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MULTI-VALUED ANALYSIS AND APPLICATIONS

.

"Learn to labour and to wait"

Longfellow

Contents

Introduction

Let X and Y be two nonempty sets and $F, G : X \to \mathcal{P}(Y)$ be multi-valued operators. An operatorial inclusion is, by definition, a problem of the following type:

(1) Find $x \in X$ satisfying the relation : $F(x) \cap G(x) \neq \emptyset$.

Such elements x are called coincidence points for F and G.

Examples:

i) If $G : X \to Y$ is defined by $G(x) = \{f(x)\}$, where $f : X \to Y$ is a single-valued map, then (1) becomes a coincidence problem for f and F or a f-fixed point problem for F:

(1_i) Find $x \in X$ such that $f(x) \in F(x)$

ii) If $G : X \to P(X)$ is defined by $G(x) = \{x\}$, then we get a fixed point problem for the multi-valued operator F:

(1_{ii}) Find $x \in X$ such that $x \in F(x)$

Moreover, a special type of fixed point problem is the following:

(1^*_{ii}) Find $x \in X$ such that $\{x\} = F(x)$

An element x having this property is called a strict fixed point for F.

iii) If $G : X \to P(Y)$ is defined by the constant operator $G(x) = \{y\}$, for each $x \in X$, then (1) is a surjectivity problem for the multi-function F:

(1_{iii}) Given $y \in Y$ find $x \in X$ such that $y \in F(x)$.

iv) If Y is a linear space and $G : X \to P(Y)$ is defined by the zero constant operator $G(x) = \{0\}$, for each $x \in X$, then (1) is a zero point problem or an equilibrium problem for the multi-function F:

(1_{iv}) Find $x \in X$ such that $0 \in F(x)$.

By definition, an operatorial anti-inclusion is the following problem:

(2) Find $x \in X$ satisfying the relation $F(x) \cap G(x) = \emptyset$.

An element $x \in X$ having this property is called an anti-coincidence point for F and G.

Examples:

i) If $F = G$ (that means $F(x) = G(x)$, for each $x \in X$), then (2) becomes a maximal element problem for F:

(2_i) Find $x \in X$ such that $F(x) = \emptyset$

ii) If $G : X \to P(Y)$ is defined by $G(x) = \{f(x)\}$, for each $x \in X$, where $f : X \to Y$ is a single-valued operator, then we obtain the following non-selection problem:

(2_{ii}) Find $x \in X$ such that $f(x) \notin F(x)$

Let us remark that if problem (2_{ii}) has no solutions, then the mapping f is a selection for F, i.e. $f(x) \in F(x)$, for each $x \in X$.

iii) If $G : X \to P(X)$ is defined by $G(x) = \{x\}$, then we get a non-fixed point problem for the multi-valued operator F:

(2_{iii}) Find $x \in X$ such that $x \notin F(x)$

Let us remark that (2_{iii}) is equivalent with the following operator inclusion: find $x \in X$ such that $x \in C_Y(F(x))$.

Similarly, we can consider the strict non-fixed point problem, the non-surjectivity problem or the non-zero point problem.

Selections, fixed points, coincidence points, zero points for multi-functions, integral and differential inclusions are several keywords and phrases which characterize this course. In fact, there are two important purposes of it. First, some abstract operatorial inclusions with focus on fixed points, coincidence points and selections are presented. Second, some applications of the abstract theory to integral and differential inclusions, dynamical systems and the theory of self-similar sets are considered.

• **Evaluation:** during the semester: two written tests (WT1, WT2) and course+seminar activity (CSA).

• The schedule for the two written tests: First written test (WT1): November 15, 2023; Second written test (WT2): December 13, 2023.

• **Final mark** $FM := 40\%\ WT1 + 40\%\ WT2 + 20\%\ CSA$.

Adrian Petruşel September 2023

Chapter 1

Multi-valued analysis

The purpose of this chapter is to report the basic theory of multi-valued operators between metric spaces. More precisely, basic properties of the Pompeiu-Hausdorff generalized metric, continuity and measurability concepts and results for multi-functions are reported. (many of them, being already classical theorems, are presented without proofs.)

1.1 Pompeiu-Hausdorff metric

The aim of this section is to present the main properties of some (generalized) functionals defined on the space of all subsets of a metric space. A special attention is paid to gap functional, excess functional and to Pompeiu-Hausdorff functional.

Let (X, d) be a metric space. Recall that a metric d for a nonempty set X is a functional $d : X \times X \to \mathbb{R}_+$ satisfying the following axioms:

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$ for every $x, y \in X$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for every $x, y, z \in X$.

In what follows, sometimes we will need to consider infinite-valued metrics, also called generalized metrics $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$, see Luxemburg [136] and Jung [116].

Throughout this book, we denote by $\mathcal{P}(X)$ the space of all subsets of a nonempty set X. If X is a metric space, $x \in X$ and $R > 0$, then $B(x, R)$ and respectively $\widetilde{B}(x, R)$ denote the open, respectively the closed ball of radius R centered in x. If X is a topological space and Y is a subset of X, then we will denote by $\overline{Y}$ the closure and by $intY$ the interior of the set Y. Also, if X is a normed space and Y is a nonempty subset of X, then $convY$ respectively $\overline{conv}Y$ denote the convex hull, respectively the closed convex hull of the set Y.

We consider, for the beginning, the diameter generalized functional defined on the space of all subsets of a metric space X.

