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Nonlinear Applied Analysis

.

"Learn to labour and to wait"

Longfellow

Contents

Introduction

The purpose of this course is to present several classes of nonlinear operators $f : X \to X$ and to discuss different properties (existence, uniqueness, data dependence, various stability properties) of the fixed point equation

$$x = f(x), x \in X$$

in metric and topological settings. Then, using fixed point approaches and techniques, existence, uniqueness, data dependence results for different types of operatorial equations are given.

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Chapter 1

Contractive-type operators and fixed points

1.1 Background in Analysis

Metric spaces

Let (X, d) be a metric space. Recall that a metric d on a nonempty set X is a functional $d : X \times X \to \mathbb{R}_+$ satisfying the following axioms

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$ for every $x, y \in X$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for every $x, y, z \in X$.

In what follows, sometimes we will need to consider infinite-valued metrics, also called generalized metrics.

Let $(X, +, \cdot, \mathbb{R})$ be a linear space. Then a functional $p : X \to \mathbb{R}_+$ is said to be a norm on X if it satisfies the following axioms

(i) $p(x) = 0$ if and only if $x = \Theta$

(ii) $p(\lambda x) = |\lambda| p(x)$, for each $x \in X$ and $\lambda \in \mathbb{R}_+$

(iii) $p(x + y) \leq p(x) + p(y)$, for each $x, y \in X$.

Usually we denote $p(x) := \|x\|$, $x \in X$.

The pair $(X, \|\cdot\|)$, where X is a nonempty set and $\|\cdot\|$ is a norm on it is called a normed space.

It is well-known that each norm induces a metric on X, by the formula

$$d(x, y) := \|x - y\|.$$

Throughout this course, we denote by $P(X)$ the space of all nonempty subsets of a nonempty set X. By $P_{cp}(X)$ we will denote the space of all nonempty compact subsets of X.

If (X, d) is a metric space, $x_0 \in X$ and $r > 0$, then

$$B_d(x_0; r) := \{x \in X | d(x_0, x) < r\} \text{ and } \widetilde{B}_d(x_0; r) := \{x \in X | d(x_0, x) \leq r\}$$

denote the open, respectively the closed ball of radius R centered in x.

If X is a topological space and Y is a subset of X, then we will denote by $\overline{Y}$ the closure and by $intY$ the interior of the set Y.

If X is a normed space and Y is a nonempty subset of X, then coY respectively $\overline{co}Y$ denote the convex hull, respectively the closed convex hull of the set Y.

Exercise. *Consider on* $\mathbb{R}$ *the functional*

$$d(x, y) = \begin{cases} 2, & \text{if } x \neq y \\ 0, & \text{otherwise} \end{cases}$$

Show that d *is a metric on* $\mathbb{R}$ *and then determine* $B_d(0; 2)$, $\widetilde{B}_d(0; 2)$ *and* $\overline{B_d(0; 2)}$.

It is well-known that each metric space is a topological space, with the topology generated by the family of all open balls from X. It is called the metric topology on X.

Moreover, if (X, d) is a metric space, then $d : X \times X \to \mathbb{R}$ is continuous in the metric topology. A similar property holds for norms.

Sequences in metric spaces

Let (X, d) be a metric space. The sequence $(x_n)_{n\in\mathbb{N}}$ is called:

(i) Cauchy if for each $\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}^*$ such that for each $n, m \geq N_\epsilon$, $m > n$ we have $d(x_n, x_m) < \epsilon$ (or, equivalently, if $d(x_n, x_{n+p}) \to 0$ as $n, p \to +\infty$ independently).

(ii) convergent to $x^* \in X$ if for each $\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}^*$ such that for each $n \geq N_\epsilon$ we have $d(x_n, x^*) < \epsilon$.

Of course, any convergent sequence in X is Cauchy in X.

A metric space (X, d) is called complete if any Cauchy sequence in X is convergent in X. Any closed subset of a complete metric space is complete.

A Banach space is a normed space having the property that it is complete with respect to the metric induced by the norm.

Theorem 9. *A uniformly continuous function maps Cauchy sequences into Cauchy sequences.*

Proof. Let $f : (X, d) \to (Y, \rho)$ be a uniformly continuous function. Let (x_n) be a Cauchy sequence in X. To see that $(f(x_n))$ is a Cauchy sequence, let $\epsilon > 0$. Then there is a $\delta > 0$ such that for every $x, y \in X$, $d(x, y) < \delta$ implies that $\rho(f(x), f(y)) < \epsilon$. Thus there exists an $N(\epsilon) \in \mathbb{N}$ such that $d(x_m, x_n) < \delta$ for any $m, n \geq N(\epsilon)$. It follows that $\rho(f(x_m), f(x_n)) < \epsilon$. for any $m, n \geq N(\epsilon)$. Hence $(f(x_n))$ is a Cauchy sequence in Y.

Remark. If f is not uniformly continuous, then the theorem may not be true. For example, $f(x) = \frac{1}{x}$ is continuous on $]0, \infty[$ and $x_n = \frac{1}{n}$ is a Cauchy sequence in $]0, \infty[$ but $f(x_n) = n$ is not a Cauchy sequence.

Remark. If $d(x_n, x_{n+1}) \leq a_n$ for every $n \in \mathbb{N}$, and $\sum\limits_{n\geq 1} a_n < \infty$, then the sequence (x_n) is Cauchy.

Examples of complete metric spaces

1) $(\mathbb{R}^n, d)$ is a complete metric space with each of the following functionals

$$d_E(x, y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$$d_M(x, y) := \sum_{i=1}^{n} |x_i - y_i|$$

$$d_C(x, y) := \max_{i \in \{1,2,\cdots,n\}} |x_i - y_i|.$$

2) $(C([a, b], \mathbb{R}^n), d)$ is a complete metric space with each of the following functionals:

$$d_C(x, y) := \max_{t \in [a,b]} d_{\mathbb{R}^n}(x(t), y(t))$$

$$d_B(x, y) := \max_{t \in [a,b]} (d_{\mathbb{R}^n}(x(t), y(t)) \cdot e^{-\tau(t-a)}).$$

3) If (X, d) is a complete metric space, then $(P_{cp}(X), H_d)$ is a complete metric space, where $H_d : P_{cp}(X) \times P_{cp}(X) \to \mathbb{R}_+$ is the so-called Pompeiu-Hausdorff metric, and it is defined by

$$H_d(A, B) := \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}.$$

Equivalence of metrics

Let X be a nonempty set and d_1, d_2 two metrics on X.

The two metrics are said to be topologically equivalent if they generate the same topology on X. There are many equivalent ways of expressing this condition. For example:

♠ a subset A of X is d_1-open if and only if it is d_2-open

♠ the identity function $I : X \to$X is both (d_1, d_2)-continuous and (d_2, d_1)-continuous.

By definition, two metric d_1 and d_2 on X are called metrically (strongly) equivalent if for each $x \in X$ there exists $c_1, c_2 > 0$ such that

$$c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y), \text{ for each } y \in X.$$

If two metrics are strongly equivalent then they also are topologically equivalent. But the reverse implication, in generally, does not hold.

Examples of Banach spaces

1) $(\mathbb{R}^n, \|\cdot\|)$ is a Banach space with each of the following functionals

$$\|x\|_E := \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$\|x\|_M := \sum_{i=1}^{n} |x_i|$$

$$\|x\|_C := \max_{i \in \{1,2,\cdots,n\}} |x_i|.$$

2) $(C([a,b],\mathbb{R}^n), d)$ is a Banach space with each of the following functionals:

$$\|x\|_C := \max_{t \in [a,b]} \|x(t)\|_{\mathbb{R}^n}$$

$$\|x\|_B := \max_{t \in [a,b]} (\|x(t)\|_{\mathbb{R}^n} \cdot e^{-\tau(t-a)}).$$

Remark. The unit interval $[0,1]$ is a complete metric space, but it is not a Banach space because it is not a linear space.

Equivalence of norms

By definition, two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are called topologically equivalent if they generate the same topology on X.

By definition, two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are called strongly equivalent if for each $x \in X$ there exists $c_1, c_2 > 0$ such that

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1.$$

If two norms are strongly equivalent then they also are topologically equivalent. But the reverse implication, in generally, does not hold.

1.2 The Banach-Caccioppoli contraction principle

Some definitions first.

Definition. If X is a nonempty set and $f : X \to X$ an operator then $x \in X$ is called a fixed point for f iff $x = f(x)$. Denote by $Fix(f)$ the fixed point set of f. Also, denote by $I(f) := \{Y \in P(X) |\ f(Y) \subset Y\}$ the set of all nonempty invariant subsets of f.

Definition. Let X be a nonempty set, $x \in X$ and $f : X \to X$ be an operator. Then the sequence of successive approximations $(x_n)_{n\in\mathbb{N}} \subset X$ for f starting from x is defined as follows:

$$x_0 = x,\ x_n = f^n(x),\ \text{for } n \in \mathbb{N},$$

where $f^0 := 1_X,\ f^1 := f, \dots, f^{n+1} = f \circ f^n,\ n \in \mathbb{N}$ are the iterate operators of f. As consequence, we also have the following recurrence relation:

$$x_0 = x,\ x_{n+1} = f(x_n),\ n \in \mathbb{N}.$$

Definition. If (X, d) is a metric space and $f : X \to X$ is an operator, then f is said to be:

i) α-Lipschitz if there is $\alpha \in \mathbb{R}_+$ such that for every $x, y \in X$ we have $d(f(x), f(y)) \leq \alpha d(x, y)$;

ii) α-contraction if it is α-Lipschitz with $\alpha \in [0, 1[$;

iii) nonexpansive if it is 1-Lipschitz;

iv) contractive if for each $x, y \in X,\ x \neq y$ we have $d(f(x), f(y)) < d(x, y)$.

The Banach-Caccioppoli contraction principle.

Let (X, d) be a complete metric space and Y a closed subset of X. Let $f : Y \to Y$ be an α-contraction. Then we have the following conclusions:

(i) $Fix(f) = \{x^*\}$;

(ii) for each $x \in Y$ the sequence of successive approximations (i.e. $x_0 = x,\ x_n := f^n(x_0),\ n \geq 1$) for f starting from x converges to x^*;

(iii) $d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha} \cdot d(x_0, f(x_0))$, *for each* $n \in \mathbb{N}$.

(iv) $d(x_{n+1}, x^*) \leq \frac{\alpha}{1-\alpha} \cdot d(x_n, x_{n+1})$, *for each* $n \in \mathbb{N}$.

Steps of the proof.

1) $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, since

$$(*) \quad d(x_n, x_{n+p}) \leq \frac{\alpha^n}{1-\alpha} \cdot d(x_0, x_1), \text{ for each } n \in \mathbb{N} \text{ and each } p \in \mathbb{N}^*.$$

2) $\lim\limits_{n\to+\infty} x_n = x^* \in Fix(f)$.

3) the uniqueness of the fixed point, by reductio ad absurdum.

4) Using $(*)$ we get (iii) and (iv). □

Exercises and examples.

1) Let $f \in C^1(\mathbb{R})$. Then

$$f \text{ is } \alpha - \text{Lipschitz} \text{ if an only if } |f'(x)| \leq \alpha, \text{ for each } x \in \mathbb{R}.$$

Hint. Use the Mean Value Theorem.

2) Let $f :]0,1[\to]0,1[,\ f(x) = \frac{x}{2}$ and $g : \mathbb{R} \to \mathbb{R},\ g(x) = \frac{\pi}{2} + x - arctan x$.

Show that f and g are fixed point free mappings. Please comment the connection with Banach-Caccioppoli theorem.

3) Let $f :]0,1[\to \mathbb{R},\ f(x) = 2x(1-x)$ (the logistic operator).

(a) Find $Fix(f)$

(b) Is f a contraction on $[0,1]$?

(c) Find a closed invariant subset $A \subset [0,1]$ such that f to be contraction on A.

4) Consider $c > 0$ and the sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ given by

$$x_0 = 1,\ x_{n+1} = \frac{1}{2}(x_n + \frac{c}{x_n}), n \in \mathbb{N}.$$

a) Show (by fixed point methods) that $\lim_{n\to+\infty} x_n = \sqrt{c}$;

b) Find $\sqrt{2}$ with an error less than 10^{-2}.

Hint. Use the Banach-Caccioppoli theorem. □

1.3 Consequences, applications and extensions

Let present first a data dependence result.

The continuous dependence of the fixed point of the Banach-Caccioppoli contraction principle.

Let (X, d) be a complete metric space and $f, g : X \to X$ such that:
(i) f is an α-contraction (denote by x_f^* its unique fixed point);
(ii) There exists $x_g^* \in Fix(g)$;
(iii) There exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for each $x \in X$.
Then $d(x_f^*, x_g^*) \leq \frac{\eta}{1-\alpha}$.

Remark. J. Hadamard (1902) defines the concept of well-posedness for a certain mathematical problem.

Remark. A.N. Tychonov define another concept of well-posed problem (in the sense of Tychonov).

Definition. Let (X, d) be a metric space and $f : X \to X$ be an operator. Consider the fixed point problem

$$x = f(x), \ x \in X. \tag{1.1}$$

We say that the fixed point problem for the operator f is well-posed if $Fix(f) = \{x^*\}$ and, if $x_n \in X$, $n \in \mathbb{N}$ is a sequence such that

$$d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty,$$

then

$$x_n \to x^* \text{ as } n \to \infty.$$

Theorem. *Let (X, d) be a complete metric space and $f : X \to X$ an α-contraction. Then, the fixed point problem for f is well-posed.*

Definition. Let (X, d) be a metric space and $f : X \to X$ be an operator. Consider the fixed point problem

$$x = f(x),\ x \in X. \tag{1.2}$$

We say that the operator f has:

(a) the limit shadowing property if for any sequence $(y_n)_{n\in\mathbb{N}} \subset X$ with the property

$$(d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

there exists $x \in X$ such that

$$d(y_n, f^n(x)) \to 0 \text{ as } n \to \infty.$$

(b) the Ostrowski's property if $Fixf = \{x^*\}$ and for any sequence $(y_n)_{n\in\mathbb{N}} \subset X$ with the property

$$(d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$y_n \to x^* \text{ as } n \to \infty.$$

For our next result we need the following auxiliary result.

Cauchy's Lemma. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of non-negative real numbers, such that $\sum_{k=0}^{+\infty} a_k < +\infty$ and $\lim_{n\to+\infty} b_n = 0$. Then*

$$\lim_{n\to+\infty}(\sum_{k=0}^{n} a_{n-k}b_k) = 0.$$

Theorem. *Let (X, d) be a complete metric space and $f : X \to X$ an α-contraction. Then, the operator f has the Ostrowski's property and the limit shadowing property.*

Proof. By the Contraction Principle, we know that $Fix(f) = \{x^*\}$.

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X such that $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$.

We shall prove first that $d(y_n, x^*) \to 0$ as $n \to +\infty$. We successively have:

$$d(x^*, y_{n+1}) \leq d(x^*, f(y_n)) + d(y_{n+1}, f(y_n)) = d(f(x^*), f(y_n)) + d(y_{n+1}, f(y_n))$$

$$\leq \alpha d(x^*, y_n) + d(y_{n+1}, f(y_n)) \leq \alpha[\alpha d(x^*, y_{n-1}) + d(y_n, f(y_{n-1}))] +$$

$$d(y_{n+1}, f(y_n)) \leq \cdots \leq \alpha^{n+1} d(x^*, y_0) + \alpha^n d(y_1, f(y_0)) + \cdots + d(y_{n+1}, f(y_n)).$$

By Cauchy's Lemma, the right hand side tends to 0 as $n \to +\infty$. Thus, $d(x^*, y_{n+1}) \to 0$ as $n \to +\infty$. This shows that f has the Ostrowski's property.

Now, for arbitrary $x \in X$, we have

$$d(y_n, f^n(x)) \leq d(y_n, x^*) + d(x^*, f^n(x)) \to 0 \text{ as } n \to +\infty.$$

Definition. Let (X, d) be a metric space and $f : X \to X$ be an operator. Consider the fixed point problem

$$x = f(x),\ x \in X. \tag{1.3}$$

We say that the operator f has the Ulam-Hyers stability property if there exists $c > 0$, such that for every $\varepsilon > 0$ and every ε-solution y^* of the fixed point problem $x = f(x)$ (which means that y^* satisfies the following relation

$$d(y^*, f(y^*)) \leq \varepsilon),$$

there exists a solution $x^* \in X$ of the fixed point problem $x = f(x)$ such that

$$d(x^*, y^*) \leq c\varepsilon.$$

Theorem. *Let (X, d) be a complete metric space and $f : X \to X$ be an α-contraction. Then, the fixed point problem $x = f(x)$ has the Ulam-Hyers property.*

The local form of the Banach-Caccioppoli contraction principle.

Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $f : B(x_0; r) \to X$ be an α-contraction such that $d(x_0, f(x_0)) < (1-\alpha)r$. Then $Fixf \neq \emptyset$.

Hint.

Consider $0 < \epsilon < r$ such that $d(x_0, f(x_0)) \leq (1-\alpha)\epsilon < (1-\alpha)r$. Show that $\widetilde{B}(x_0; \epsilon) \in I(f)$ and apply the Banach-Caccioppoli theorem. $\Box$

An application of the above results are the so-called domain invariance principles.

Let E be a Banach space and $X \subset E$. Let $f : X \to E$ be an operator. Then the operator $g : X \to E$ defined by $g(x) = x - f(x)$ is said to be the field generated by f.

We have the following result.

Theorem.

Let E be a Banach space and U an open subset of it. Let $f : U \to E$ be an α-contraction.

Then the following conclusions hold:

i) the field g generated by f is an open operator, i.e., the image of any open set is open too;
ii) $g(U)$ is open in E;
iii) $g : U \to g(U)$ is a homeomorphism.

Sketch of the proof.

i) g is open operator if and only if for any V an open subset of U the set $g(V)$ is open in E too. For, it's enough to prove that for any $y \in g(V)$ there exists W an open neighborhood of y such that $W \subset g(V)$.

In order to get the conclusion, one can prove first that the following implication holds:

$$\text{for } u \in V \text{ and each } B(u; r) \subset V \Rightarrow B(g(u); (1-\alpha)r) \subset g(B(u; r)).$$

In order to prove it, we will apply the local form of the Banach-Caccioppoli principle. Indeed, let $u \in V$ and $y \in B(g(u);(1-\alpha)r)$, i.e., $\|y - g(u)\| < (1-\alpha)r$. We have to show that $y \in g(B(u;r))$, which means that there exists $x \in B(u;r)$ such that $y = g(x)$. This means that we are looking for $x \in B(u;r)$ such that $y + f(x) = x$. For this conclusion, it is enough to apply the local form of the Banach-Caccioppoli principle for $h : B(u;r) \to X$, $h(x) = y + f(x)$. We can do it, since h is an α-contraction and $d(u, h(u)) = \|u - h(u)\| = \|u - f(u) - y\| = \|g(u) - y\| < (1-\alpha)r$.

Now, let V be an open subset of U and take any $y \in g(V)$. Then, there exists $u \in V$ such that $y = g(u)$. Notice that, since V is open, there exists $B(u;r) \subset V$. Take $W := B(g(u);(1-\alpha)r)$. Hence, by the above proof, we have $W \subset g(B(u;r)) \subset g(V)$.

(ii) Apply (i) for $U = V$.

(iii) $g : U \to g(U)$ is surjective and continuous. Moreover, it is also injective since

$$\|g(x) - g(y)\| = \|x - f(x) - y + f(y)\| = \|(x-y) - (f(x) - f(y))\| \geq$$

$$\|x - y\| - \|f(x) - f(y)\| \geq (1-\alpha)\|x - y\|.$$

Then, if $g(x) = g(y)$, then $x = y$. Additionally, g^{-1} is continuous too, since for any open set $V \subset g(U)$ we have that $(g^{-1})^{-1}(V) = g(V)$ is open (by (i)). Thus, g it is a homeomorphism. □

Other local fixed point theorems are the following.

Theorem.

Let E be a Banach space and let $f : \widetilde{B}(0;r) \to E$ be an α-contraction, such that $f(\partial \widetilde{B}(0;r)) \subset \widetilde{B}(0;r)$. Then $Fix(f) = \{x^*\}$.

Proof. Let us define, for $x \in \widetilde{B}(0;r)$

$$G(x) := \frac{1}{2}(x + f(x)).$$

Then we have:

(i) $Fix(f) = Fix(G)$;

(ii) $G : \widetilde{B}(0;r) \to \widetilde{B}(0;r)$ (take $G(x) := \frac{1}{2}(x + f(u) + f(x) - f(u))$, where $u := \frac{r}{\|x\|}x$); Indeed, we have

$$\|G(x)\| = \|\frac{1}{2}(x+f(u)+f(x)-f(u))\| \leq \frac{1}{2}(\|x\|+\|f(u)\|+\|f(x)-f(u)\|) \leq$$

$$\frac{1}{2}(\|x\|+r+\alpha\|x-u\|) \leq \frac{1}{2}\left[\|x\| + r + \alpha(r - \|x\|)\right] = \frac{1}{2}\left[(1+\alpha)r + (1-\alpha)\|x\|\right] \leq$$

$$\frac{1}{2}\left[(1-\alpha)r + (1+\alpha)r\right] = r.$$

(iii) G is $\frac{1+\alpha}{2}$-contraction.

Theorem.

Let E be a Banach space and let $f : \widetilde{B}(0;r) \to E$ be an α-contraction, such that $f(-x) = -f(x)$, for each $x \in \partial\widetilde{B}(0;r)$. Then $Fix(f) = \{x^*\}$.

Proof. Show that $f(-x) = -f(x)$, for each $x \in \partial\widetilde{B}(0;r)$ implies that $f(\partial\widetilde{B}(0;r)) \subset \widetilde{B}(0;r)$. (indeed, for $x \in \partial\widetilde{B}(0;r)$ we have:

$$2\|f(x)\| = \|f(x) - f(-x)\| \leq 2\alpha\|x\|.$$

Thus, $\|f(x)\| \leq \alpha\|x\| = \alpha r < r$.

Exercise. *Show, by fixed point methods, that for each $y \in \mathbb{R}$ the equation $x - \frac{1}{3}sinx = y$ has a unique solution in $\mathbb{R}$.*

Continuation results for contractions

Let (X, d) be a complete metric space and Y a closed subset such that $intY \neq \emptyset$. Denote by $CR(Y,X)$ the family of all contractions from Y to X. Let (J, ρ) the metric space of parameters.

Definition. The family $(H_\lambda)_{\lambda\in J} \subset CR(Y,X)$ is said to be α-contractive if $\alpha \in [0,1[$ and there is $M > 0$ and $p \in]0,1]$ such that:

(i) $d(H_\lambda(x_1), H_\lambda(x_2)) \leq \alpha d(x_1, x_2)$, for each $x_1, x_2 \in Y$ and $\lambda \in J$;

(ii) $d(H_\lambda(x), H_\mu(x)) \leq M[\rho(\lambda,\mu)]^p$, for each $x \in Y$ and $\lambda, \mu \in J$.

Denote by $A := \partial Y$, $U := intY$ and by $CR_A(Y,X) := \{f \in CR(Y,X) : f_{|A} : A \to X$ is fixed point free $\}$.

Theorem. *Let (X,d) be a complete metric space and Y a closed subset such that $intY \neq \emptyset$. Let (J,ρ) be a connex metric space and $(H_\lambda)_{\lambda\in J}$ be an α-contractive family from $CR_A(Y,X)$. Then:*

(i) if there exists $\lambda_0^* \in J$ such that the equation $H_{\lambda_0^*}(x) = x$ has a solution then the equation $H_\lambda(x) = x$ has a unique solution for every $\lambda \in J$;

(ii) if $H_\lambda(x_\lambda) = x_\lambda$ for $\lambda \in J$ then the operator $j : J \to intY$ given by $j(\lambda) = x_\lambda$ is continuous.

Kannan's fixed point theorem

The following result is a fixed point theorem for operators which are not necessarily continuous.

Theorem. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be a Kannan type contraction, i.e. there exists* $\alpha \in]0, \frac{1}{2}[$ *such that*

$$d(f(x), f(y)) \leq \alpha[d(x, f(x)) + d(y, f(y))], \text{ for each } x, y \in X.$$

Then we have the following conclusions:

(i) $Fix(f) = \{x^*\}$*;*

(ii) for each $x \in Y$ *the sequence of successive approximations (i.e.* $x_0 \in X,\ x_n := f^n(x_0),\ n \geq 1$*) for* f *starting from* x *converges to* x^**.*

Steps of the proof.

Let $x_0 \in X$ be arbitrary and $x_n := f^n(x_0)$, for $n \geq 1$. Thus $x_{n+1} = f(x_n)$, $n \in \mathbb{N}$. Then we have:

1) $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence; Notice first that

$$d(x_n, x_{n+1}) \leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n) \leq k^n d(x_0, x_1), \text{ for each } n \geq 1,$$

where $k := \frac{\alpha}{1-\alpha} < 1$. Thus, as a consequence, we obtain

$$d(x_n, x_{n+p}) \leq \frac{k^n}{1-k} \cdot d(x_0, x_1), \text{ for each } n \in \mathbb{N} \text{ and each } p \in \mathbb{N}^*.$$

2) $\lim\limits_{n\to+\infty} x_n = x^* \in X$, by the completeness of the space.

3) $x^* \in Fix(f)$ since we can write $d(x^*, f(x^*)) \leq d(x^*, f(x_n)) + d(f(x_n), f(x^*)) \leq d(x^*, x_{n+1}) + \alpha\left(d(x_n, x_{n+1}) + d(x^*, f(x^*))\right)$. Thus

$$d(x^*, f(x^*)) \leq \frac{\alpha}{1-\alpha}\left[d(x^*, x_{n+1}) + \alpha d(x_n, x_{n+1})\right] \to 0, \text{ as } n \to \infty.$$

4) the uniqueness of the fixed point follows by contradiction. □

Exercise. *Let* $f : [0, 1] \to [0, 1]$ *be defined by*

$$f(x) := \begin{cases} \frac{x}{4}, & x \in [0, \frac{1}{2}[\\ \frac{x}{5}, & x \in [\frac{1}{2}, 1], \end{cases}$$

Show that f is not a contraction, but f is a Kannan type contraction with $\alpha = \frac{4}{9}$.

Exercise. *Show that $f : [0,1] \to [0,1]$ $f(x) = \frac{x}{3}$ is a $\frac{1}{3}$-contraction, but it is not a Kannan type contraction (Take $x = \frac{1}{3}$ and $y = 0$).*

Exercise. *Under the assumptions of Kannan's fixed point theorem show that the fixed point problem $x = f(x)$ is well-posed and has the Ulam-Hyers property.*

A generalization of both Contraction Principle and Kannan's fixed point theorem is the following result proved by L. Ćirić.

Ćirić's fixed point theorem

Theorem. *Let (X, d) be a complete metric space and $f : Y \to Y$ be a Ćirić-type contraction, i.e. there exists $\alpha \in]0,1[$ such that for each $x, y \in X$ we have:*

$$d(f(x), f(y)) \leq \alpha \cdot \max\{d(x,y), d(x,f(x)), d(y,f(y)), \frac{1}{2}[d(x,f(y)) + d(y,f(x))]\}.$$

Then we have the following conclusions:

(i) $Fix(f) = \{x^*\}$;

(ii) for each $x \in Y$ the sequence of successive approximations (i.e. $x_0 = x,\ x_n := f^n(x_0),\ n \geq 1$) for f starting from x converges to x^*.

1.4 The Nemitki-Edelstein fixed point principle

Recall that if (X, d) is a metric space, then an operator $f : X \to X$ is called contractive if:

$$x, y \in X,\ x \neq y \text{ implies } d(f(x), f(y)) < d(x, y).$$

Theorem.

Let (X, d) be a compact metric space and $f : X \to X$ be a contractive operator. Then $Fix(f) = \{x^*\}$ and for each $x \in Y$ the sequence of successive approximations (i.e. $x_0 = x,\ x_n := f^n(x_0),\ n \geq 1$) for f starting from x converges to x^*.

Sketch of the proof. Since X is compact and the functional $h(x) = d(x, f(x))$ is continuous from X to $\mathbb{R}$ there exists $x^* \in X$ such that $h(x^*) = \inf\limits_{x \in X} h(x)$. Next, show, by reductio ad absurdum, that $x^* \in Fix(f)$. Indeed, suppose $x^* \neq f(x^*)$. Then, $h(f(x^*)) = d(f(x^*), f^2(x^*)) < d(x^*, f(x^*)) = h(x^*)$, which is a contradiction. The uniqueness is an easy consequence of the contractive condition. Hence $Fix(f) = \{x^*\}$.

For the convergence property, notice first that $x_{n+1} = f(x_n)$. Since X is compact, there exists a convergent subsequence x_{n_k}. Moreover, since $x_{n_k+1} = f(x_{n_k})$, using the continuity of f, we get (by passing to limit) that x_{n_k} converges to $x^* \in Fix(f)$.

Let us consider now any convergent subsequence (x_{n_p}) of x_n. Suppose (x_{n_p}) conveges to some $l \in X$. Then, since $x_{n_p+1} = f(x_{n_p})$, we obtain again that $l \in Fix(f)$ and so $l = x^*$. Thus x_{n_p} converges to x^*, as $p \to +\infty$. Now, since each convergent subsequence of (x_n) has the fixed point x^* of f as the limit point, it follows (by a well-known result in functional analysis) that the whole sequence (x_n) converges to x^*. $\square$

Exercise. Show that even $f : \mathbb{R} \to \mathbb{R},\ f(x) = \ln(1+e^x)$ is contractive, nevertheless $Fix(f) = \emptyset$. Why ?

1.5 Meir-Keeler Fixed Point Theorem

Let (X, d) be a metric space. An operator $f : X \to X$ is called a Meir-Keeler operator if for each $\epsilon > 0$ there exists $\delta > 0$ such that the following implication holds:

$$x, y \in X \ \epsilon \leq d(x, y) < \epsilon + \delta \ \Rightarrow \ d(f(x), f(y)) < \epsilon.$$

Remark. Any Meir-Keeler operator is contractive and, hence, continuous. Indeed, if we chose $x, y \in X$ with $x \neq y$, then, by taking $\epsilon := d(x, y) > 0$ we obtain that $d(f(x), f(y)) < \epsilon = d(x, y)$.

Meir-Keeler Theorem. *Let (X, d) be a complete metric space and $f : X \to X$ be a Meir-Keeler operator. Then:*

(i) $F_f = \{x^*\}$;

(ii) the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges to x^*, for each $x \in X$.

Proof.

Step 1. The Meir-Keeler condition implies that $d(x_n, x_{n+1}) \to 0$ as $n \to +\infty$, where $x_n := f^n(x)$, $x \in X$.

Indeed, if we denote $c_n := d(x_n, x_{n+1})$, $n \in \mathbb{N}$, then, since (c_n) is deacresing and positive, it is convergent to a certain $\eta \geq 0$. Suppose that $\eta > 0$. Then there exists $\delta > 0$ such that the following implication holds:

$$\eta \leq c_n < \eta + \delta \text{ implies } c_{n+1} < \eta.$$

This is a contradiction with the fact that (c_n) is deacreasing to η. Hence Step 1 is proved.

Step 2. The sequence $x_n := f^n(x)$, $x \in X$ is Cauchy.

Indeed, suppose by contradiction that there exists $x_0 \in X$ such that the sequence $x_n := f^n(x_0)$ is not Cauchy. Then, there exists $\epsilon > 0$ such that $\lim\limits_{n,m \to +\infty} d(x_m, x_n) > 2\epsilon$. By Meir-Keeler condition, there exists $\delta > 0$ such that

$$x, y \in X \ \epsilon \leq d(x, y) < \epsilon + \delta \ \Rightarrow \ d(f(x), f(y)) < \epsilon.$$

Choose $\delta' := \min\{\delta, \epsilon\}$. By Step 1, we get that there exists $M \in \mathbb{N}^*$ such that $c_M < \frac{\delta'}{\epsilon}$. Let $m, n > M, n > m$ be such that $d(x_m, x_n) > 2\epsilon$. For any $j \in [m, n]$ we have

$$|d(x_m, x_j) - d(x_m, x_{j+1}| \leq c_j = d(x_j, x_{j+1}) < \frac{\delta'}{3}.$$

Now, since $d(x_m, x_{m+1}) < \epsilon$ and $d(x_m, x_n) > 2\epsilon = \epsilon + \epsilon \geq \epsilon + \delta'$, there exists $j \in [m, n]$ such that

$$\epsilon + \frac{2\delta'}{3} < d(x_m, x_j) < \epsilon + \delta'.$$

Thus,

$$\epsilon \leq \epsilon + \frac{2\delta'}{3} < d(x_m, x_j) < \epsilon + \delta' < \epsilon + \delta.$$

For all m and j we have that $d(x_m, x_j) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{j+1}) + d(x_{j+1}, x_j)$ and, therefore, by the above estimations we get that $d(x_m, x_j) \leq d(x_m, x_{m+1}) + d(f(x_m), f(x_j)) + d(x_{j+1}, x_j) \leq c_m + \epsilon + c_j < \frac{\delta'}{3} + \epsilon + \frac{\delta'}{3} = \frac{2\delta'}{3} + \epsilon$, a contradiction.

