}{n(\mu_1 - \mu_0)} + \frac{\mu_1 + \mu_0}{2} \;=\; K_\alpha.$$

Then we use the test statistic $TS = \overline{X}$, for which we found the rejection region

$$RR \;=\; \{\overline{X} \geq K_\alpha\}.$$

But

$$\begin{aligned}
\alpha &= P\Big(\overline{X} \geq K_\alpha \mid \mu = \mu_0\Big) \\
&= P\Big(\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \geq \frac{K_\alpha - \mu_0}{\sigma/\sqrt{n}} \mid \mu = \mu_0\Big) \\
&= P\Big(Z_0 \geq \frac{K_\alpha - \mu_0}{\sigma/\sqrt{n}} \mid Z_0 \in N(0,1)\Big) \\
&= P\Big(Z_0 \geq z_{1-\alpha}\Big),
\end{aligned}$$

since $Z_0 = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \in N(0,1)$.

Then we must have

$$\frac{K_\alpha - \mu_0}{\sigma/\sqrt{n}} = z_{1-\alpha}, \quad K_\alpha = \mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}},$$

so K_α is independent of μ_1.

Thus, the test with $RR = \{\overline{X} \geq K_\alpha\}$ is a *uniformly* most powerful test for testing

$$H_0: \quad \mu = \mu_0$$
$$H_1: \quad \mu > \mu_0,$$

at the significance level α.

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Remark 6.8.

In a similar manner, we can find a uniformly most powerful test for the left-tailed case

$$H_0: \quad \mu = \mu_0$$
$$H_1: \quad \mu < \mu_0.$$