Definition 1.1.1. Let (X, d) be a metric space. The diameter generalized functional, $diam : \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by:

$$diam(Y) = \begin{cases} \sup\{d(a,b) |\ a \in Y,\ b \in Y\}, & \text{if } Y \neq \emptyset \\ 0, & \text{if } Y = \emptyset \end{cases}$$

Definition 1.1.2. The subset Y of X is said to be bounded if and only if $diam(Y) < \infty$.

Lemma 1.1.3. *Let (X, d) be a metric space and Y, Z nonempty bounded subsets of X. Then:*

i) $diam(Y) = 0$ if and only if $Y = \{y_0\}$.

ii) If $Y \subset Z$ then $diam(Y) \leq diam(Z)$.

iii) $diam(\overline{Y}) = diam(Y)$.

iv) If $Y \cap Z \neq \emptyset$ then $diam(Y \cup Z) \leq diam(Y) + diam(Z)$.

v) If X is a normed space then:

a) $diam(x + Y) = diam(Y)$, for each $x \in X$.

b) $diam(\alpha Y) = |\alpha| diam(Y)$, where $\alpha \in \mathbb{R}$.

c) $diam(Y) = diam(conv\, Y)$.

d) $diam(Y) \leq diam(Y+Z) \leq diam(Y) + diam(Z)$.

Proof. iii) Because $Y \subseteq \overline{Y}$ we have $diam(Y) \leq diam(\overline{Y})$. For the reverse inequality, let consider $x, y \in \overline{Y}$. Then there exist $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} \subset Y$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. It follows that $d(x_n, y_n) \stackrel{\mathbb{R}}{\to} d(x,y)$. Because $d(x_n, y_n) \leq diam(Y)$, for all $n \in \mathbb{N}$ we get by passing to limit $d(x,y) \leq diam(Y)$. Hence $diam(\overline{Y}) \leq diam(Y)$.

iv) Let $u, v \in Y \cup Z$. We have the following cases:

a) If $u, v \in Y$ then $d(u,v) \leq diam(Y) \leq diam(Y) + diam(Z)$ ąnd so $diam(Y \cup Z) \leq diam(Y) + diam(Z)$.

b) If $u, v \in Z$ then by an analogous procedure we have $d(u,v) \leq diam(Z) \leq diam(Y) + diam(Z)$ and so $diam(Y \cup Z) \leq diam(Y) + diam(Z)$.

c) If $u \in Y$ and $v \in Z$ then choosing $t \in Y \cap Z$ we have that $d(u,v) \leq d(u,t) + d(t,v) \leq diam(Y) + diam(Z)$. Hence, $diam(Y \cup Z) \leq diam(Y) + diam(Z)$.

v) c) Let us prove that $diam(convY) \leq diam(Y)$. Let $x, y \in convY$. Then there exist $x_i, y_j \in Y$, $\lambda_i, \mu_j \in \mathbb{R}_+$, such that

$$x = \sum_{i=1}^{n} \lambda_i x_i, \quad y = \sum_{j=1}^{m} \mu_j y_j, \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \sum_{j=1}^{m} \mu_j = 1.$$

From these relations we have:

$$\|x-y\| = \left\| \sum_{i=1}^{n} \lambda_i x_i - \sum_{j=1}^{m} \mu_j y_j \right\| = \left\| \left(\sum_{j=1}^{m} \mu_j \right) \sum_{i=1}^{n} \lambda_i x_i - \left(\sum_{i=1}^{n} \lambda_i \right) \sum_{j=1}^{m} \mu_j y_j \right\|$$
$$\leq \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i \mu_j \|x_i - y_j\| \leq \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i \mu_j \right) diam(Y) = diam(Y).$$

□

Let us consider now the following spaces of subsets of a metric space (X, d):

$$P(X) = \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\};\ P_b(X) = \{Y \in P(X) |\ diam(Y) < \infty\};$$

$P_{op}(X) = \{Y \in P(X) | \; Y$ is open $\}$; $P_{cl}(X) = \{Y \in P(X) | \; Y$ is closed $\}$;

$P_{b,cl}(X) = P_b(X) \cap P_{cl}(X)$; $P_{cp}(X) = \{Y \in P(X) | \; Y$ is compact $\}$;

$P_{cn}(X) = \{Y \in P(X) | \; Y$ is connex$\}$.

If X is a normed space, then we denote:

$$P_{cv}(X) = \{Y \in P(X) | \; Y \text{ convex}\};\; P_{cp,cv}(X) = P_{cp}(X) \cap P_{cv}(X).$$

Let us define the following generalized functionals:

(1) $D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$$D(A,B) = \begin{cases} \inf\{d(a,b) | \; a \in A, \; b \in B\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

the so-called distance between the sets A and B or the gap functional.

In particular, $D(x_0, B) = D(\{x_0\}, B)$ (where $x_0 \in X$) is called the distance from the point x_0 to the set B.

(2) $\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\delta(A,B) = \begin{cases} \sup\{d(a,b) | \; a \in A, \; b \in B\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{otherwise} \end{cases}$$

(3) $\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\rho(A,B) = \begin{cases} \sup\{D(a,B) | \; a \in A\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset \\ +\infty, & \text{if } B = \emptyset \neq A \end{cases}$$

the so-called excess functional.

(4) $H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$H(A,B) = \begin{cases} \max\{\rho(A,B), \rho(B,A)\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

the so-called Pompeiu-Hausdorff generalized functional.