Step 3. $Fix(f) = \{x^*\}$ and $\lim\limits_{n \to +\infty} f^n(x) = x^*$ for each $x \in X$.

Indeed, denote first $x^*(x) := \lim\limits_{n \to +\infty} f^n(x)$ for $x \in X$. Next, by the above Remark we know that f is contractive and hence continuous. Thus $x^*(x) \in Fix(f)$. By the contractive condition we obtain the uniqueness of the fixed point.

The proof is now complete.

1.6 Krasnoselskii's Theorem

Theorem. (Cantor) *Let (X, d) be a complete metric space and Y_n, $n \in \mathbb{N}$ be nonempty closed subsets of X such that $Y_{n+1} \subset Y_n$, $n \in \mathbb{N}$ and $\delta(Y_n) \to 0$ as $n \to \infty$.*

Then, $\bigcap\limits_{n \in \mathbb{N}} Y_n = \{x^*\}$.

Using Cantor's theorem we have:

Theorem (1972) *Let* (X,d) *be a complete metric space and* $f : X \to X$ *be an operator. Suppose that for each* $0 < a \le b < +\infty$ *there is* $l(a,b) \in [0,1[$ *such that*

$$x, y \in X,\ a \le d(x,y) \le b \text{ implies } d(f(x), f(y)) \le l(a,b)d(x,y).$$

Then we have:

(i) $Fixf = Fixf^n = \{x^*\}$*;*

(ii) the sequence $(f^n(x))_{n\in\mathbb{N}}$ *converges to* x^**, for each* $x \in X$*.*

Proof. Step 1. We prove that for each $r > 0$ there exists $\tilde{B}(x;r) \subset X$ such that $\tilde{B}(x;r) \in I(f)$.

We will use the "reductio ad absurdum" method: suppose there exists $r > 0$ such that fo each $x \in X$ one have $\tilde{B}(x;r) \notin I(f)$. Then, there exists $x_1 \in X$ such that $d(x,x_1) \le r$ and $d(x, f(x_1)) > r$. We have two cases:

a) $d(x,x_1) \le \frac{r}{2}$. Then $d(x,x_1) \le d(x,x_1) \le \frac{r}{2}$ implies $d(f(x), f(x_1)) \le l(d(x,x_1), \frac{r}{2})d(x,x_1) < \frac{r}{2}$. Thus, $d(x,f(x)) \ge d(x,f(x_1)) - d(f(x_1), f(x)) \ge r - \frac{r}{2} = \frac{r}{2}$.

b) $d(x,x_1) > \frac{r}{2}$. Then $\frac{r}{2} < d(x,x_1) \le r$ implies $d(f(x), f(x_1)) \le l(\frac{r}{2}, r)d(x,x_1)$. Hence $d(x,f(x)) \ge d(x,f(x_1)) - d(f(x_1), f(x)) \ge r - l(\frac{r}{2}, r)d(x,x_1) \ge r - l(\frac{r}{2}, r)\cdot r = r[1 - l(\frac{r}{2}, r)]$.

Thus, in both cases we have:

$$(*)\ \ d(x, f(x)) \ge \min\{\frac{r}{2}, r[1 - l(\frac{r}{2}, r)]\} := a.$$

On the other hand, for each $x_0 \in \tilde{B}(x;r)$ we have that $a \le d(x_0, f(x_0)) \le d(x_0, f(x_0)) := b$ implies $d(f(x_0), f^2(x_0)) \le l(a,b)d(x_0, f(x_0)) < d(x_0, f(x_0))$. Thus

$$a \le d(f^k(x_0), f^{k+1}(x_0)) \le l^k(a,b)\cdot d(x_0, f(x_0)) \to 0, \text{ as } k \to +\infty.$$

Hence $u_k := d(f^k(x_0), f^{k+1}(x_0)) \to 0$, $k \to +\infty$. As consequence, for each $\epsilon > 0$ there is $k(\epsilon) \in \mathbb{N}^*$ such that for each $k \ge k(\epsilon)$ one have that $u_k < a$.

In particular, for $\epsilon := a$ there is a $k^* \in \mathbb{N}^*$ such that for each $k \geq k^*$ we have $u_{k^*} < a$. Hence, $u_{k^*} = d(f^{k^*}(x_0), f^{k^*+1}(x_0)) = d(x, f(x)) < a$ (where $x := f^{k^*}(x_0)$). The contradiction shows that Step 1 is proved.

STEP 2. There exists a decreasing sequence $B_1, B_2, \cdots, B_n, \cdots$ of closed balls such that $diam B_n \to 0$, as $n \to +\infty$.
Indeed, let $B_1 := \tilde{B}(x, 1) \in I(f)$. For $f : B_1 \to B_1$ we can apply Step 1 and we get that there exists $B_2 \in B_1$ ($B_2 := \tilde{B}(x, \frac{1}{2})$) such that $B_2 \in I(f)$. By this procedure we also get $B_n := \tilde{B}(x, \frac{1}{n}) \in I(f), \cdots$. Since $diam B_n \to 0$ as $n \to +\infty$, we obtain, by Cantor's theorem, that $\bigcap_{n \in \mathbb{N}^*} B_n = \{x^*\} \in I(f)$. Thus $x^* \in Fix(f)$.

STEP 3. The uniqueness of the fixed point.
Suppose that $x^*, y^* \in Fixf$. Then $d(x^*, y^*) = d(f(x^*), f(y^*)) \leq l(d(x^*, y^*), d(x^*, y^*))d(x^*, y^*) < d(x^*, y^*)$, which represents a contradiction.

STEP 4. Let $x \in X$ with $x \neq x^*$. We have $d(f^n(x), x^*) \to 0$, as $n \to +\infty$.
Indeed, since the sequence $(d(f^n(x), x^*))_{n \in \mathbb{N}}$ is decreasing, it is convergent too. If, by contradiction $d(f^n(x), x^*) \to u > 0$ as $n \to +\infty$, then $d(f^n(x), x^*) \leq l(u, d(x, x^*))^n \cdot d(x, x^*) \to 0$ as $n \to +\infty$. Thus $d(f^n(x), x^*) \to 0$ as $n \to +\infty$.

Finally, notice that from (ii) we obtain $Fixf^n = Fixf = \{x^*\}$.
The proof is now complete. □

Remark. *i) Let (X, d) be a compact metric space and $f : X \to X$ be a contractive operator. Then f is a generalized contraction in Krasnoselskii' sense.*

ii) Let (X, d) be a metric space, $f : X \to X$ and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous mapping such that $\gamma(t) > 0$ and for each $t > 0$. Suppose that for each $x, y \in X$ the following assertion is satisfied:

$$d(f(x), f(y)) \leq d(x, y) - \gamma(d(x, y)).$$

Then f is a generalized contraction in Krasnoselskii' sense.

A local result is the following:

Theorem. *Let E be a Banach space and $f : B := \tilde{B}(0; r) \to E$ be a generalized contraction in Krasnoselskii' sense. Suppose $f(\partial B) \subset B$. Then $Fixf = \{x^*\}$.*

1.7 Graphic Contraction Principle

Let (X, d) be a metric space and $f : X \to X$ be an operator. We denote by

$$Graph(f) := \{(x, f(x)) : x \in X\}$$

the graph of the operator f.

The contraction condition means that there exists $\alpha \in]0, 1[$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y), \text{ for every } (x, y) \in X \times X.$$

If the above condition is assumed not for all $(x, y) \in X \times X$, but only for $(x, y) \in Graph(f) := \{(x, f(x)) : x \in X\}$, then we obtain a weaker assumption on f. The problem is now if we can obtain existence, uniqueness, data dependence of the fixed points of f under this weaker assumption on f. The following result is an existence theorem for the solution of the fixed point equation $x = f(x)$ under the assumption that f is a graphic α-contraction.

Theorem (1972). *Let (X, d) be a complete metric space, $f : X \to X$ and $\alpha \in [0, 1[$. We suppose that:*

(a) f is a graphic α-contraction, i.e., $d(f(x), f^2(x)) \leq \alpha d(x, f(x))$, for all $x \in X$;

(b) the operator f has closed graph, i.e., the set $Graph(f)$ is closed in $X \times X$.

Then:

(i) $Fix(f) \neq \emptyset$;

(ii) $f^n(x) \to f^\infty(x)$ as $n \to \infty$, and $f^\infty(x) \in Fix(f)$, $\forall\, x \in X$;

(iii) $d(x, f^\infty(x)) \leq \dfrac{1}{1-\alpha} d(x, f(x))$, for all $x \in X$.

Proof. (i)+(ii). Let $x \in X$ be arbitrary chosen. By (a), we have that $x_n := f^n(x)$, for $n \in \mathbb{N}$ is a Cauchy sequence. Indeed, for any $x \in X$, we have

$$d(x_n, x_{n+1}) = d(f^n(x), f^{n+1}(x)) \leq \alpha d(f^{n-1}(x), f^n(x)) \leq \cdots \leq \alpha^n d(x, f(x)).$$

Then

$$d(x_n, x_{n+p}) \le \frac{\alpha^n}{1-\alpha} \cdot d(x, f(x)), \text{ for each } n \in \mathbb{N} \text{ and each } p \in \mathbb{N}^*.$$

Thus, $d(x_n, x_{n+p}) \to 0$ as $n, p \to \infty$. This shows that (x_n) is Cauchy.

Since (X, d) is a complete metric space it follows that $(f^n(x))_{n\in\mathbb{N}}$ is convergent and we denote by $f^\infty(x)$ its limit. By (b), since $x_{n+1} = f(x_n)$ for each $n \in \mathbb{N}$, we have that $f^\infty(x) \in Fix(f)$, i.e., $Fix(f) \neq \emptyset$.

(iii) We can write that

$$d(x, f^{n+1}(x)) \le d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^n(x), f^{n+1}(x))$$

$$\le (1 + \alpha + \alpha^2 + \cdots + \alpha^n) d(x, f(x)).$$

Then, letting $n \to \infty$, we have

$$d(x, f^\infty(x)) \le \frac{1}{1-\alpha} d(x, f(x)), \quad \text{for all } x \in X.$$

□

Exercise. *Show that, in the conditions of the Graphic Contraction Principle, we have that* $Fix(f^n) = Fix(f)$, *for every* $n \in \mathbb{N}$.

Exercise. *Let* $f : [0,1] \to [0,1]$ *be defined by*

$$f(x) := \begin{cases} 0, & x \in [0, \frac{1}{2}[\\ 1, & x \in [\frac{1}{2}, 1], \end{cases}$$

Show that f *is not a contraction, but* f *is a discontinuous graphic* k*-contraction (with any* $k \in]0,1[$*) and* $Fix(f) = \{0, 1\}$. *Moreover, show that* $f^n(x) \to 0$ *as* $n \to \infty$, *for every* $x \in [0, \frac{1}{2}[$ *and* $f^n(x) \to 1$ *as* $n \to \infty$, *for every* $x \in [\frac{1}{2}, 1]$.

Exercise. *Let* $X := [0,1] \cup [2,3]$ *and* $f : X \to X$ *be defined by*

$$f(x) := \begin{cases} \frac{1}{2}x, & x \in [0,1] \\ \frac{1}{2}x + \frac{3}{2}, & x \in [2,3]. \end{cases}$$

Show that f is a continuous graphic $\frac{1}{2}$-contraction and $Fix(f) = \{0, 3\}$.

Exercise. *Let $f : [-1, 1] \to [-1, 1]$ be defined by*

$$f(x) := \begin{cases} \frac{x}{2}, & x \neq 0 \\ \frac{1}{2}, & x = 0, \end{cases}$$

Show that f is a graphic $\frac{1}{2}$-contraction, $f^n(x) \to 0$ as $n \to \infty$, for every $x \in [-1, 1]$ and $Fix(f) = \emptyset$. Why it happens this ?

1.8 Caristi-Browder's Theorem

Theorem (1976). *Let (X, d) be a complete metric space, $f : X \to X$ be an operator and $\varphi : X \to \mathbb{R}_+$ be a functional. We suppose that:*

(a) $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$, for all $x \in X$;

(b) the operator f has closed graph.

Then:

(i) $Fix(f) \neq \emptyset$;

(ii) $f^n(x) \to f^\infty(x)$ as $n \to \infty$, and $f^\infty(x) \in Fix(f)$, $\forall\, x \in X$;

(iii) if there is $\alpha \in \mathbb{R}_+^*$ such that $\varphi(x) \leq \alpha d(x, f(x))$, then

$$d(x, f^\infty(x)) \leq \alpha d(x, f(x)), \quad \textit{for all } x \in X.$$

Proof. (i)+(ii). Let $x \in X$ be arbitrary chosen. For $n \in \mathbb{N}$, let us denote $a_{n+1} := \sum_{k=0}^{n} d(f^k(x), f^{k+1}(x))$, $n \in \mathbb{N}$. From (a) it follows that, for every $n \in \mathbb{N}$, we have

$$a_{n+1} = \sum_{k=0}^{n} d(f^k(x), f^{k+1}(x)) \leq \varphi(x) - \varphi(f^{n+1}(x)) \leq \varphi(x).$$

On the other hand,

$$a_{n+1} - a_n = d(f^n(x), f^{n+1}(x)) \geq 0, \text{ for every } n \in \mathbb{N}.$$

By the above two relation, we get that the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent. Hence, $(a_n)_{n\in\mathbb{N}}$ is also Cauchy. Thus, for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that, for every $n, m \geq n(\varepsilon)$, we have that $|a_m - a_n| < \varepsilon$. On the other hand, for every $n, m \geq n(\varepsilon)$ with $m > n$ we have

$$d(f^n(x), f^m(x)) \leq \sum_{k=0}^{m-1} d(f^k(x), f^{k+1}(x)) - \sum_{k=0}^{n-1} d(f^k(x), f^{k+1}(x)) =$$

$$= a_m - a_n < \varepsilon.$$

This implies that $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence and, hence, it is convergent in X. Let us denote by $f^\infty(x)$ its limit. From (b) we have that $f^\infty(x) \in Fix(f)$.

(iii) $d(x, f^{n+1}(x)) \leq \sum_{k=0}^{n} d(f^k(x), f^{k+1}(x)) \leq \varphi(x) \leq \alpha d(x, f(x))$.

So, $d(x, f^\infty(x)) \leq \alpha d(x, f(x))$, for all $x \in X$. □

Exercise. Show that, under the conditions of the above theorem, we have $Fix(f) = Fix(f^n)$, for every $n \in \mathbb{N}$.

Exercise. Let (X, d) be a complete metric space and $f : X \to X$ be an α-contraction. Show that f satisfies the Caristi condition:

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \text{ for all } x \in X,$$

with a function φ which should be indicated.

1.9 Picard and weakly Picard operators

Let (X, d) be a metric space. An operator $f : X \to X$ is called weakly Picard operator (WPO) if the sequence of successive approximations $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and its limit (which generally depend on x) is a fixed point of f. If an operator f is WPO with a unique fixed point, i.e., $Fix(f) = \{x^*\}$, then, by definition, f is called Picard operator (PO).

If $f : X \to X$ is a WPO, we can define the operator

$$f^\infty : X \to Fix(f), \text{ given by } f^\infty(x) := \lim_{n\to\infty} f^n(x).$$

Notice that, $f^\infty(X) = Fix(f)$ and the restriction of f^∞ to $Fix(f)$ is the identity, i.e., f^∞ is a set retraction of X on $Fix(f)$. Notice that in the case of a Picard operator with $Fix(f) = \{x^*\}$, then $f^\infty(x) = x^*$, for every $x \in X$.

In this context, if (X, d) is a metric space, $f : X \to X$ is a WPO and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function, then by definition f is a ψ-WPO if the following conditions hold:

(a) ψ is increasing, continuous at 0 and $\psi(0) = 0$;

(b) $d(x, f^\infty(x)) \leq \psi(d(x, f(x)))$, $\forall\, x \in X$.