Let us prove now that the functional H is a metric on the space $P_{b,cl}(X)$. First we will prove the following auxiliary result:

Lemma 1.1.5. *$D(b, A) = 0$ if and only if $b \in \overline{A}$.*

Proof. We shall prove that $\overline{A} = \{x \in X |\ D(x, A) = 0\}$. For this aim, let $x \in \overline{A}$ be arbitrarily. It follows that for each $r > 0$ and for each $B(x, r) \subset X$ we have $A \cap B(x, r) \neq \emptyset$. Then for each $r > 0$ there exists $a_r \in A$ such that $d(x, a) < r$. It follows that for each $r > 0$ we have $D(x, A) < r$ and hence $D(x, A) = 0$. □

Lemma 1.1.6. *Let (X, d) be a metric space. Then the pair $(P_{b,cl}(X), H)$ is a metric space.*

Proof. We shall prove that the axioms of the metric hold:

a) $H(A, B) \geq 0$, for all $A, B \in P_{b,cl}(X)$ is obviously.

$H(A, B) = 0$ is equivalent with $\rho(A, B) = 0$ and $\rho(B, A) = 0$, that means $\sup\limits_{a \in A} D(a, B) = 0$ and $\sup\limits_{b \in B} D(b, A) = 0$. Hence $D(a, B) = 0$, for each $a \in A$ and $D(b, A) = 0$, for each $b \in B$. Using Lemma 1.1.5. we obtain that $a \in B$, for all $a \in A$ and $b \in A$, for all $b \in B$, proving that $A \subseteq B$ and $B \subseteq A$.

b) $H(A, B) = H(B, A)$ is quite obviously.

c) For the third axiom of the metric, let consider $A, B, C \in P_{b,cl}(X)$. For each $a \in A$, $b \in B$ and $c \in C$ we have $d(a, c) \leq d(a, b) + d(b, c)$. It follows that $\inf\limits_{c \in C} d(a, c) \leq d(a, b) + \inf\limits_{c \in C} d(b, c)$, for all $a \in A$ and $b \in B$. We get $D(a, C) \leq d(a, b) + D(b, C)$, for all $a \in A$, $b \in B$. Hence $D(a, C) \leq D(a, B) + H(B, C)$, for all $a \in A$ and so $D(a, C) \leq H(A, B) + H(B, C)$, for all $a \in A$. In conclusion, we have proved that $\rho(A, C) \leq H(A, B) + H(B, C)$. Similarly, we get $\rho(C, A) \leq H(A, B) + H(B, C)$, and so $H(A, C) \leq H(A, B) + H(B, C)$. □

Remark 1.1.7. H is called the Pompeiu-Hausdorff metric induced by the metric d. Occasionally, we will denote by H_d the Pompeiu-Hausdorff functional generated by the metric d of the space X.

Remark 1.1.8. H is a generalized metric on $P_{cl}(X)$.

Lemma 1.1.9. *Let (X, d) a metric space. Then we have:*

i) $D(\cdot, Y) : (X, d) \to \mathbb{R}_+$, $x \mapsto D(x, Y)$, (where $Y \in P(X)$) is nonexpansive.

ii) $D(x, \cdot) : (P_{cl}(X), H) \to \mathbb{R}_+$, $Y \mapsto D(x, Y)$, (where $x \in X$) is nonexpansive.

Proof. i) We shall prove that for each $Y \in P(X)$ we have

$$|D(x_1, Y) - D(x_2, Y)| \le d(x_1, x_2), \text{for all } x_1, x_2 \in X.$$

Let $x_1, x_2 \in X$ be arbitrarily. Then for all $y \in Y$ we have

$d(x_1, y) \le d(x_1, x_2) + d(x_2, y)$. Then $\inf\limits_{y \in Y} d(x_1, y) \le d(x_1, x_2) + \inf\limits_{y \in Y} d(x_2, y)$ and so $D(x_1, Y) \le d(x_1, x_2) + D(x_2, y)$. We have proved that $D(x_1, y) - D(x_2, Y) \le d(x_1, x_2)$. Interchanging the roles of x_1 and x_2 we obtain $D(x_2, Y) - D(x_1, Y) \le d(x_1, x_2)$, proving the conclusion.

ii) We shall prove that for each $x \in X$ we have:

$$|D(x, A) - D(x, B)| \le H(A, B), \text{ for all } A, B \in P_{cl}(X).$$

Let $A, B \in P_{cl}(X)$ be arbitrarily. Let $a \in A$ and $b \in B$. Then we have $d(x, a) \le d(x, b) + d(b, a)$. It follows $D(x, A) \le d(x, b) + D(b, A) \le d(x, b) + H(B, A)$ and hence $D(x, A) - D(x, B) \le H(A, B)$. By a similar procedure we get $D(x, B) - D(x, A) \le H(A, B)$ and so $|D(x, A) - D(x, B)| \le H(A, B)$, for all $A, B \in P_{b,cl}(X)$. □

Lemma 1.1.10. *Let (X, d) be a metric space. Then the generalized functional diam : $(P_{cl}(X), H) \to \mathbb{R}_+ \cup \{+\infty\}$ is continuous.*

Lemma 1.1.11. *Let (X, d) be a metric space. Then we have:*

i) $D(\overline{Y},\overline{Z}) = D(Y,Z)$, *for all* $Y,Z \in P(X)$.

ii) $D(Y,Z) \leq D(Y,W) + D(W,Z) + diam(W)$, *for all* $Y,Z,W \in P(X)$.

iii) $D(Y, Z\cup W) = \min\{D(Y,Z), D(Y,W)\}$, *for all* $Y,Z,W \in P(X)$.

iv) If $Y,Z \in P(X)$ *such that* $Y \subset Z \subset \overline{Y}$ *then*

$$D(x_0,Y) = D(x_0,Z) = D(x_0,\overline{Y}), \textit{for all } x_0 \in X.$$

Proof. i) Because $Y \subseteq \overline{Y}$ and $Z \subseteq \overline{Z}$ the inequality $D(\overline{Y},\overline{Z}) \leq D(Y,Z)$ is obviously. For the reverse inequality let us consider $u \in \overline{Y}$, $v \in \overline{Z}$. Then there exists $(x_n)_{n\in\mathbb{N}} \subset Y$ and $(y_n)_{n\in\mathbb{N}} \subset Z$ such that $\lim\limits_{n\to\infty} x_n = u$, $\lim\limits_{n\to\infty} y_n = v$. Because $D(Y,Z) \leq d(x_n,y_n) \leq d(x_n,u)+d(u,v)+d(v,y_n)$ it follows, for $n\to\infty$, that: $D(Y,Z) \leq d(u,v)$, for all $u \in \overline{Y}, v \in \overline{Z}$. Hence $D(Y,Z) \leq D(\overline{Y},\overline{Z})$.

ii) We have $d(y,z) \leq d(y,w_1)+d(w_1,w_2)+d(w_2,z)$, for all $y \in Y, z \in Z$, and for all $w_1,w_2 \in W$. We get $D(y,Z) \leq d(y,w_1) + d(w_1,w_2) + D(w_2,Z)$, for all $y \in Y, w_1,w_2 \in W$. Then $D(Y,Z) \leq D(y,Z) \leq d(y,w_1) + d(w_1,w_2) + D(w_2,Z)$, for all $y \in Y$ and $w_1,w_2 \in W$. We have now $D(Y,Z) \leq d(y,w_1)+diam(W)+D(w_2,Z)$, for all $y \in Y, w_1,w_2 \in W$. So $D(Y,Z) \leq D(y,W) + diam(W) + D(W,Z)$, for all $y \in Y$. Finally $D(Y,Z) \leq D(Y,W) + D(W,Z) + diam(W)$. □

Let us define now the notion of neighborhood for a nonempty set.

Definition 1.1.12. Let (X,d) be a metric space, $Y \in P(X)$ and $\varepsilon > 0$. An open neighborhood of radius ε for the set Y is the set denoted $V^0(Y,\varepsilon)$ and defined by $V^0(Y,\varepsilon) = \{x \in X|\ D(x,Y) < \varepsilon\}$. We also consider the closed neighborhood for the set Y, defined by $V(Y,\varepsilon) = \{x \in X|\ D(x,Y) \leq \varepsilon\}$.

Remark 1.1.13. From the above definition we have that, if (X,d) is a metric space, $Y \in P(X)$ then:

a)$\bigcup\{B(y,r) : y \in Y\} = V^0(Y,r)$, where $r > 0$.

b)$\bigcup\{\widetilde{B}(y,r) : y \in Y\} \subset V(Y,r)$, where $r > 0$.

c)$V^0(Y, r+s) \supset V^0(V^0(Y,s), r)$, where $r, s > 0$.

d) $diam(V^0(Y,r)) \leq diam(Y) + 2r$, for all $Y \in P_b(X)$ and for all $r > 0$.

e) If (X,d) is a normed space, then:

i) $V^0(Y, r+s) = V^0(V^0(Y,s), r)$, where $r, s > 0$

ii) $V^0(Y,r) = Y + int(r\widetilde{B}(0,1))$.

Proof. d) Let $\varepsilon > 0$ and $x, y \in V(Y,r)$. From the definition of $V^0(Y,r)$ there exist $u, v \in Y$ such that $d(x,u) < r + \varepsilon$, $d(x,v) < r + \varepsilon$.

Hence we have $d(x,y) \leq d(x,u) + d(u,v) + d(v,x) \leq \delta(Y) + 2r + 2\varepsilon$, for all $x, y \in Y$.

Hence $diam(V(Y,r)) \leq diam(Y) + 2r + 2\varepsilon$, for all $\varepsilon > 0$. □

Remark 1.1.14. If (X,d) is a metric space and $Y, Z \in P(X)$ then $D(Y,Z) = \inf\{\varepsilon > 0 | \; Y \bigcap V(Z,\varepsilon) \neq \emptyset\}$.

Lemma 1.1.15. *a) Let (X,d) be a metric space and $Y, Z \in P(X)$. Then $D(Y,Z) = \inf\limits_{x \in X} D(x,Y) + D(x,Z)$.*

b) Let (X,d) be a metric space and $(A_{i)})_{i \in I}$, B nonempty subsets of X. Then $D(\bigcup\limits_{i \in I} A_i, B) = \inf\limits_{i \in I} D(A_i, B)$

c) Let X be a normed space and $A, B, C \in P(X)$. If A is a convex set, then we have:

$$D(\lambda B + (1-\lambda)C, A) \leq \lambda D(B,A) + (1-\lambda)D(C,A), \textit{ for each } \lambda \in [0,1].$$

Proof. a) We denote by $u = \inf\{D(x,Z) + D(x,Y) : x \in X\}$. Because $D(Y,Z) = \inf\{D(x,Y) + D(x,Z) : x \in Y\}$ we have that $u \leq D(Y,Z)$. For the reverse inequality, let $x \in X$ and $y \in Y, z \in Z$ having the property $d(x,y) \leq D(x,Y) + \varepsilon$ and $d(x,z) \leq D(x,Z) + \varepsilon$. Then we have: $D(Y,Z) \leq d(y,z) \leq D(x,Y) + D(x,Z) + 2\varepsilon$. But ε was arbitrarily chosen, and so $D(Y,Z) \leq u$. □

Lemma 1.1.16. *Let (X, d) a metric space. Then we have:*

i) If $Y, Z \in P(X)$ then $\delta(Y, Z) = 0$ if and only if $Y = Z = \{x_0\}$

ii) $\delta(Y, Z) \leq \delta(Y, W) + \delta(W, Z)$, for all $Y, Z, W \in P_b(X)$.

iii) Let $Y \in P_b(X)$ and $q \in]0, 1[$. Then, for each $x \in X$ there exists $y \in Y$ such that $q\delta(x, Y) \leq d(x, y)$.