In particular, if $\psi(t) = ct$ for all $t \in \mathbb{R}_+$ (for some $c > 0$), then f is called a c-WPO.

Exercise. (i) Show that any α-contraction and any Kannan type contraction on a complete metric space are c-PO; Find the corresponding value of c in each case.

(ii) Show that any graphic contraction with closed graph and any Caristi-Browder operator on a complete metric spaces are WPO. In this context, is a graphic contraction or a Caristi-Browder operator a ψ-WPO ? Motivation. Find ψ if the answer is positive.

1.10 Gronwall type inequalities

Let X be a nonempty set, $d : X \times X \to \mathbb{R}_+$ and $\preceq$ be a binary relation on X. Then the triple $(X, d, \preceq)$ is called an ordered metric space if:

(i) (X, d) is a metric space;

(ii) $(X, \preceq)$ is an ordered set, i.e., $\preceq$ is an order relation (reflexive, transitive and antisymmetric) on X;

(iii) If $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ are sequences in X such that $x_n \preceq y_n$ for every $n \in \mathbb{N}$, $x_n \to x, y_n \to y$ as $n \to \infty$, then $x \preceq y$.

Theorem. (Gronwall type lemma for Picard operators) *Let* $(X, d, \preceq)$ *be an ordered metric space and* $f : X \to X$ *be an operator. We suppose:*

(a) f *is increasing;*

(b) f *is a Picard operator (we denote by* x^* *its unique fixed point).*

Then the following conclusions hold:

(1) if $x \in X$ *with* $x \preceq f(x)$ *then* $x \preceq x^*$*;*

(2) if $x \in X$ *with* $x \succeq f(x)$ *then* $x \succeq x^*$*.*

Proof. (1) Let $x \in X$ such that $x \preceq f(x)$. Then, by (a), we have

$$x \preceq f(x) \preceq f^2(x) \preceq \cdots \preceq f^n(x), \forall n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$ and using (iii) from the above definition and the fact that f is a Picard operator, we get that $x \preceq x^*$.

Corollary. (Gronwall type lemma for contractions) *Let* $(X, d, \preceq)$ *be a complete ordered metric space and* $f : X \to X$ *be an operator. We suppose:*

(a) f *is increasing;*

(b) f *is a contraction (we denote by* x^* *its unique fixed point).*

Then the following conclusions hold:

(1) if $x \in X$ *with* $x \preceq f(x)$*, then* $x \preceq x^*$*;*

(2) if $x \in X$ *with* $x \succeq f(x)$*, then* $x \succeq x^*$*.*

1.11 Comparison theorems for weakly Picard operators

In the case of weakly Picard operators we have the following result.

Theorem. (Comparison theorem for two weakly Picard operators) *Let $(X, d, \preceq)$ be an ordered metric space and $f, g : X \to X$ be two given operators. We suppose:*

(a) g is increasing;

(b) $f(x) \preceq g(x)$, for every $x \in X$;

(c) f, g are weakly Picard operators.

Then, if $x \preceq y$ then $f^\infty(x) \preceq g^\infty(y)$.

Proof. Let $x, y \in X$ such that $x \preceq y$. Then, by (b) and (a), we have $f(x) \preceq g(x) \preceq g(y)$. Then, we have

$$f^2(x) \preceq g(f((x)) \preceq g(g(x)) = g^2(x) \preceq g(g(y)) = g^2(y).$$

Inductively, we have

$$f^n(x) \preceq g^n(x) \preceq g^n(y), \forall n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$ and using (iii) from the definition of a ordered metric space and the fact that f is a weakly Picard operator, we get that $f^\infty(x) \preceq g^\infty(y)$.

Exercise. Write and prove a comparison theorem for the case of a graphic contraction.

Theorem. (Comparison theorem for three weakly Picard operators) *Let $(X, d, \preceq)$ be an ordered metric space and $f, g, h : X \to X$ be three given operators. We suppose:*

(a) g is increasing;

(b) $f(x) \preceq g(x) \preceq h(x)$, for every $x \in X$;

(c) f, g, h are weakly Picard operators.

Then, if $x \preceq y \preceq z$ then $f^\infty(x) \preceq g^\infty(y) \preceq h^\infty(z)$.

1.12 Maia-Rus's fixed point theorem

The following result was proved by Maia in the paper M.G. Maia: *Un'osservatione sulle contrazioni metriche*, Rend. Sem. Mat. Univ. Padova, 40(1968), 139-143.

Theorem (Maia). *Let X be a nonempty set, d and ρ be two metrics on X and $f : X \to X$ be an operator. We suppose that:*

(1) *there exists $c > 0$ such that, $d(x, y) \leq c\rho(x, y)$, $\forall\ x, y \in X$;*

(2) *(X, d) is a complete metric space;*

(3) *$f : (X, d) \to (X, d)$ is continuous;*

(4) *$f : (X, \rho) \to (X, \rho)$ is an α-contraction.*

Then:

(i) *$Fix(f) = \{x^*\}$;*

(ii) *$f : (X, d) \to (X, d)$ is a PO.*

Proof. Let $x_0 \in X$ be arbitrary, and consider the sequence of successive approximations starting from x_0, i.e., $x_n := f^n(x_0)$, for $n \in \mathbb{N}^*$. Then $x_{n+1} = f(x_n)$, for every $n \in \mathbb{N}$. The proof is organized in some steps:

I. By (4), it follows (in a similar way to the Contraction Principle) that the sequence (x_n) is Cauchy in (X, ρ).

II. By (1) and I. it follows that the sequence (x_n) is Cauchy in (X, d) too, since $d(x_n, x_{n+p}) \leq c\rho(x_n, x_{n+p}) \to 0$ as $n.p \to \infty$.

III. By (2) and II. it follows that the sequence (x_n) is convergent in (X, d). We denote by x^* its limit, i.e., $x_n \to x^*$ with respect to d, as $n \to \infty$.

IV. By (3) and III., using the fact that $x_{n+1} = f(x_n)$, for every $n \in \mathbb{N}$, we obtain that $x^* = f(x^*)$.

V. By (4) and IV. we obtain (by reductio ad absurdum) that the fixed point is unique. This complete the proof. □

Remark. *Maia's Theorem remains true (see the paper I.A. Rus:* On a fixed point theorem of Maia*, Studia Univ. Babeş-Bolyai Math., 22(1977), 40-42) if we replace the condition* (1) *with the following one:*

$(1')$ *there exists* $c > 0$ *such that,* $d(f(x), f(y)) \leq c\rho(x,y)$, $\forall\ x, y \in X$.

Hence, we obtain the so-called Rus's variant of Maia's fixed point theorem or Maia-Rus's Theorem.

Exercise. Suppose that all the conditions in Maia's theorem are satisfied. Is f a ψ-PO with respect to d or with respect to ρ ? Motivation. Find ψ if the answer is positive.

1.13 Applications to operatorial equations

1.13.1 Integral equations

Let us consider first the following system of Volterra integral equations:

$$x(t) = \int_a^t K(t, s, x(s))ds + g(t),\ t \in [a, b],$$

where $g \in C([a, b], \mathbb{R}^n)$ and $K \in C([a, b] \times [a, b] \times \mathbb{R}^n, \mathbb{R}^n)$.

By a solution of the system we understand a map $x \in C([a, b], \mathbb{R}^n)$ which satisfies the system for every $t \in [a, b]$.

We also suppose that the following Lipschitz condition holds: there exists $L_K > 0$ such that

$$\|K(t, s, u) - K(t, s, v)\| \leq L_K \cdot \|u - v\|,$$

for each $(t, s, u), (t, s, v) \in [a, b] \times [a, b] \times \mathbb{R}^n$, where $\| \cdot \|$ denotes a norm in $\mathbb{R}^n$.

Notice first that, if we introduce the operator

$$A : C([a, b], \mathbb{R}^n) \to C([a, b], \mathbb{R}^n),\ x \longmapsto Ax,$$

defined by

$$Ax(t) := \int_a^t K(t, s, x(s))ds + g(t),\ t \in [a, b],$$

then the above system of Volterra integral equations can be written as a fixed point equation of the form

$$x = Ax, x \in X,$$

where $X := C([a, b], \mathbb{R}^n)$ will be endowed by the following Bielecki type norm

$$\|x\|_B := \max_{t \in [a,b]} \left(\|x(t)\| e^{-\tau(t-a)} \right),\ \text{where } \tau > 0.$$

Since $(C([a,b],\mathbb{R}^n), \|\cdot\|_B)$ is a Banach space, in order to apply the Contraction principle for the above fixed point problem, we need to prove that A is a contraction. Indeed, we have:

$$\|Ax(t)-Ay(t)\| \leq \int_a^t \|K(t,s,x(s))-K(t,s,y(s))\|ds \leq L_K \int_a^t \|x(s)-y(s)\|ds$$

$$= L_K \int_a^t \|x(s)-y(s)\|e^{-\tau(s-a)}e^{\tau(s-a)}ds \leq L_K\|x-y\|_B \int_a^t e^{\tau(s-a)}ds$$

$$\leq \frac{L_K}{\tau}\|x-y\|_B e^{\tau(t-a)}, \text{ for each } t \in [a,b].$$

Thus, after multiplying with $e^{-\tau(t-a)}$ and taking $\max\limits_{t\in[a,b]}$ we obtain that

$$\|Ax-Ay\|_B \leq \frac{L_K}{\tau}\|x-y\|_B, \text{ for every } x,y \in X.$$

Since τ is arbitrary, we can choose $\tau > L_K$ and thus $L_A := \frac{L_K}{\tau} < 1$. This shows that A is a contraction (with constant L_A) on the Banach space X. By the Contraction Principle, the fixed point equation $x = Ax$ has a unique solution $x^* \in X$. Moreover, this solution can be approximate by the sequence of successive approximations of A.

Hence, we proved the following result.

Theorem. (existence, uniqueness and approximation for the solution of a system of Volterra integral equations)

Let us consider the following system of Volterra integral equations:

$$x(t) = \int_a^t K(t,s,x(s))ds + g(t), \ t \in [a,b].$$

We suppose:

(i) $g \in C([a,b],\mathbb{R}^n)$ *and* $K \in C([a,b]\times[a,b]\times\mathbb{R}^n,\mathbb{R}^n)$;

(ii) there exists $L_K > 0$ *such that*

$$\|K(t,s,u)-K(t,s,v)\| \leq L_K \cdot \|u-v\|,$$

for each $(t,s,u),(t,s,v)\in[a,b]\times[a,b]\times\mathbb{R}^n$, *where* $\|\cdot\|$ *denotes a norm in* $\mathbb{R}^n$.

Then, the above system has a unique solution $x^*\in C([a,b],\mathbb{R}^n)$ *and the sequence* $(x_n)_{n\in\mathbb{N}}$ *defined by*

$$x_0\in C([a,b],\mathbb{R}^n), x_{n+1}(t):=\int_a^t K(t,s,x_n(s))ds+g(t),\ t\in[a,b],n\in\mathbb{N}$$

converges (uniformly) in $C([a,b],\mathbb{R}^n)$ *to* x^*.

Exercise. *Let us consider the following system of Fredholm integral equations:*

$$x(t)=\int_a^b K(t,s,x(s))ds+g(t),\ t\in[a,b],$$

where $g\in C([a,b],\mathbb{R}^n)$ *and* $K\in C([a,b]\times[a,b]\times\mathbb{R}^n,\mathbb{R}^n)$.

By a solution of the above system we understand $x\in C([a,b],\mathbb{R}^n)$ *which satisfies the system for every* $t\in[a,b]$.

We also suppose that the following Lipschitz condition holds: there exists $L_K>0$ *such that*

$$\|K(t,s,u)-K(t,s,v)\|\le L_K\cdot\|u-v\|,$$

for each $(t,s,u),(t,s,v)\in[a,b]\times[a,b]\times\mathbb{R}^n$, *where* $\|\cdot\|$ *denotes a norm in* $\mathbb{R}^n$.

Prove an existence, uniqueness and approximation result for the above system of Fredholm integral equations, working in the Banach space $X:=C([a,b],\mathbb{R}^n)$ *endowed by the following Cebîsev type norm*

$$\|x\|_C:=\max_{t\in[a,b]}\|x(t)\|.$$

1.13.2 The Cauchy problem for a system of differential equations

Consider the following initial value problem (Cauchy problem):

(1) $x'(t) = f(t, x(t)), t \in [a, b]$

(2) $x(t_0) = x^0$,

where $t_0 \in [a, b]$, $x^0 \in \mathbb{R}^n$ are given and $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function such that $f(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is L_f-Lipschitz.

By a solution of the above Cauchy problem we understand a map $x \in C^1([a, b], \mathbb{R}^n)$ which satisfies (1) for every $t \in [a, b]$ and (2).

Lemma. *Let us consider the Cauchy problem* $(1) + (2)$. *We suppose that* $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function. Then* $(1) + (2)$ *is equivalent with the following Volterra integral equation*

$$(3)\ x(t) = \int_{t_0}^{t} f(s, x(s))ds + x^0,\ t \in [a, b].$$

By the main theorem of the above section, we have:

Theorem (existence, uniqueness and approximation result for the solution of the Cauchy problem $(1) + (2)$)

Let us consider the Cauchy problem $(1) + (2)$. *We suppose:*

(i) $f \in C([a, b] \times \mathbb{R}^n, \mathbb{R}^n)$;

(ii) there exists $L_f > 0$ *such that*

$$\|f(t, u) - f(t, v)\| \le L_f \|u - v\|,\ \textit{for every } t \in [a, b], u, v \in \mathbb{R}^n,$$

where $\|\cdot\|$ *denotes a norm in* $\mathbb{R}^n$.

Then, the Cauchy problem (1)+(2) has a unique solution x^* *and the sequence* $(x_n)_{n \in \mathbb{N}} \subset C([a, b], \mathbb{R}^n)$, *given by*

$$x_{n+1}(t) = \int_{t_0}^{t} f(s, x_n(s))ds + x^0,\ t \in [a, b],\ n \in \mathbb{N}$$

converges (uniformly) to x^*, *for every* $x_0 \in C([a, b], \mathbb{R}^n)$.

Proof. The result follows by applying the existence, uniqueness and approximation for the solution of the following system of Volterra integral equations

$$(3)\ x(t) = \int_{t_0}^{t} f(s, x(s))ds + x^0,\ t \in [a, b].$$

Here the operator which must be considered is

$$Ax(t) := \int_{t_0}^{t} f(s, x(s))ds + x^0,\ t \in [a, b].$$

1.13.3 The Dirichlet problem for a differential equation of second order

Consider the following boundary value problem (Dirichlet problem):

(1) $x''(t) = f(t, x(t)), t \in [a, b]$

(2) $x(a) = x(b) = 0,$

where $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function such that the map $f(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is L_f-Lipschitz.

By a solution of the above Dirichlet problem we understand a map $x \in C^2([a, b], \mathbb{R}^n)$ which satisfies (1) for every $t \in [a, b]$ and (2).

The following result is important in our approach.

Lemma. *Let us consider the Dirichlet problem* $(1) + (2)$. *We suppose that* $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function. Then* $(1) + (2)$ *is equivalent with the following Fredholm integral equation*

$$(3)\ x(t) = -\int_{a}^{b} G(t, s) f(s, x(s))ds,\ t \in [a, b],$$

where $G : [a, b] \times [a, b] \to \mathbb{R}_+$ *is the following Green function associated to this problem*

$$G(t, s) := \begin{cases} \frac{(b-t)(s-a)}{b-a}, & a \le s \le t \le b \\ \frac{(b-s)(t-a)}{b-a}, & a \le t < s \le b. \end{cases}$$

Remark. *We have the following properties of* G*:*

(a) G *is continuous on* $[a,b]\times[a,b]$*;*

(b) G *is positive and symmetric;*

(c) $G(t,s)\in[0,\frac{b-a}{4}]$*, for every* $t,s\in[a,b]$*;*

(d) $\int_a^b G(t,s)ds\in[0,\frac{(b-a)^2}{8}]$*, for every* $t\in[a,b]$*;*

(e) $\int_a^b |\frac{\partial G(t,s)}{\partial t}|ds\in[0,\frac{b-a}{2}]$*, for every* $t\in[a,b]$*.*

Let us consider the operator

$$B:(C([a,b],\mathbb{R}^n),\|\cdot\|_C)\to(C([a,b],\mathbb{R}^n),\|\cdot\|_C),\ x\longmapsto Bx,$$

defined by

$$Bx(t):=-\int_a^b G(t,s)f(s,x(s))ds,\ t\in[a,b].$$

Then, the above Fredholm integral equation (3) (and, as a consequence of the Lemma, the Dirichlet problem (1) + (2)) is equivalent to the fixed point equation

$$x=Bx,\ x\in X,$$

where $X:=C([a,b],\mathbb{R}^n),\|\cdot\|_C)$ is a Banach space. The main problem now is to prove that (under some additional assumptions) B is a contraction.

We have the following existence, uniqueness and approximation result for the solution of the above Dirichlet problem.

Theorem. (existence, uniqueness and approximation result for the solution of the Dirichlet problem (1) + (2))

Consider the above Dirichlet problem (1) + (2)*. We suppose:*

(i) $f:[a,b]\times\mathbb{R}^n\to\mathbb{R}^n$ *is continuous;*

(ii) there exists $L_f>0$ *such that*

$$\|f(t,u)-f(t,v)\|\le L_f\|u-v\|,\ \textit{for every}\ t\in[a,b],u,v\in\mathbb{R}^n,$$

where $\|\cdot\|$ *denotes a norm in* $\mathbb{R}^n$*;*

(iii) $\frac{L_f(b-a)^2}{8}<1$*.*

Then, the Dirichlet problem has a unique solution x^* which can be approximated by the following sequence of successive approximations

$$x_0 \in C([a,b],\mathbb{R}^n),\ \ x_{n+1}(t) = -\int_a^b G(t,s)f(s,x_n(s))ds,\ \ t \in [a,b], n \in \mathbb{N}.$$

Proof. Under the above assumption the operator

$$B : (C([a,b],\mathbb{R}^n), \|\cdot\|_C) \to (C([a,b],\mathbb{R}^n), \|\cdot\|_C),\ \ x \longmapsto Bx,$$

defined by

$$Bx(t) := -\int_a^b G(t,s)f(s,x(s))ds,\ \ t \in [a,b]$$

is a contraction with constant $L_B := \frac{L_f(b-a)^2}{8}$. The conclusion follows by the Contraction Principle. □

1.13.4 Nonlinear alternative with an application to a Cauchy problem

By teh Continuation Theorem we can obtain the following result which is useful in applications.

Theorem. (Nonlinear Alternative for Contractions) *Let E be a Banach space, $X \in P_{cl,cv}(E)$ and U an open subset of X such that $0 \in U$. Let $f : \overline{U} \to X$ be an α-contraction such that $f(\overline{U})$ is bounded. Then f has at least one of the following properties:*

(i) f has a unique fixed point

(ii) there exist $y_0 \in \partial U$ and $\lambda_0 \in]0,1[$ such that $y_0 = \lambda_0 f(y_0)$.

Proof. For $(\lambda, x) \in [0,1] \times \overline{U}$ we define: $H_\lambda(x) := \lambda \cdot f(x)$. Then, $(H_\lambda)_{\lambda\in[0,1]}$ is an α-contractive family of contractions with $p = 1$. Hence $(H_\lambda)_{\lambda\in[0,1]} \subset CR(\overline{U}, X)$.

a) if $(H_\lambda)_{\lambda\in[0,1]} \subset CR_{\partial U}(\overline{U}, X)$ then, since $H_0(0) = 0$, we can apply the continuation theorem for contractions and we get that $H_1 = f$ has a fixed point in U.

b) if $(H_\lambda)_{\lambda\in[0,1]}$ is not in $CR_{\partial U}(\overline{U}, X)$ then, $H_\lambda = \lambda \cdot f$ has a fixed point in ∂U for some $\lambda \in [0,1]$. Of course, $\lambda \neq 0$ (since, if $\lambda = 0$ then, because $0 = H_0(0)$ we have $0 \in \partial U$, a contradiction with $0 \in U$). Hence, in this case, f has a fixed point in ∂U (for $\lambda = 1$) or (ii) holds. □

Consider the following initial value problem:
(1) $x'(t) = f(t, x(t)), t \in [0, T]$
(2) $x(0) = 0$,

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.
Suppose that:
(a) for each $r > 0$ there is $a_r \in \mathbb{R}$ such that $|f(t, x) - f(t, y)| \leq a_r|x - y|$, for each $t \in [0, T]$ and each $x, y \in [-r, r]$;
(b) There exists a monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that $|f(t, x)| \leq \varphi(|x|)$, for each $t \in [0, T]$ and each $x \in \mathbb{R}$;
(c) $T < \int_0^{+\infty} \frac{ds}{\varphi(s)}$;

Then the problem (1) + (2) has a unique solution $x \in C^1[0, T]$.
Proof. Consider, for $\lambda \in [0, 1]$, the following family of initial value problems:
(1_λ) $x'(t) = \lambda f(t, x(t)), t \in [0, T]$
(2_λ) $x(0) = 0$,

Let $M > 0$ such that $T < \int_0^M \frac{ds}{\varphi(s)} < \int_0^{+\infty} \frac{ds}{\varphi(s)}$.

Step 1. For each solution x of $(1_\lambda)+(2_\lambda)$ we have $|x(t)| < M$, for each $t \in [0, T]$.
Since $|x'(t)| \leq |\lambda|\varphi(|x(t)|)$, for each $t \in [0, T]$, we obtain, by integrating from 0 to t that

$$\int_0^t \frac{|x'(s)|}{\varphi(|x(s)|)} ds \leq \int_0^t \lambda ds.$$

If we change variables ($v := |x(s)|$) then

$$\int_0^{|x(t)|} \frac{1}{\varphi(v)} dv \le \lambda t \le \lambda T \le T < \int_0^M \frac{ds}{\varphi(s)}.$$

Thus $|x(t)| < M$, for each $t \in [0, T]$.

Let $L := a_M > 0$. Consider on $C[0, T]$ the Bielecki type norm:

$$\|x\|_B := \max_{t\in[0,T]} (|x(t)|e^{-Lt}).$$

Define

$$U := \{x \in C[0, T] | |x(t)| < M, \forall t \in [0, T]\}.$$

Then $0 \in U$ and U is open in $C[0, T]$.
Define $G : \overline{U} \to C[0, T]$, $x \mapsto Gx$, where

$$Gx(t) := \int_0^t f(s, x(s))ds.$$

Step 2. We show that G is a contraction.
Let $x.y \in \overline{U}$. Then:
$|Gx(t) - Gy(t)| \le \int_0^t |f(s, x(s)) - f(s, y(s))|ds \le$
$L \int_0^t |x(s) - y(s)|e^{-Ls}e^{Ls}ds \le (e^{Lt} - 1)\|x - y\|_B$.

Choose $\alpha < 1$ such that $e^{Lt} - 1 \le \alpha e^{Lt}$, for each $t \in [0, T]$ (for example any $\alpha \ge 1 - e^{-LT}$). Then:
$|Gx(t) - Gy(t)| \le \alpha\|x - y\|_B e^{Lt}$, for any $t \in [0, T]$. As consequence, $\|Gx - Gy\|_B \le \alpha\|x - y\|_B$.

Step 3. We prove that λG is fixed point free on ∂U, i.e., $\nexists x \in \partial U$ such that $x = \lambda Gx$.

By contradiction, suppose that there is $x \in \partial U$ such that $x = \lambda Gx$. Then

$$x(t) = \lambda \int_0^t f(s, x(s))ds,\ t \in [0, T].$$

Hence $x'(t) = \lambda f(t, x(t)), t \in [0, T]$ and $x(0) = 0$, showing that x is a solution for $(1_\lambda) + (2_\lambda)$. Then, by Step 1, we get that $|x(t)| < M$, for all $t \in [0, T]$. Thus $x \in U$, which is a contradiction with $x \in \partial U$.

Step 4. We prove that $G(\overline{U})$ is bounded.

We have:

$$|Gx(t)| \leq \int_0^t |f(s, x(s))| ds \leq \int_0^t \varphi(|x(s)|) ds \leq \varphi(M) \cdot T,$$

for each $t \in [0, T]$. Hence $\|G(x)\| \leq \varphi(M) \cdot T$.

Then from the Nonlinear Alternative we get that G has a unique fixed point in $x^* \in \overline{U}$. This x^* is a solution of the problem (1) + (2). $\square$.

Chapter 2

Topological Fixed Point Theorems

2.1 Multivalued Analysis

The aim of this section is to present the main properties of some (generalized) functionals defined on the space of all subsets of a metric space. A special attention is paid to gap functional, excess functional and to Pompeiu-Hausdorff functional.

Let (X, d) be a metric space. Sometimes we will need to consider infinite-valued metrics, also called generalized metrics $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$.

We denote by $\mathcal{P}(X)$ the set of all subsets of a nonempty set X. Recall that, if X is a metric space, $x \in X$ and $R > 0$, then $B(x, R)$ and respectively $\widetilde{B}(x, R)$ denote the open, respectively the closed ball of radius R centered in x. Also, if X is a topological space and Y is a subset of X, then we will denote by $\overline{Y}$ the closure and by $int(Y)$ the interior of the set Y. Also, if X is a normed space and Y is a nonempty subset of X, then $co(Y)$ respectively $\overline{co}(Y)$ denote the convex hull, respectively the closed convex hull of the set Y.

We consider, for the beginning, the generalized diameter functional defined on the space of all subsets of a metric space X.

Definition. Let (X, d) be a metric space. The generalized diameter functional $diam : \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by:

$$diam(Y) = \begin{cases} \sup\{d(a,b) \mid a \in Y,\ b \in Y\}, & \text{if } Y \neq \emptyset \\ 0, & \text{if } Y = \emptyset \end{cases}$$

Definition. The subset Y of X is said to be bounded if and only if $diam(Y) < \infty$.

Lemma. *Let (X, d) be a metric space and Y, Z nonempty bounded subsets of X. Then:*

i) $diam(Y) = 0$ if and only if $Y = \{y_0\}$.

ii) If $Y \subset Z$ then $diam(Y) \leq diam(Z)$.

iii) $diam(\overline{Y}) = diam(Y)$.

iv) If $Y \cap Z \neq \emptyset$ then $diam(Y \cup Z) \leq diam(Y) + diam(Z)$.

v) If X is a normed space then:

a) $diam(x + Y) = diam(Y)$, for each $x \in X$.

b) $diam(\alpha Y) = |\alpha| diam(Y)$, where $\alpha \in \mathbb{R}$.

c) $diam(Y) = diam(co(Y))$.

d) $diam(Y) \leq diam(Y + Z) \leq diam(Y) + diam(Z)$.

Proof. iii) Because $Y \subseteq \overline{Y}$ we have $diam(Y) \leq diam(\overline{Y})$. For the reverse inequality, let consider $x, y \in \overline{Y}$. Then there exist $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} \subset Y$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. It follows that $d(x_n, y_n) \stackrel{\mathbb{R}}{\to} d(x, y)$. Because $d(x_n, y_n) \leq diam(Y)$, for all $n \in \mathbb{N}$ we get by passing to limit $d(x, y) \leq diam(Y)$. Hence $diam(\overline{Y}) \leq diam(Y)$.

iv) Let $u, v \in Y \cup Z$. We have the following cases:

a) If $u, v \in Y$ then $d(u, v) \leq diam(Y) \leq diam(Y) + diam(Z)$ and so $diam(Y \cup Z) \leq diam(Y) + diam(Z)$.

b) If $u, v \in Z$ then by an analogous procedure we have $d(u,v) \leq diam(Z) \leq diam(Y) + diam(Z)$ and so $diam(Y \cup Z) \leq diam(Y) + diam(Z)$.

c) If $u \in Y$ and $v \in Z$ then choosing $t \in Y \cap Z$ we have that $d(u,v) \leq d(u,t) + d(t,v) \leq diam(Y) + diam(Z)$. Hence, $diam(Y \cup Z) \leq diam(Y) + diam(Z)$.

v) c) Let us prove that $diam(co(Y)) \leq diam(Y)$. Let $x, y \in co(Y)$. Then there exist $x_i, y_j \in Y$, $\lambda_i, \mu_j \in \mathbb{R}_+$, such that

$$x = \sum_{i=1}^{n} \lambda_i x_i, \quad y = \sum_{j=1}^{m} \mu_j y_j, \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \sum_{j=1}^{m} \mu_j = 1.$$

From these relations we have:

$$\|x - y\| = \left\| \sum_{i=1}^{n} \lambda_i x_i - \sum_{j=1}^{m} \mu_j y_j \right\| = \left\| \left(\sum_{j=1}^{m} \mu_j \right) \sum_{i=1}^{n} \lambda_i x_i - \left(\sum_{i=1}^{n} \lambda_i \right) \sum_{j=1}^{m} \mu_j y_j \right\|$$
$$\leq \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i \mu_j \|x_i - y_j\| \leq \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i \mu_j \right) diam(Y) = diam(Y).$$

□

Let us consider now the following sets of subsets of a metric space (X, d):

$$P(X) = \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\};\ P_b(X) = \{Y \in P(X) |\ diam(Y) < \infty\};$$

$$P_{op}(X) = \{Y \in P(X) |\ Y \text{ is open}\};\ P_{cl}(X) = \{Y \in P(X) |\ Y \text{ is closed}\};$$

$$P_{b,cl}(X) = P_b(X) \cap P_{cl}(X);\ P_{cp}(X) = \{Y \in P(X) |\ Y \text{ is compact}\};$$

$$P_{cn}(X) = \{Y \in P(X) |\ Y \text{ is connex}\}.$$

If X is a normed space, then we denote:

$$P_{cv}(X) = \{Y \in P(X) |\ Y \text{ convex}\};\ P_{cp,cv}(X) = P_{cp}(X) \cap P_{cv}(X).$$

Let us define the following generalized functionals:

(1) $D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$$D(A,B) = \begin{cases} \inf\{d(a,b) |\ a \in A,\ b \in B\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

D is called the gap functional between A and B.

In particular, $D(x_0, B) = D(\{x_0\}, B)$ (where $x_0 \in X$) is called the distance from the point x_0 to the set B.

(2) $\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\delta(A,B) = \begin{cases} \sup\{d(a,b) |\ a \in A,\ b \in B\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{otherwise} \end{cases}$$

(3) $\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\rho(A,B) = \begin{cases} \sup\{D(a,B) |\ a \in A\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset \\ +\infty, & \text{if } B = \emptyset \neq A \end{cases}$$

ρ is called the excess functional of A over B.

(4) $H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$H(A,B) = \begin{cases} \max\{\rho(A,B), \rho(B,A)\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

H is called the generalized Pompeiu-Hausdorff functional of A and B.

Let us prove now that the functional H is a metric on the space $P_{b,cl}(X)$. First we will prove the following auxiliary result:

Lemma. $D(b, A) = 0$ *if and only if* $b \in \overline{A}$.

Proof. We shall prove that $\overline{A} = \{x \in X|\ D(x, A) = 0\}$. For this aim, let $x \in \overline{A}$ be arbitrarily. It follows that for each $r > 0$ and for each $B(x, r) \subset X$ we have $A \cap B(x, r) \neq \emptyset$. Then for each $r > 0$ there exists $a_r \in A$ such that $d(x, a) < r$. It follows that for each $r > 0$ we have $D(x, A) < r$ and hence $D(x, A) = 0$. □

Theorem. *Let (X, d) be a metric space. Then the pair $(P_{b,cl}(X), H)$ is a metric space.*

Proof. We shall prove that the three axioms of the metric hold:

a) $H(A, B) \geq 0$, for all $A, B \in P_{b,cl}(X)$ is obviously.

$H(A, B) = 0$ is equivalent with $\rho(A, B) = 0$ and $\rho(B, A) = 0$, that means $\sup_{a\in A} D(a, B) = 0$ and $\sup_{b\in B} D(b, A) = 0$. Hence $D(a, B) = 0$, for each $a \in A$ and $D(b, A) = 0$, for each $b \in B$. Using Lemma 1.4. we obtain that $a \in B$, for all $a \in A$ and $b \in A$, for all $b \in B$, proving that $A \subseteq B$ and $B \subseteq A$.

b) $H(A, B) = H(B, A)$ is quite obviously.

c) For the third axiom of the metric, let consider $A, B, C \in P_{b,cl}(X)$. For each $a \in A$, $b \in B$ and $c \in C$ we have $d(a, c) \leq d(a, b) + d(b, c)$. It follows that $\inf_{c\in C} d(a, c) \leq d(a, b) + \inf_{c\in C} d(b, c)$, for all $a \in A$ and $b \in B$. We get $D(a, C) \leq d(a, b) + D(b, C)$, for all $a \in A$, $b \in B$. Hence $D(a, C) \leq D(a, B) + H(B, C)$, for all $a \in A$ and so $D(a, C) \leq H(A, B) + H(B, C)$, for all $a \in A$. In conclusion, we have proved that $\rho(A, C) \leq H(A, B) + H(B, C)$. Similarly, we get $\rho(C, A) \leq H(A, B) + H(B, C)$, and so $H(A, C) \leq H(A, B) + H(B, C)$. □

Remark. H (or H_d if necessary) is called the Pompeiu- Hausdorff metric induced by the metric d on $P_{b,cl}(X)$. Notice also that H is a generalized metric (in the sense that $H(A, B) \in \mathbb{R}_+ \cup +\infty$) on $P_{cl}(X)$.