Proof. ii) Let $Y, Z, W \in P_b(X)$. Then we have:

$d(y, z) \leq d(y, w) + d(w, z)$, for all $y \in Y, z \in Z, w \in W$. Then $\sup\limits_{z \in Z} d(y, z) \leq d(y, w) + \sup\limits_{z \in Z} d(w, z)$, for all $y \in Y, w \in W$. So $\delta(y, Z) \leq \delta(y, w) + \delta(w, Z) \leq \delta(y, W) + \delta(W, Z)$ and hence $\delta(Y, Z) \leq \delta(Y, W) + \delta(W, Z)$.

iii) Suppose, by absurdum, that there exists $x \in X$ and there exists $q \in]0, 1[$ such that for all $y \in Y$ to have $q\delta(x, Y) > d(x, y)$. It follows that $q\delta(x, Y) \geq \sup\limits_{y \in Y} d(x, y)$ and hence $q\delta(x, Y) \geq \delta(x, Y)$. In conclusion, $q \geq 1$, a contradiction. □

Lemma 1.1.17. *Let (X, d) be a metric space, $Y, Z, W \in P(X)$. Then:*

i) $\rho(Y, Z) = 0$ if and only if $Y \subset \overline{Z}$

ii) $\rho(Y, Z) \leq \rho(Y, W) + \rho(W, Z)$

iii) If $Y, Z \in P(X)$ and $\varepsilon > 0$ then:

a) $\rho(Y, Z) \leq \varepsilon$ if and only if $Y \subset V(Z; \varepsilon)$.

b) $\rho(Y, Z) = \inf\{\varepsilon > 0 |\ Y \subset V^0(Z, \varepsilon)\}$. (we consider $\inf \emptyset = \infty$)

c) If Y is closed, then $\rho(Y, Z) = \sup\limits_{x \in X} D(x, Z) - D(x, Y)$

d) $\rho(Y, Z) = \rho(\overline{Y}, \overline{Z})$

iv) Let $\varepsilon > 0$. If $Y, Z \in P(X)$ such that for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq \varepsilon$ then $\rho(Y, Z) \leq \varepsilon$.

v) Let $\varepsilon > 0$ and $Y, Z \in P(X)$. Then for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq \rho(Y, Z) + \varepsilon$.

vi) Let $q > 1$ and $Y, Z \in P(X)$. Then, for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq q\rho(Y, Z)$.

Proof. i) Suppose that $\rho(Y, Z) = 0$ and let $y \in Y$ be arbitrary. Then

$0 \leq \inf\{d(y,z) |\ z \in Z\} = D(y,Z) \leq \rho(Y,Z) = 0$ implies that there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset Z$ such that $d(y,z_n) \to 0$, when $n \to \infty$. It follows $z_n \to y$ when $n \to \infty$ and so $y \in \overline{Z} \ \Rightarrow\ Y \subset \overline{Z}$.

Reversely, suppose that $Y \subset \overline{Z}$ with $\alpha = \dfrac{1}{2}\rho(Y,Z) > 0$. Then there exists $y_0 \in Y$ with $D(y_0,Z) > \alpha$. For $y_0 \in Y \subset \overline{Z}$ we find a sequence $(z_n)_{n\in\mathbb{N}} \subset Z$ such that $z_n \to y_0$, when $n \to \infty$. Hence there exists $n_0 \in \mathbb{N}$ such that $d(z_n,y_0) \leq \alpha$, for all $n \geq n_0$, a contradiction with: for all $n \geq n_0 :\ \alpha \geq d(z_n,y_0) \geq \inf\{d(z,y_0) |\ z \in Z\} = D(y_0,Z) > \alpha$.

ii) Let $\varepsilon > 0$ and $y \in Y$. Because $D(y,W) = \inf\{d(y,w) |\ w \in W\}$ we have that there exists $w \in W$ such that $d(y,w) < D(y,W) + \varepsilon$. For each $z \in Z$ we have: $D(y,Z) \leq d(y,z) \leq d(y,w) + d(w,z) < d(w,z) + D(y,W) + \varepsilon$.

So $D(y,Z) - D(y,W) - \varepsilon < d(z,w)$, for all $z \in Z$ proving that $D(y,Z) - D(y,W) - \varepsilon \leq D(w,Z)$.

Hence $D(y,Z) \leq \rho(W,Z) + \rho(Y,W) + \varepsilon$, for all $y \in Y$.

Finally, $\rho(Y,Z) \leq \rho(Y,W) + \rho(W,Z) + \varepsilon$ and so we get the desired conclusion.

iii) a) $\rho(Y,Z) \leq \varepsilon$ is equivalent with: for all $y \in Y D(y,Z) \leq \varepsilon$ and equivalent with $Y \subset V(Z,\varepsilon)$.