Lemma. *Let the open balls $A := B(x_0; r), B := B(y_0; s) \subset \mathbb{R}^n$, where $x_0, y_0 \in \mathbb{R}^n$ and $r, s > 0$. Then*

$$H(A, B) = \|x_0 - y_0\|_E + |r - s|,$$

where $\|\cdot\|_E$ *denotes the Euclidean norm in* $\mathbb{R}^n$.

Lemma. *Let* (X,d) *a metric space. Then we have:*

i) $D(\cdot,Y):(X,d)\to\mathbb{R}_+$, $x\mapsto D(x,Y)$, *(where* $Y\in P(X)$*) is nonexpansive.*

ii) $D(x,\cdot):(P_{cl}(X),H)\to\mathbb{R}_+$, $Y\mapsto D(x,Y)$, *(where* $x\in X$*) is nonexpansive.*

Proof. i) We shall prove that for each $Y\in P(X)$ we have

$$|D(x_1,Y)-D(x_2,Y)|\le d(x_1,x_2), \text{for all } x_1,x_2\in X.$$

Let $x_1,x_2\in X$ be arbitrarily. Then for all $y\in Y$ we have

$d(x_1,y)\le d(x_1,x_2)+d(x_2,y)$. Then $\inf\limits_{y\in Y} d(x_1,y)\le d(x_1,x_2)+\inf\limits_{y\in Y} d(x_2,y)$ and so $D(x_1,Y)\le d(x_1,x_2)+D(x_2,y)$. We have proved that $D(x_1,y)-D(x_2,Y)\le d(x_1,x_2)$. Interchanging the roles of x_1 and x_2 we obtain $D(x_2,Y)-D(x_1,Y)\le d(x_1,x_2)$, proving the conclusion.

ii) We shall prove that for each $x\in X$ we have:

$$|D(x,A)-D(x,B)|\le H(A,B), \text{ for all } A,B\in P_{cl}(X).$$

Let $A,B\in P_{cl}(X)$ be arbitrarily. Let $a\in A$ and $b\in B$. Then we have $d(x,a)\le d(x,b)+d(b,a)$. It follows $D(x,A)\le d(x,b)+D(b,A)\le d(x,b)+H(B,A)$ and hence $D(x,A)-D(x,B)\le H(A,B)$. By a similar procedure we get $D(x,B)-D(x,A)\le H(A,B)$ and so $|D(x,A)-D(x,B)|\le H(A,B)$, for all $A,B\in P_{b,cl}(X)$. □

Lemma. *Let* (X,d) *be a metric space. Then the generalized functional* $diam:(P_{cl}(X),H)\to\mathbb{R}_+\cup\{+\infty\}$ *is continuous.*

Let us define now the notion of neighborhood for a nonempty set.

Definition. Let (X,d) be a metric space, $Y\in P(X)$ and $\varepsilon>0$. An open neighborhood of radius ε for the set Y is the set denoted $V^0(Y,\varepsilon)$ and defined by $V^0(Y,\varepsilon)=\{x\in X|\ D(x,Y)<\varepsilon\}$. We also consider the closed neighborhood for the set Y, defined by $V(Y,\varepsilon) = \{x \in X|\ D(x,Y) \leq \varepsilon\}$.

Remark. From the above definition we have that, if (X,d) is a metric space, $Y \in P(X)$ then:

a)$\bigcup\{B(y,r) : y \in Y\} = V^0(Y,r)$, where $r > 0$.

b)$\bigcup\{\widetilde{B}(y,r) : y \in Y\} \subset V(Y,r)$, where $r > 0$.

c)$V^0(Y, r+s) \supset V^0(V^0(Y,s), r)$, where $r, s > 0$.

d)$V^0(Y,r)$ is an open set, while $V(A,r)$ is a closed set.

e) If (X,d) is a normed space, then:

i) $V^0(Y, r+s) = V^0(V^0(Y,s), r)$, where $r, s > 0$

ii) $V^0(Y,r) = Y + int(r\widetilde{B}(0,1))$.

Proof. d) $V^0(Y,r) = f^{-1}(]-\infty, r[)$ and $V(Y,r) = f^{-1}([0,r])$, where $f(x) = D(x,Y)$, $x \in X$ is a continuous function.

Lemma. *Let (X,d) a metric space. Then we have:*

i) If $Y, Z \in P(X)$ then $\delta(Y,Z) = 0$ if and only if $Y = Z = \{x_0\}$

ii) $\delta(Y,Z) \leq \delta(Y,W) + \delta(W,Z)$, for all $Y, Z, W \in P_b(X)$.

iii) Let $Y \in P_b(X)$ and $q \in]0,1[$. Then, for each $x \in X$ there exists $y \in Y$ such that $q\delta(x,Y) \leq d(x,y)$.

Proof. ii) Let $Y, Z, W \in P_b(X)$. Then we have:

$d(y,z) \leq d(y,w) + d(w,z)$, for all $y \in Y, z \in Z, w \in W$. Then $\sup_{z \in Z} d(y,z) \leq d(y,w) + \sup_{z \in Z} d(w,z)$, for all $y \in Y, w \in W$. So $\delta(y,Z) \leq \delta(y,w) + \delta(w,Z) \leq \delta(y,W) + \delta(W,Z)$ and hence $\delta(Y,Z) \leq \delta(Y,W) + \delta(W,Z)$.

iii) Suppose, by reductio ad absurdum, that there exists $x \in X$ and there exists $q \in]0,1[$ such that for all $y \in Y$ to have $q\delta(x,Y) > d(x,y)$. It follows that $q\delta(x,Y) \geq \sup_{y \in Y} d(x,y)$ and hence $q\delta(x,Y) \geq \delta(x,Y)$. In conclusion, $q \geq 1$, a contradiction. □

Lemma. *Let (X,d) a metric space. Let $Y, Z \in P(X)$ and $q > 1$. Then, for each $y \in Y$ there exists $z \in Z$ such that $d(y,z) \leq qH(Y,Z)$.*

Some very important properties of the metric space $(P_{cl}(X), H_d)$ are contained in the following result:

Theorem. *i) If (X, d) is a complete metric space, then $(P_{cl}(X), H_d)$ is a complete metric space.*

ii) If (X, d) is a totally bounded metric space, then $(P_{cl}(X), H_d)$ is a totally bounded metric space.

iii) If (X, d) is a compact metric space, then $(P_{cl}(X), H_d)$ is a compact metric space.

iv) If (X, d) is a separable metric space, then $(P_{cp}(X), H_d)$ is a separable metric space.

v) If (X, d) is a ε-chainable metric space, then $(P_{cp}(X), H_d)$ is also an ε-chainable metric space.

Proof. i) We will prove that each Cauchy sequence in $(P_{cl}(X), H_d)$ is convergent. Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(P_{cl}(X), H_d)$. Let us consider the set A defined as follows:

$$A = \bigcap_{n=1}^{\infty} \left(\overline{\bigcup_{m=n}^{\infty} A_m} \right).$$

We have two steps in the proof:

1) $A \neq \emptyset$.

In this respect, consider $\varepsilon > 0$. Then for each $k \in \mathbb{N}$ there is $N_k \in \mathbb{N}$ such that for all $n, m \geq N_k$ we have $H(A_n, A_m) < \frac{\varepsilon}{2^{k+1}}$. Let $(n_k)_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers such that $n_k \geq N_k$. Let $x_0 \in A_{n_0}$. Let us construct inductively a sequence $(x_k)_{k\in\mathbb{N}}$ having the following properties:

α) $x_k \in A_{n_k}$, for each $k \in \mathbb{N}$

β) $d(x_k, x_{k+1}) < \frac{\varepsilon}{2^{k+1}}$, for each $k \in \mathbb{N}$.

Suppose that we have $x_0, x_1, \ldots, x_k$ satisfying α) and β) and we will generate x_{k+1} in the following way.

We have:

$$D(x_k, A_{n_{k+1}}) \leq H(A_{n_k}, A_{n_{k+1}}) < \frac{\varepsilon}{2^{k+1}}.$$

It follows that there exists $x_{k+1} \in A_{n_{k+1}}$ such that $d(x_k, x_{k+1}) < \frac{\varepsilon}{2^{k+1}}$.

Hence, we have proved that there exist a sequence $(x_k)_{k\in\mathbb{N}}$ satisfying α) and β).

From β) we get that $(x_k)_{k\in\mathbb{N}}$ is Cauchy in (X, d). Because (X, d) is complete it follows that there exists $x \in X$ such that $x = \lim\limits_{k\to\infty} x_k$. I need to show now that $x \in A$. Since $(n_k)_{k\in\mathbb{N}}$ is an increasing sequence it follows that for $n \in \mathbb{N}^*$ there exists $k_n \in \mathbb{N}^*$ such that $n_{k_n} \geq n$. Then $x_k \in \bigcup\limits_{m\geq n} A_m$, for $k \geq k_n$, $n \in \mathbb{N}^*$ implies that $x \in \overline{\bigcup\limits_{m\geq n} A_m}$, $n \in \mathbb{N}^*$. Hence $x \in A$.

2) In the second step of the proof, we will establish that $H(A_n, A) \to 0$ as $n \to \infty$.

The following inequalities hold:

$$d(x_k, x_{k+p}) \leq d(x_k, x_{k+1}) + \cdots + d(x_{k+p-1}, x_{k+p}) <$$

$$< \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+2}} + \cdots + \frac{\varepsilon}{2^{k+p}} < \varepsilon\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^k} + \dots\right) =$$

$$= \varepsilon\frac{1}{1 - \frac{1}{2}} = 2\varepsilon, \text{ for all } p \in \mathbb{N}^*.$$

If in $d(x_k, x_{k+p}) < 2\varepsilon$ we are letting $p \to \infty$ we obtain $d(x_k, x) < 2\varepsilon$, for each $k \in \mathbb{N}$. In particular $d(x_0, x) < 2\varepsilon$. So, for each $n_0 \in \mathbb{N}$, $n_0 \geq N_0$ and for $x_0 \in A_{n_0}$ there exists $x \in A$ such that $d(x_0, x) \leq 2\varepsilon$, which imply

$$\rho(A_{n_0}, A) \leq 2\varepsilon, \text{ for all } n_0 \geq N_0 \qquad (1).$$

On the other side, because the sequence $(A_n)_{n\in\mathbb{N}}$ is Cauchy, it follows that there exists $N_\varepsilon \in \mathbb{N}$ such that for $m, n \geq N_\varepsilon$ we have $H(A_n, A_m) < \varepsilon$. Let $x \in A$ be arbitrarily. Then $x \in \overline{\bigcup\limits_{m=n}^{\infty} A_m}$, for $n \in \mathbb{N}^*$, which implies that there exist $n_0 \in \mathbb{N}$, $n_0 \geq N_\varepsilon$ and $y \in A_{n_0}$ such that $d(x, y) < \varepsilon$.

Hence, there exists $m \in \mathbb{N}$, $m \geq N_\varepsilon$ and there is $y \in A_m$ such that $d(x,y) < \varepsilon$.

Then, for $n \in \mathbb{N}^*$, with $n \geq N_\varepsilon$ we have:

$$D(x,A_n) \leq d(x,y) + D(y,A_n) \leq d(x,y) + H(A_m,A_n) < \varepsilon + \varepsilon = 2\varepsilon.$$

So,

$$\rho(A,A_n) < 2\varepsilon, \text{ for each } n \in \mathbb{N} \text{ with } n \geq N_\varepsilon. \quad (2)$$

From (1) and (2) and choosing $n_\varepsilon := \max\{N_0, N_\varepsilon\}$ it follows that $H(A_n,A) < 2\varepsilon$, for each $n \geq n_\varepsilon$. Hence $H(A_n,A) \to 0$ as $n \to \infty$.

v) (X,d) being an ε-chainable metric space (where $\varepsilon > 0$) it follows, by definition, that for all $x, y \in X$ there exists a finite subset (the so-called ε-net) of X, let say $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_{k-1}, x_k) < \varepsilon$, for all $k = 1, 2, \ldots, n$.

Let $y \in X$ arbitrary and $Y = \{y\}$. Obviously, $Y \in P_{cp}(X)$. Because the ε-chainability property is transitive, it is sufficient to prove that for all $A \in P_{cp}(X)$ there exist an ε-net in $P_{cp}(X)$ linking Y with A. We have two steps in our proof:

a) Let suppose first that A is a finite set, let say $A = \{a_1, a_2, ..., a_n\}$ We will use the induction method after the number of elements of A. If $n = 1$ then $A = \{a\}$ and the conclusion follows from the ε-chainability of (X,d). Let suppose now that the conclusion holds for each subsets of X consisting of at most n elements. Let A be a subset of X with $n+1$ points, $A = \{x_1, x_2, \ldots, x_{n+1}\}$. Using the ε-chainability of the space (X,d) it follows that there exist an ε-net in X, namely $x_1 = u_0, u_1, \ldots, u_m = x_2$ linking the points x_1 and x_2. We obtain that the following finite set: A, $\{u_1, x_2, \ldots, x_{n+1}\}, \ldots,$ $\{u_{m-1}, x_2, \ldots, x_{n+1},$ $\{x_2, \ldots, x_{n+1}\}$ is an ε-net in $P_{cp}(X)$ from A to $B := \{x_2, \ldots, x_{n+1}\}$. But, from the hypothesis B is ε-chainable with Y, and hence A is ε-chainable with Y in $P_{cp}(X)$.

b) Let consider now $A \in P_{cp}(X)$ be arbitrary.

A being compact, there exists a finite family of nonempty compact subsets of A, namely $\{A_k\}_{k=1}^{n}$, having $diam(A_k) < \varepsilon$ such that $A =$

$\bigcup_{k=1}^{n} A_k$. For each $k = 1, 2, \ldots n$ we can choose $x_k \in A_k$ and define $C = \{x_1, \ldots, x_n\}$. Then for all $z \in A$ there exists $k \in \{1, 2, \ldots, n\}$ such that $D(z, C) \leq \delta(A_k)$. We obtain:

$$H(A, C) = \max\left\{\sup_{z \in A} D(z, C), sup_{y \in C} D(y, A)\right\} =$$

$$= \sup_{z \in A} D(z, C) \leq \max_{i \leq k \leq n} \delta(A_k) < \varepsilon,$$

meaning that A is ε-chainable by C in $P_{cp}(X)$. Using the conclusion a) of this proof, we get that C is ε-chainable by Y in $P_{cp}(X)$ and so we have proved that A is ε-chainable by Y in $P_{cp}(X)$. $\square$

Exercise. 1) Let $A = [0, 3], B = [1, 5]$ and $C = [4, \infty[$. Find:
a) $diam(A)$, $diam(C)$;
b) $D(0, B)$, $D(B, C)$, $D(A, C)$;
c) $\rho(A, C)$, $\rho(C, A)$, $\rho(A, B)$;
d) $H(A, B), H(A, C), H(B, C)$.

2) Let us consider the closed balls $A := \tilde{B}((0, 1); 2), B := \tilde{B}((1, 3); 1)$ in $\mathbb{R}^2$. Find $H(A, B)$ and $D((3, 4); A)$.

2.2 Nadler's Contraction Principle for Multi-valued Operators

Let X, Y be two nonempty sets. By a multivalued operator $F : X \multimap Y$ we understand an operator $F : X \to \mathcal{P}(Y)$ which assign (by a given rule) to every point $x \in X$ a set $F(x) \subset Y$. Usually, we are working with multi-valued operators with nonempty values, i.e., $F : X \to P(Y)$.

The graph of the multi-valued operator F is the set

$$Graph(F) := \{(x, y) \in X \times Y : y \in F(x)\}.$$

If $Y = X$, then a fixed point for F is an element $x^* \in X$ with $x^* \in F(x^*)$, while a strict fixed point for F is an $x^* \in X$ with $F(x^*) = \{x^*\}$. We denote by $Fix(F)$ the fixed point set and by $SFix(F)$ the strict fixed point set of F.

For a multi-valued operator $F : X \to P(X)$, the sequence of the successive approximations starting from $x_0 \in X$ is a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_{n+1} \in F(x_n)$, for every $n \in \mathbb{N}$.

The following theorem was proved by Nadler in 1969 for multi-valued operators with nonempty, bounded and closed values and it was improved by Covitz and Nadler in 1970, for the case of multi-valued operators with closed values.

Theorem. (Nadler's Contraction Principle) *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multi-valued k-contraction, i.e., $k \in [0, 1[$ and*

$$H(F(x), F(y)) \leq kd(x, y), \text{ for every } x, y \in X.$$

Then, the following conclusions hold:

(a) $Fix(F) \neq \emptyset$;

(b) for every $(x_0, x_1) \in Graph(F)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations starting from $x_0 \in X$ which converges to a fixed point of F.

Proof. Let $x_0 \in X$ be arbitrary and choose $x_1 \in F(x_0)$ also arbitrary. Let $1 < q < \frac{1}{k}$. Then, by the second Lemma on page 51, for $x_1 \in F(x_0)$ there exists $x_2 \in F(x_1)$ such that

$$d(x_1, x_2) \leq qH(F(x_0), F(x_1)).$$

Thus, by the contraction condition, we obtain

$$d(x_1, x_2) \leq qH(F(x_0), F(x_1)) \leq qkd(x_0, x_1).$$

let us denote by $K := qk < 1$. By an iterative procedure, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for F starting from $x_0 \in X$

such that

$$d(x_n, x_{n+1}) \leq K^n d(x_0, x_1), \text{ for every } n \in \mathbb{N}.$$

By a standard procedure we can show that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d). Thus, by the completeness of the space (X, d) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to an element $x^* = x^*(x_0) \in X$. We will show now that x^* is a fixed point of F. We can estimate

$$0 \leq D(x^*, F(x^*)) \leq d(x^*, x_{n+1}) + D(x_{n+1}, F(x^*)) \leq$$

$$d(x^*, x_{n+1}) + H(F(x_n), F(x^*)) \leq d(x^*, x_{n+1}) + kd(x_n, x^*), \text{ for every } n \in \mathbb{N}.$$

Letting $n \to \infty$ we obtain that $D(x^*, F(x^*)) = 0$, so using the fact that F has closed values and Lemma on page 48 we get that $x^* \in F(x^*)$. □

Exercise. *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multi-valued k-contraction. We suppose that $SFix(F) \neq \emptyset$. Show that $Fix(F) = SFix(F) = \{x^*\}$.*

2.3 Schauder's Fixed Point Theorems

2.3.1 K^2M operators

Let X be a linear space over $\mathbb{R}$. A subset A of X is called a linear subspace if for all $x, y \in A$ we have that $x + y \in A$ and for all $x \in X$ and each $\lambda \in \mathbb{R}$ we have $\lambda \cdot x \in A$.

Let A be a nonempty subset of X.

Then, the linear hull (or the span) of A, denoted by *span*(A) is, by definition, the intersection of all subspaces which contains A, i.e., the smallest linear subspace containing A. We have the following characterization of the linear hull.

$$span(A) = \{x \in X | x = \sum_{i=1}^{n} \lambda_i \cdot x_i, \text{ with } x_i \in A, \lambda_i \in \mathbb{R}, n \in \mathbb{N}\}.$$

If $A \subset \mathbb{R}^2$ and $A = \{p\}$ with $p \neq 0$, then *span*(A) is the line through p and the origin.

Similarly, the affine hull, denoted by $aff(A)$ is defined by

$$aff(A) = \{x \in X | x = \sum_{i=1}^{n} \lambda_i \cdot x_i, \ \sum_{i=1}^{n} \lambda_i = 1, x_i \in A, \lambda_i \in \mathbb{R}, n \in \mathbb{N}\}.$$

If $A = \{x_1, x_2\} \subset \mathbb{R}^2$, then $aff(A)$ is the line through x_1 and x_2.

Finally, we define the convex hull of A, denoted by $co(A)$, as the intersection of all convex subsets of X which contains A, i.e., $co(A)$ is the smallest convex set which contains A. We have the following characterization of coA.

$$co(A) = \{x \in X | x = \sum_{i=1}^{n} \lambda_i \cdot x_i, \sum_{i=1}^{n} \lambda_i = 1, x_i \in A, \lambda_i \in [0,1], n \in \mathbb{N}\}.$$

Of course, $co(A) \subset aff(A) \subset span(A)$.

Similarly, we denote by $\overline{co}(A)$ is the intersection of all convex and closed subsets of X which contains A, i.e., $\overline{co}(A)$ is the smallest convex and closed set which contains A.

Also, a k-dimensional flat (or a linear k-variety) in X is a subset L of X with $dimL = k$ such that for each $x, y \in L$, with $x \neq y$, the whole line joining x and y is included in L, i.e.,

$$(1 - \lambda) \cdot x + \lambda \cdot y \in L, \text{ for each } \lambda \in \mathbb{R}.$$

The basic fixed point theorem in a topological setting was given by Bohl-Brouwer-Hadamard in 1904-1909-1910.

Brouwer's Fixed Point Theorem. *Let Y be a compact convex subset of a finite dimensional Banach space X and $f : Y \to Y$ be a continuous operator. Then there exists at least one fixed point for f.*

Definition. A subset A of a linear space X is said to be finitely closed if its intersection with any finite-dimensional flat $L \subset X$ is closed in the Euclidean topology of L.

If X is a linear topological space, then any closed subset of X is finitely closed.

Definition. A family $\{A_i|\ i \in I\}$ of sets is said to have the finite intersection property if the intersection of each finite sub-family is not empty.

We present now the concept of KKM operator, using the definition given by Ky Fan.

Definition. Let X be a linear space and Y be a nonempty subset of X. The multivalued operator $G : Y \to P(X)$ is called a Kuratowski-Knaster-Mazurkiewicz operator (briefly K^2M operator) if and only if

$$co\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i),$$

for each finite subset $\{x_1, \ldots, x_n\} \subset Y$.

The main property of K^2M operators is given in the following theorem. We have here the Ky Fan variant (1961) of the originally KKM principle (1929).

Theorem. (K^2M principle) *Let X be a linear space, Y be a nonempty subset of X and $G : Y \to P(X)$ be a K^2M operator such that $G(x)$ is finitely closed (or, in particular, closed), for each $x \in Y$. Then the family $\{G(x)|\ x \in Y\}$ of sets has the finite intersection property.*

Proof. We argue by contradiction: assume that there exist $\{x_1, \ldots, x_n\} \subset Y$ such that $\bigcap_{i=1}^{n} G(x_i) = \emptyset$. Denote by L the finite dimensional flat spanned by $\{x_1, \ldots, x_n\}$, i.e., $L = span\{x_1, \cdots, x_n\}$. Let us denote by d the Euclidean metric in L and by $C := co\{x_1, \ldots, x_n\} \subset L$.

Because $L \cap G(x_i)$ is closed in L, for all $i \in \{1, 2, \ldots, n\}$ we have that:

$$D_d(x, L \cap G(x_i)) = 0 \ \Leftrightarrow \ x \in L \cap G(x_i), \text{ for all } i = \overline{1,n}.$$

Since $\bigcap_{i=1}^{n} [L \cap G(x_i)] = \emptyset$ it follows that the map $\lambda : C \to \mathbb{R}$ given by

$$\lambda(c) = \sum_{i=1}^{n} D_d(c, L \cap G(x_i)) \neq 0, \text{ for each } c \in C.$$

Hence we can define the continuous map $f : C \to C$ by the formula

$$f(c) = \frac{1}{\lambda(c)} \sum_{i=1}^{n} D_d(c, L \cap G(x_i))x_i.$$

By Brouwer's fixed point theorem there is a fixed point $c_0 \in C$ of f, i.e., $f(c_0) = c_0$. Let

$$I = \{i |\ D_d(c_0, L \cap G(x_i)) \neq 0\}.$$

Then

$$c_0 \notin \bigcup_{i \in I} G(x_i).$$

Indeed, arguing by contradiction, if $c_0 \in \bigcup_{i \in I} G(x_i)$, then c_0 is in at least one $G(x_i), i \in I$. This implies that $c_0 \in G(x_i) \cap L$, a contradiction with the fact that $D_d(c_0, L \cap G(x_i)) \neq 0$.

On the other side:

$$c_0 = f(c_0) \in co\{x_i |\ i \in I\} \subset \bigcup_{i \in I} G(x_i),$$

where last inclusion follows by the K^2M assumption of G. This is a contradiction, which proves the result. □

As an immediate consequence we obtain the following theorem:

Corollary. (Ky Fan) *Let X be a linear topological space, Y be a nonempty subset of X and $G : Y \to P_{cl}(X)$ be a K^2M operator. If, for $x \in X$, at least one of the sets $G(x)$ is compact, then*

$$\bigcap_{x \in Y} G(x) \neq \emptyset.$$

2.3.2 First Schauder's Fixed Point Theorem

One of the simplest application of K^2M principle is the well-known best approximation theorem of Ky Fan.

Lemma. (Ky Fan-Best approximation theorem) *Let X be a normed space, Y be a compact convex subset of X and $f : Y \to X$ be a continuous operator. Then there exists at least one $y_0 \in Y$ such that*

$$\|y_0 - f(y_0)\| = \inf_{x \in Y} \|x - f(y_0)\|.$$

Proof. Define $G : Y \to P(X)$ by

$$G(x) = \{y \in Y | \|y - f(y)\| \leq \|x - f(y)\|\}.$$

Because f is continuous, the sets $G(x)$ are closed in Y and therefore compact. We verify that G is a K^2M operator. For, let $y \in co\{x_1, \ldots, x_n\} \subset Y$. Suppose, by contradiction, that $y \notin \bigcup_{i=1}^{n} G(x_i)$. Then $\|y - f(y)\| > \|x_i - f(y)\|$ for $i \in \{1, 2, \cdots, n\}$. This shows that all the points x_i lie in an open ball of radius $\|y - f(y)\|$ centered at $f(y)$. Therefore, the convex hull of it is also there and in particular y. Thus $\|y - f(y)\| > \|y - f(y)\|$, which is a contradiction. By the compactness of $G(x)$ we find a point y_0 such that $y_0 \in \bigcap_{x \in Y} G(x)$ and hence $\|y_0 - f(y_0)\| \leq \|x - f(y_0)\|$, for all $x \in Y$. This clearly implies $\|y_0 - f(y_0)\| = \inf_{x \in Y} \|x - f(y_0)\|$ and the proof is complete. □

Theorem. *Let Y be a compact convex subset of a Banach space X. Let $f : Y \to X$ be a continuous operator such that for each $x \in Y$ with $x \neq f(x)$, the line segment $[x, f(x)]$ contains at least two points of Y. Then f has at least a fixed point.*

Proof. By the previous Lemma, we obtain an element $y_0 \in Y$ with $\|y_0 - f(y_0)\| = \inf_{x \in Y} \|x - f(y_0)\|$. We will show that y_0 is a fixed point of f. The segment $[y_0, f(y_0)]$ must contain a point of Y other than y_0, let say x. Then $x = ty_0 + (1 - t)f(y_0)$, with some $t \in]0, 1[$. Then $\|y_0 - f(y_0)\| \leq t\|y_0 - f(y_0)\|$ and since $t < 1$, we must have $\|y_0 - f(y_0)\| = 0$. □

Corollary. (Schauder I) *Let Y be a compact convex subset of a Banach space X. Let $f : Y \to Y$ be a continuous operator. Then f has at least a fixed point.*

2.3.3 Second and Third Schauder's Fixed Point Theorem

Definition. Let X, Y be two Banach spaces, $K \subseteq X$ and $f : K \to Y$. Then f is called:

1) continuous, if $x_n \in K$, $n \in \mathbb{N}$ with $x_n \to x \in K$ as $n \to +\infty$ implies $f(x_n) \to f(x)$ as $n \to +\infty$;

2) with closed graph, if $x_n \in K$, $n \in \mathbb{N}$ with $x_n \to x$ and $f(x_n) \to y$ as $n \to +\infty$ implies $x \in K$ and $y = f(x)$;

3) bounded, if for each bounded subset A of K, implies $f(A)$ is bounded in Y;

4) compact, if for each bounded subset A of K the set $f(A)$ is relatively compact in Y;

5) completely continuous, if f is continuous and compact;

6) with relatively compact range, if f is continuous and $f(K)$ is relatively compact in Y (i.e., $\overline{f(K)}$ is compact);

Two well-known results are:

Lemma. *a) Let $f : K \to Y$ be a continuous function and $A \subset K$ be compact. Then $f(A)$ is compact too.*

b) If M is a compact set and $Z \subseteq M$, then Z is relatively compact.

Remark. i) If $f : K \to Y$ is with relatively compact range, then f is completely continuous.

ii) Suppose X is a finite dimensional space, $K \subseteq X$ is closed and $f : K \subset X \to Y$. Then f is completely continuous if and only if f is continuous.

Proof. i) If f is with relatively compact range, then $\overline{f(K)}$ is compact in Y. Let $A \subset K$ be bounded. Then, $f(A) \subset f(K) \subset \overline{f(K)}$. Hence $f(A)$

is relatively compact.

ii) Let A be a bounded subset of K. Then $\overline{A}$ is a compact set (since the space X is finite dimensional). Then, $f(\overline{A})$ is a compact set. From $f(A) \subset f(\overline{A})$ and by the previous Lemma b), we get that $f(A)$ is relatively compact. □

Recall now a very important theorem in functional analysis.

Mazur's Theorem. *a) Let X be a Banach space and M be a relatively compact subset of it. Then $co(M)$ is relatively compact.*

b) Let X be a Banach space and M be a relatively compact subset of it. Then $\overline{co}(M)$ is compact.

c) If X is a finite dimensional normed space and $M \subset X$ is compact, then $co(M)$ is compact too.

The next result (Schauder' second theorem) is very useful for applications.

Theorem. (Schauder II) *Let Y be a bounded closed convex subset of a Banach space X. Let $f : Y \to Y$ be a completely continuous operator. Then f has at least a fixed point.*

Proof. Since f is completely continuous and Y is bounded we have that $f(Y)$ is relatively compact in X. From Mazur's theorem we know that the closed convex hull of a relatively compact subset of a Banach space is compact. Hence $K := \overline{co}(f(Y))$ is compact and convex. Since Y is closed and convex and $f(Y) \subset Y$ we get that $K \subset Y$. Then

$$f(K) \subseteq f(Y) \subseteq K := \overline{co}(f(Y)).$$

Thus $f : K \to K$. Also, $K \in P_{cp.cv}(X)$. By Schauder I, we obtain that $Fix(f) \neq \emptyset$. □

By Schauder I we immediately obtain the following result.

Theorem. (Schauder III) *Let Y be a closed convex subset of a Banach space X. Let $f : Y \to Y$ be an operator with relatively compact range. Then f has at least a fixed point.*

Proof. Since f is with relatively compact range, we have that f is continuous and $f(Y)$ is relatively compact in X. From Mazur's theorem we get that $K := \overline{co}(f(Y))$ is compact and convex. Since Y is closed and convex and $f(Y) \subset Y$ we get that $K \subset Y$. Thus $f : K \to K$. By Schauder I we get the conclusion.

2.4 Applications

Let (X, d) be a compact metric space and denote

$$C(X, \mathbb{R}) := \{f : X \to \mathbb{R} | f \text{ continuous } \}.$$

Then $(C(X, \mathbb{R}), \|\cdot\|_{C,B})$ is a Banach space.

Definition. A subset $Y \subset C(X, \mathbb{R})$ is said called:

i) bounded if there is $M > 0$ such that $|u(x)| \le M$, for each $u \in Y$ and each $x \in X$;

ii) echicontinuous if for each $\epsilon > 0$ there is $\delta > 0$ such that the following implication holds:

$$d(x_1, x_2) < \delta \quad \Rightarrow \quad |u(x_1) - u(x_2)| < \epsilon, \quad \forall\, u \in Y.$$

Theorem. (Ascoli-Arzela) *$Y \subset C(X, \mathbb{R})$ is relatively compact if and only if Y is bounded and echicontinuous.*

Theorem. (The Fredholm integral operator)
Let $K : [a, b] \times [a, b] \times [-R, R] \to \mathbb{R}$ continuous. Consider the Fredholm integral operator

$$T : C[a, b] \to C[a, b], \qquad u \longmapsto Tu$$

given by

$$Tu(x) := \int_a^b K(x, s, u(s))ds,\ x \in [a, b].$$

Then T is completely continuous.