If Z is compact, then $Y \subset V(Z,\varepsilon)$ is equivalent with the fact that for all $y \in Y$ we have $D(y,Z) \leq \varepsilon$ and equivalent with: for all $y \in Y$ there exists $z_0 \in Z$ such that $d(y,z_0) \leq \varepsilon$, meaning that for all $y \in Y$ there exists $z_0 \in Z \cap \widetilde{B}(y;\varepsilon)$ and hence for all $y \in Y :\ Z \cap \widetilde{B}(y,\varepsilon) \neq \emptyset$.

c) Denote $u = \sup\limits_{x\in X} D(x,Z) - D(x,Y)$. We shall prove that $\rho(Y,Z) \leq u$. If $u = \infty$ then the inequality is obviously. Let us consider $u < \infty$. Let $y \in Y$ and $v > u$. We have: $D(y,Z) = D(y,Z) - D(y,Y) \leq u < v$ and so $y \in V^0(Z,v)$. Hence we have proved that $Y \subseteq V^0(Z,v)$ and so we get that $\rho(Y,Z) \leq u$. We will prove now that $\rho(Y,Z) \geq u$. Let $\varepsilon > 0$ and $x \in X$. We can choose $y \in Y$ such that $d(x,y) < D(x,Y) + \varepsilon$. Let $z \in Z$ be such that $d(y,z) < D(y,Z) + \varepsilon \leq \rho(Y,Z) + \varepsilon$. We have

$D(x, Z) \le d(x, z) \le d(x, y) + d(y, z) < D(x, Y) + \rho(Y, Z) + 2\varepsilon$ and so $D(x, Z) - D(x, Y) \le \rho(Y, Z) + 2\varepsilon$. Because x was arbitrarily we obtain that $\sup_{x \in X} D(x, Z) - D(x, Y) \le \rho(Y, Z) + 2\varepsilon$. For $\varepsilon \searrow 0$, we have $u \le \rho(Y, Z)$. $\square$

Lemma 1.1.18. *Let (X, d) be a metric space, $A, B \in P(X)$ and $(A_{i)})_{i \in I}$ a family of nonempty subsets of X. Then:*

a) $\rho(\bigcup_{i \in I} A_i, B) = \sup_{i \in I} \rho(A_i, B)$

b) If $A \in P_{cl}(X)$ then:

i) $\rho(A, \cdot) : (P_{cl}(X), H) \to \mathbb{R}_+$ *is nonexpansive.*

ii) $\rho(\cdot, A) : (P_{cl}(X), H) \to \mathbb{R}_+$ *is nonexpansive.*

Proof. b) ii) Let us consider $B, C \in P_{cl}(X)$ with $H(B, C) < +\infty$. Then $\rho(B, A) \le \rho(B, C) + \rho(C, A)$ and $\rho(C, A) \le \rho(C, B) + \rho(B, A)$. Since $\rho(C, B) < +\infty$ it is clear that $\rho(B, A) = +\infty$ if and only if $\rho(C, A) = +\infty$. If both are finite then $|\rho(C, A) - \rho(B, A)| \le max\{\rho(B, C), \rho(C, B)\} = H(B, C)$. $\square$

Lemma 1.1.19. *Let X be a normed space, A, B, C be nonempty bounded, convex subsets of X and $r \in [0, 1]$. Then:*

a) $\rho(\overline{conv} A, B) = \rho(A, B)$

b) $\rho(rB + (1 - r)C, A) \le r\rho(B, A) + (1 - r)\rho(C, A)$

c) $\rho(A, rB + (1 - r)C) \le r\rho(A, B) + (1 - r)\rho(A, C)$

If (X, d) is a metric space, we have defined the Pompeiu-Hausdorff generalized functional $H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ and we have shown that H is a generalized metric on $P_{cl}(X)$. Other important properties of the functional H are as follows.

Lemma 1.1.20. *Let (X, d) be a metric space and $Y, Z, V, W \in P(X)$. Then we have:*

i) $H(Y, Z) = 0$ *if and only if* $\overline{Y} = \overline{Z}$

ii) $H(Y,Z) = H(Y,\overline{Z}) = H(\overline{Y},Z) = H(\overline{Y},\overline{Z})$.

iii) $H(Y \bigcup V, Z \bigcup W) \leq max\{H(Y,Z), H(V,W)\}$.

Proof. iii) From the definition of ρ we have:

$$\rho(Y \cup V, Z \cup W) = \sup\{D(x, Z \cup W)|\ x \in Y \cup V\} =$$

$$= \max\{\rho(Y, Z \cup W), \rho(V, Z \cup W)\} \leq \max\{\rho(Y,Z), \rho(V,W)\}.$$

By a similar procedure we also get:

$$\rho(Z \cup W, Y \cup V) \leq \max\{\rho(Z,Y), \rho(W,V)\}.$$

Hence

$$H(Y \cup V, Z \cup W) \leq \max\{\rho(Y,Z), \rho(V,W), \rho(Z,Y), \rho(W,V)\}$$

$= \max\{H(Y,Z), H(V,W)\}$. $\Box$

Lemma 1.1.21. *Let (X,d) be a metric space. Then we have:*

i) Let $Y, Z \in P(X)$. Then $H(Y,Z) = \sup\limits_{x \in X} D(x,Y) - D(x,Z)$

ii) The operator $I(x) = \{x\}$ is an isometry of (X,d) into $(P_{cl}(X), H_d)$

iii) Let $Y, Z \in P(X)$ and $\varepsilon > 0$.Then for each $y \in Y$ there exists $z \in Z$ such that $d(y,z) \leq H(Y,Z) + \varepsilon$.

iv) Let $Y, Z \in P(X)$ and $q > 1$. Then for each $y \in Y$ there exists $z \in Z$ such that $d(y,z) \leq qH(Y,Z)$.

v) If $Y, Z \in P_{cp}(X)$ then for each $y \in Y$ there exists $z \in Z$ such that $d(y,z) \leq H(Y,Z)$.

vi) If $Y, Z \in P(X)$. If. for each $y \in Y$ there exists $z \in Z$ such that $d(y,z) \leq \varepsilon$ and for each $z \in Z$ there exists $y \in Y$ with $d(y,z) \leq \varepsilon$, then $H(Y,Z) \leq \varepsilon$.

vii) Let $\varepsilon > 0$. If $Y, Z \in P(X)$ are such that $H(Y,Z) < \varepsilon$ then for each $y \in Y$ there exists $z \in Z$ such that $d(y,z) < \varepsilon$.

Proof. iii) Supposing contrary, there exists $\varepsilon > 0$ and exists $y \in Y$ such that for all $z \in Z$ we have $d(y,z) > H(Y,Z) + \varepsilon$. It follows that

$D(y, Z) \geq H(Y, Z) + \varepsilon$ and so $H(Y, Z) \geq D(y, Z) \geq H(Y, Z) + \varepsilon$, proving that $\varepsilon \leq 0$, a contradiction.

iv) Supposing again contrary: there exists $q > 1$ and there exists $y \in Y$ such that for all $z \in Z$ we have $d(y, z) > qH(Y, Z)$. Then we have: $D(y, Z) \geq qH(Y, Z)$. But $H(Y, Z) \geq D(Y, Z) \geq qH(Y, Z)$. Hence $q \leq 1$, a contradiction. □

Remark 1.1.22. Using the above result (vi) it follows that the Pompeiu-Hausdorff functional can be also defined by the following formula:

$$H(A, B) = \inf\{\varepsilon > 0 |\ A \subset V(B, \varepsilon) \text{ and } B \subset V(A, \varepsilon)\},$$

for all $A, B \in P(X)$.

Lemma 1.1.23. *Let X be a Banach space. Then:*

i) $H(Y_1 + \cdots + Y_n, Z_1 + \cdots + Z_n) \leq H(Y_1, Z_1) + \cdots + H(Y_n, Z_n)$, for all $Y_i, Z_i \in P(X)$, $i = 1, 2, \ldots, n$ ($n \in \mathbb{N}^*$)

ii) $H(Y + Z, Y + W) \leq H(Z, W)$, for all $Y, Z, W \in P(X)$

iii) $H(Y + Z, Y + W) = H(Z, W)$, for all $Y \in P_b(X)$ and for all $Z, W \in P_{b,cl,cv}(X)$

iv) $H(conv\, Y, conv\, Z) \leq H(Y, Z)$, for all $Y, Z \in P_b(X)$

v) $H(\overline{conv}\, Y, \overline{conv}\, Z) \leq H(Y, Z)$, for all $Y, Z \in P_{b,cl}(X)$

vi) $H(A, rB + sC)) \leq rH(A, B) + sH(A, C)$, for each $A, B.C \in P_{b,cv}(X)$.

Proof. i) Let $\varepsilon > 0$. From the definition of H it follows that there exists $(y_1 + \cdots + y_n) \in Y_1 + \cdots + Y_n$ such that $D(y_1 + \cdots + y_n, Z_1 + \cdots + Z_n) \geq H(Y_1 + \cdots + Y_n, Z_1 + \cdots + Z_n) - \varepsilon$ or exists $(z_1 + \cdots + z_n) \in Z_1 + \cdots + Z_n$ such that $D(z_1 + \cdots + z_n, Y_1 + \cdots + Y_n) \geq H(Y_1 + \cdots + Y_n, Z_1 + \cdots + Z_n) - \varepsilon$. Let us consider the first situation.

For $y_1, \ldots, y_n$ we get $z_1 \in Z_1, \ldots, z_n \in Z_n$ such that $\|y_1 - z_1\| \leq H(Y_1, Z_1) + \frac{\varepsilon}{4}, \ldots, \|y_n - z_n\| \leq H(Y_n, Z_n) + \frac{\varepsilon}{4}$. Then

$$\|(y_1 + \cdots + y_n) - (z_1 + \cdots + z_n)\| \leq \|y_1 - z_1\| + \cdots + \|y_n - z_n\| \leq$$

$$\leq H(Y_1, Z_1) + \cdots + H(Y_n, Z_n) + \varepsilon.$$

Because

$$H(Y_1 + \cdots + Y_n, Z_1 + \cdots + Z_n) - \varepsilon \leq D(y_1 + \cdots + y_n, z_1 + \cdots + z_n) \leq$$

$$\leq \|(y_1 + \cdots + y_n) - (z_1 + \cdots + z_n)\|$$

we obtain that

$$H(Y_1 + \cdots + Y_n, Z_1 + \cdots + Z_n) - \varepsilon \leq H(Y_1, Z_1) + \cdots + H(Y_n, Z_n) + \varepsilon,$$

proving the desired inequality.

iii) From ii) we have $H(Y + Z, Y + W) \leq H(Z, W)$. For the equality, let us suppose contrary: $H(Y + Z, Y + W) < H(Z, W)$. Let $t \in \mathbb{R}^*_+$ such that $H(Y + Z, Y + W) < t < H(Z, W)$. Then

$$Y + Z \subset Y + W + B_X(0; t) \subset Y + \overline{W + B_X(0; t)}$$

$$Y + W \subset Y + Z + B_X(0; t) \subset Y + \overline{Z + B_X(0; t)}.$$

Because $\overline{W + B_X(0; t)}$, $\overline{Z + B_X(0; t)} \in P_{cl,cv}(X)$ and $Y \in P_m(X)$ it follows from Lemma 4.1.7(i) that

$$Z \subset \overline{W + B_X(0; t)} \quad \text{and} \quad W \subset \overline{Z + B_X(0; t)}.$$