Proof. 1) The continuity of T.
Let $u_0 \in C[a,b]$ be arbitrary and $\varepsilon > 0$. We will prove that T is continuous in u_0, i.e., for $\varepsilon > 0$ there exists $\delta(u_0, \varepsilon) > 0$ such that, if $u \in C[a,b]$ with $\|u - u_0\| \leq \delta$ implies $\|Tu - Tu_0\| \leq \varepsilon$.

Since K is continuous on the compact set $W := [a,b] \times [a,b] \times [-R,R]$, it is uniformly continuous with respect to the third variable. Hence, there exists $\delta_1(\varepsilon) > 0$ such that for any $p, q \in [-R,R]$ with $|p - q| < \delta_1(\varepsilon)$ implies $|K(x,s,p) - K(x,s,q)| < \frac{\varepsilon}{b-a}$, for each $(x,s) \in [a,b] \times [a,b]$.

Then, there exists $\delta(u_0, \varepsilon) := \delta_1(\varepsilon) > 0$ such that for each $u \in C[a,b]$ with $\|u - u_0\| \leq \delta$ we have $|K(x,s,u(s)) - K(x,s,u_0(s))| < \frac{\varepsilon}{b-a}$, for each $(x,s) \in [a,b] \times [a,b]$.

Thus, $|Tu(x) - Tu_0(x)| \leq \int_a^b |K(x,s,u(s)) - K(x,s,u_0(s))| ds \leq \varepsilon$, for each $x \in [a,b]$. Takig the $\sup\limits_{x \in [a,b]}$ we get that $\|Tu - Tu_0\| \leq \varepsilon$.

2) We will prove now that T is compact, i.e. for each bounded subset Y of $C[a,b]$ the set $\overline{T(Y)}$ is compact.

By Ascoli-Arzela, it is enough to prove that $T(Y)$ is bounded and equicontinuous.

i) We prove first that $T(Y)$ is bounded, i.e., there exists $M > 0$ such that $\|v\| \leq M$, for every $v \in T(Y)$.
We have: $|Tu(x)| \leq \int_a^b |K(x,s,u(s))| ds \leq M_K(b-a) := M$ (where $M_K := \max\limits_{(x,s,p) \in W} |K(x,s,p)|$). By taking $\sup\limits_{x \in [a,b]}$, we get $v := \|Tu\| \leq M$, for each $u \in Y$.

ii) We prove now that $T(Y)$ is equicontinuous.
Since K is uniformly continuous on $W := [a,b] \times [a,b] \times [-R,R]$ with respect to the first variable we can write that there exists $\delta_2(\varepsilon) > 0$ such that for any $x_1, x_2 \in [a,b]$ with $|x_1 - x_2| < \delta_2(\varepsilon)$ implies $|K(x_1,s,p) - K(x_2,s,p)| < \frac{\varepsilon}{b-a}$, for each $(s,p) \in [a,b] \times [-R,R]$. Hence there exists $\delta_2(\varepsilon) > 0$ such that for any $x_1, x_2 \in [a,b]$ with $|x_1 - x_2| < \delta_2(\varepsilon)$ and any $u \in Y$ we have $|K(x_1,s,u(s)) - K(x_2,s,u(s))| < \frac{\varepsilon}{b-a}$, for each $s \in [a,b]$.

Thus, $|Tu(x_1) - Tu(x_2)| \leq \int_a^b |K(x_1, s, u(s)) - K(x_2, s, u(s))| \leq \frac{\varepsilon}{b-a}(b - a) = \varepsilon$. As a conclusion, $T(Y)$ is equicontinuous. □

Remark. *a) Let $K : [a, b] \times [a, b] \times [-R, R] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous. Consider the Fredholm-type integral operator*

$$T : C[a, b] \to C[a, b] \qquad u \longmapsto Tu$$

given by

$$Tu(x) := \int_a^b K(x, s, u(s))ds + g(x),\ x \in [a, b].$$

Then T is completely continuous.

b) Let $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ be continuous. Consider the Fredholm integral operator

$$T : C[a, b] \to C[a, b]$$

$$u \longmapsto Tu$$

given by

$$Tu(x) := \int_a^b K(x, s, u(s))ds,\ x \in [a, b].$$

Then T is completely continuous.

An existence result for Fredholm integral equation is:

Theorem. *Let $K : [a, b] \times [a, b] \times [-R, R] \to \mathbb{R}$ be continuous. Consider the Fredholm integral equation:*

$$u(x) = \lambda \int_a^b K(x, s, u(s))ds,\ x \in [a, b].$$

Suppose that $|\lambda| \leq \frac{R}{M_K(b-a)}$, where $M_K := \max\limits_{(x,s,p)\in W} |K(x, s, p)|$ (here $W := [a, b] \times [a, b] \times [-R, R]$).

Then there at least one $u^* \in \tilde{B}(0; R) \subset C[a, b]$ a solution to the above Fredholm integral equation.

Proof. Using the previous theorem we have that the Fredholm integral operator

$$T : \tilde{B}(0; R) \subset C[a, b] \to C[a, b]$$
$$u \longmapsto Tu$$

given by

$$Tu(x) := \lambda \int_a^b K(x, s, u(s))ds, \ x \in [a, b]$$

is completely continuous.

We will prove now that the set $\tilde{B}(0; R)$ is invariant with respect to T. Indeed, let $u \in \tilde{B}(0; R)$. We will show that $Tu \in \tilde{B}(0; R)$.

We have: $|Tu(x)| \leq |\lambda| \int_a^b |K(t, s, u(s))|ds \leq |\lambda| M_K(b - a) \leq R$. By taking $\max\limits_{x \in [a,b]}$, we get that $\|Tu\| \leq R$, for each $u \in \tilde{B}(0; R)$.

Hence we have that $T : \tilde{B}(0; R) \subset C[a, b] \to \tilde{B}(0; R)$ is completely continuous on the bounded, closed and convex subset $\tilde{B}(0; R)$ of the Banach space $C[a, b]$. By Schauder II, there exists at least one fixed point $u^* \in \tilde{B}(0; R)$ for T. This fixed point is a solution of the above Fredholm integral equation. □

Theorem. (The Volterra integral operator)
Let $K : [a, b] \times [a, b] \times [-R, R] \to \mathbb{R}$ continuous. Consider the Volterra integral operator

$$T : C[a, b] \to C[a, b]$$
$$u \longmapsto Tu$$

given by

$$Tu(t) := \int_a^t K(t, s, u(s))ds, \ t \in [a, b].$$

Then T is completely continuous.

Remark. *a) Let $K : [a, b] \times [a, b] \times [-R, R] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous. Consider the Volterra-type integral operator*

$$T : C[a, b] \to C[a, b]$$

$$u \longmapsto Tu$$

given by

$$Tu(t) := \int_a^t K(t,s,u(s))ds + g(t),\ t \in [a,b].$$

Then T is completely continuous.

b) Let $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ be continuous. Consider the Volterra integral operator

$$T : C[a,b] \to C[a,b]$$

$$u \longmapsto Tu$$

given by

$$Tu(x) := \int_a^t K(x,s,u(s))ds,\ x \in [a,b].$$

Then T is completely continuous.

An existence result for a Volterra-type equation is:

Theorem. *Let $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ continuous, such that there exist $\alpha, \beta > 0$ such that $|K(t,s,p)| \leq \alpha \cdot |p| + \beta$, for each $(t,s) \in [a,b]$ and $p \in \mathbb{R}$. Consider $g \in C[a,b]$.*

Then there exists at least one solution of the following Volterra-type integral equation:

$$u(t) = \int_a^t K(t,s,u(s))ds + g(t),\ t \in [a,b].$$

Proof. Consider on $C[a,b]$ the Bielecki-type norm, with arbitrary $\tau > 0$, i.e.,

$$\|u\|_B := \max_{t \in [a,b]} |u(t)| \cdot e^{-\tau(t-a)}.$$

Let $R > 0$ be arbitarily chosen and $\tilde{B}(0;R) \subset (C[a,b], \|\cdot\|_B)$.

STEP 1. The operator

$$T : \tilde{B}(0;R) \subset C[a,b] \to C[a,b]$$

$$u \longmapsto Tu$$

given by

$$Tu(t) := \int_a^t K(t,s,u(s))ds + g(t),\ t \in [a,b]$$

is completely continuous, by the above theorem on Volterra integral operators.

STEP 2. We prove that $\tilde{B}(0;R) \subset (C[a,b], \|\cdot\|_B)$ is invariant with respect to T.
Let $u \in \tilde{B}(0;R)$. Then, we have:

$|Tu(t)| \le \int_a^t |K(t,s,u(s))|ds + \|g\| \le \alpha \int_a^t |u(s)|ds + \beta(b-a) + \|g\| = \alpha \int_a^t |u(s)|e^{-\tau(s-a)}e^{\tau(s-a)}ds + \beta(b-a) + \|g\| \le \alpha\|u\|_B \cdot \frac{1}{\tau}e^{\tau(t-a)} + \beta(b-a) + \|g\|$.

Hence, $|Tu(t)|e^{-\tau(t-a)} \le \frac{\alpha}{\tau}\|u\|_B + \beta(b-a) + \|g\|$, for each $t \in [a,b]$.

We choose $\tau > 0$ such that $\frac{\alpha}{\tau}R + \beta(b-a) + \|g\| \le R$. Thus, $\|Tu\|_B \le R$ and then $T : \tilde{B}(0;R) \subset (C[a,b], \|\cdot\|_B) \to \tilde{B}(0;R) \subset (C[a,b], \|\cdot\|_B)$.

The conclusion follows now by Schauder II. □

Application to differential equations.

A. Peano's Theorem.

Consider the Cauchy problem:

$$u'(t) = f(t,u(t)), \quad u(t_0) = u^0,$$

where $f : D \to \mathbb{R}$ *is continuous.*
(here $D := \{(t,u) \in \mathbb{R}^2 | t \in [t_0 - a, t_0 + a] \times [u^0 - b, u^0 + b]\}$*).*
Then the Cauchy problem has at least one solution in $C[t_0 - h, t_0 + h]$*, where* $h := \min\{a, \frac{b}{M}\}$ *(with* $M = \max_D |f(t,u)|$*).*

Proof. Denote

$$X := (C[t_0 - h, t_0 + h], \|\cdot\|) \text{ and } Y := \tilde{B}(u^0; b) \subset X.$$

Define $T : \tilde{B}(u^0;b) \subset X \to X,\ x \longmapsto Tx$, where

$$Tu(t) := \int_{t_0}^t f(s,u(s))ds + u^0, \quad \text{for } t \in [t_0 - h, t_0 + h].$$

Notice that the Cauchy problem is now equivalent to the following fixed point problem: $u = Tu$.

We have:

1) $T : Y \to Y$

Indeed, we will prove that if $u \in Y$, then $Tu \in Y$. We have $|Tu(t) - u^0| \leq \int_{t_0}^{t} |f(s, u(s))| ds \leq M(t - t_0) \leq Mh \leq M\frac{b}{M} = b$, for each $t \in [t_0 - h, t_0 + h]$. By taking the maximum of $t \in [t_0 - h, t_0 + h]$, we get that

$$\|Tu - u^0\| \leq b, \text{ for every } u \in Y.$$

Thus $T(u) \in Y$, for every $u \in Y$.

2) T is completely continuous from the above theorem on Volterra operators.

Hence, by Schauder II, we get that T has at least one fixed point in Y. This fixed point is a solution for our Cauchy problem. □

B. Boundary Value Problem of Dirichlet-type.

Consider the following boundary value problems of Dirichlet-type:

I.
$(I1)$ $x''(t) = 0,\ t \in [a, b]$
$(I2)$ $x(a) = \alpha,\ x(b) = \beta.$

II.
$(II1)$ $x''(t) = f(t),\ t \in [a, b]$
$(II2)$ $x(a) = 0,\ x(b) = 0,$

where $f : [0, T] \to \mathbb{R}$ is a continuous function.

III.
$(III1)$ $x''(t) = f(t),\ t \in [a, b]$
$(III2)$ $x(a) = \alpha,\ x(b) = \beta,$

where $f : [a, b] \to \mathbb{R}$ is a continuous function.

IV.
$(IV1)$ $x''(t) = f(t, x(t)),\ t \in [a, b]$
$(IV2)$ $x(a) = 0,\ x(b) = 0,$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

V.
$(V1)$ $x''(t) = f(t, x(t)),\ t \in [a, b]$
$(V2)$ $x(a) = \alpha,\ x(b) = \beta,$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The purpose is to solve or to give existence results for these problems.

• The unique solution to problem $(I1)$ and $(I2)$ is:

$$x_I(t) = \frac{t-a}{b-a}\beta + \frac{b-t}{b-a}\alpha,\ t \in [a,b].$$

• The unique solution to problem $(II1)$ and $(II2)$ is:

$$x_{II}(t) = -\int_a^b G(t,s)f(s)ds,\ t \in [a,b],$$

where $G : [a,b] \times [a,b] \to \mathbb{R}$ is the Green function corresponding to the problem (II), i.e.,

$$G(t,s) := \begin{cases} \frac{(s-a)(b-t)}{b-a}, & \text{if } s \leq t \\ \frac{(t-a)(b-s)}{b-a}, & \text{if } s \geq t \end{cases}$$

• The unique solution to problem $(III1)$ and $(III2)$ is:

$$x_{III}(t) = -\int_a^b G(t,s)f(s)ds + \frac{t-a}{b-a}\beta + \frac{b-t}{b-a}\alpha,\ t \in [a,b].$$

• Problem $(IV1)$ and $(IV2)$ is equivalent to the following Fredholm integral equation:

$$x(t) = -\int_a^b G(t,s)f(s,x(s))ds,\ t \in [a,b].$$

• Problem $(V1)$ and $(V2)$ is equivalent to the following Fredholm-type integral equation:

$$x(t) = -\int_a^b G(t,s)f(s,x(s))ds + \frac{t-a}{b-a}\beta + \frac{b-t}{b-a}\alpha,\ t \in [a,b].$$

Consider the problem $(IV1)$ and $(IV2)$.
$(IV1)$ $x''(t) = f(t,x(t)),\ t \in [a,b]$
$(IV2)$ $x(a) = 0,\ x(b) = 0.$

We have the following existence result.

Theorem. *Let* $f : [a,b] \times [-R,R] \to \mathbb{R}$ *be a continuous function, where* $R > 0$ *is such that if* $x \in \tilde{B}(0;R) \subset C[a,b]$ *then* $x(t) \in [-R,R]$*, for each* $t \in [a,b]$*. Suppose* $M(b-a) \le R$*, where* $M := \max_{(t,u)\in[a,b]\times[-R,R]} |G(t,s)| \cdot |f(t,u)|$.

Then, the problem $(IV1)$ *and* $(IV2)$ *has at least one solution in* $\tilde{B}(0;R) \subset C[a,b]$.

Proof. STEP 1. The problem $(IV1)$ and $(IV2)$ is equivalent to the following Fredholm integral equation:

$$x(t) = -\int_a^b G(t,s)f(s,x(s))ds,\ t \in [a,b].$$

STEP 2. The operator $T : \tilde{B}(0;R) \subset C[a,b] \to C[a,b],\ \ x \longmapsto Tx$, where

$$Tx(t) := -\int_a^b G(t,s)f(s,x(s))ds,\ t \in [a,b]$$

is completely continuous by the corresponding result for Fredholm operators.

STEP 3. We prove that $T : \tilde{B}(0;R) \subset C[a,b] \to \tilde{B}(0;R)$.
Indeed, let $x \in \tilde{B}(0;R) \subset C[a,b]$. Then $|Tx(t)| \le \int_a^b |G(t,s)| \cdot |f(s,x(s))|ds \le M(b-a) \le R$, for each $t \in [a,b]$. Hence $\|Tx\| \le R$, for every $x \in \tilde{B}(0;R)$.

By Schauder II, we get that T has at least one fixed point in $x^* \in \tilde{B}(0;R) \subset C[a,b]$. This fixed point is clearly a solution to problem $(IV1)$ and $(IV2)$. □

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