On the other side,

$$\overline{W + B_X(0; t)} = \bigcap_{n=1}^{n} [(W + B_X(0; t) + 2^{-n} B_X(0; 1)]$$

$$\overline{Z + B_X(0; t)} = \bigcap_{n=1}^{n} [(Z + B_X(0; t) + 2^{-n} B_X(0; 1)]$$

and choosing n such that $t + 2^{-n} < H(Z, W)$ we get

$$Z \subset W + (t + 2^{-n}) B_X(0; 1) \quad \text{and} \quad W \subset Z + (t + 2^{-n}) B_X(0; 1).$$

Hence we obtain $H(Z,W) \leq t + 2^{-n}$, a contradiction.

iv) Because $Y \subseteq conv\, Y$ it follows that $D(z, conv\, Y) \leq D(z,Y)$, for all $z \in Z$. Let $A = \{a \in X|\ D(a, conv\, Y) \leq H(Y,Z)\}$. Of course A is convex and $A \supseteq Z$. we can write $conv\, Z \subset A$ and hence for all $v \in conv\, Z$ we have $D(v, conv\, Y) \leq H(Y,Z)$. A similar procedure produces that for all $u \in conv\, Y$ we have $D(u, conv\, Z) \leq H(Y,Z)$. In conclusion: $H(conv\, Y, conv\, Z) \leq H(Y,Z)$.

v) Let $Y, Z \in P_{m,cl}(X)$ and $\varepsilon > 0$. Let $p \in \overline{conv}\, Y$. Then there exist $y_1, y_2, \ldots, y_n \in Y$ and $\lambda_1, \ldots, \lambda_n \in [0,1]$ with $\sum\limits_{i=1}^{n} \lambda_i = 1$ such that

$$\left\| p - \sum_{i=1}^{n} \lambda_i y_i \right\| < \frac{\varepsilon}{2}.$$

For each $i = 1, 2, \ldots, n$ and $y_1, \ldots, y_n \in A$ there exist (see Lemma 1.1.21. iii)) $z_1, \ldots, z_n \in Z$ such that $\|y_i - z_i\| \leq H(Y,Z) + \frac{\varepsilon}{2}$. Let $q = \sum\limits_{i=1}^{n} \lambda_i z_i$. Obviously $q \in \overline{conv}\, Z$ and we also have:

$$\|p - q\| \leq \left\| p - \sum_{i=1}^{n} \lambda_i g_i \right\| + \left\| \sum_{i=1}^{n} \lambda_i y_i - \sum_{i=1}^{n} \lambda_i z_i \right\| <$$

$$< \frac{\varepsilon}{2} + \sum_{i=1}^{n} \lambda_i \|y_i - z_i\| < H(Y,Z) + \varepsilon.$$

Hence

$$p \in V(\overline{conv}\, Z; H(Y,Z) + \varepsilon) \;\Rightarrow\; \overline{conv}\, Y \subseteq V(\overline{conv}\, Z, H(Y,Z) + \varepsilon).$$

Similarly, we can be prove $\overline{conv}\, Z \subseteq V(\overline{conv}\, Y, H(Y,Z) + \varepsilon)$. In conclusion we obtain that $H(\overline{conv}\, Y, \overline{conv}\, Z) \leq H(Y,Z) + \varepsilon$, proving the conclusion.

vi) By the definition of Pompeiu-Hausdorff metric, it suffices to prove that for any $\epsilon > 0$ with $B \subset V^0(A, \epsilon)$, any $\mu > 0$ with $A \subset V^0(B, \mu)$ and $C \subset V^0(A, \mu)$ and $A \subset V^0(C, \mu)$ we have that:

$$A \subset V^0(rB + sC, r\epsilon + s\mu) \text{ and } rB + sC \subset V^0(A, r\epsilon + s\mu).$$

For any $a \in A$ there exist $b \in B$ and $c \in C$ such that $d(a,b) < \epsilon$ and $d(a,c) < \mu$. Since $r + s = 1$ it follows that

$$d(a, rb + sc) = \|a - rb - sc\| \leq r\|a - b\| + s\|a - c\| \leq r\epsilon + s\mu.$$

This implies that $A \subset V^0(rB + sC, r\epsilon + s\mu)$. By the convexity of B and C and the similar argument used above, we can also prove that $rB + sC \subset V(A, r\epsilon + s\mu)$. $\square$

Remark 1.1.24. Let X be a normed space and $A \in P_{cp}(X)$. We denote $\|A\| = H(A, \{0\})$.

Example 1.1.25. a) $H([a_1, a_2], [b_1, b_2]) = \max\{|b_1 - a_1|, |b_2 - a_2|\}$ (where $a_1, a_2, b_1, b_2 \in \mathbb{R}$).

b) If $B(x_i, r_i)$ are two ball in $\mathbb{R}^n$ (where $x_1, x_2 \in \mathbb{R}^n$ and $r_1, r_2 \in \mathbb{R}_+^*$), then $H(B(x_1, r_1), B(x_1, r_2)) = ||x_1 - x_2|| + |r_1 - r_2|$.

Let us recall that a metric space (X, d) is said to be ϵ-chainable (where $\epsilon > 0$ is fixed) if and only if given $a, b \in X$ there is an ϵ-chain from a to b, that is a finite set of points $x_0, x_1, \ldots, x_n$ in X such that $x_0 = a$, $x_n = b$ and $d(x_{i-1}, x_i) < \epsilon$, for all $i \in \{1, 2, \ldots, n\}$.

Some very important properties of the metric space $(P_{cl}(X), H_d)$ are contained in the following result:

Theorem 1.1.26. *i) If (X, d) is a complete metric space, then $(P_{cl}(X), H_d)$ is a complete metric space.*

ii) If (X, d) is a totally bounded metric space, then $(P_{cl}(X), H_d)$ is a totally bounded metric space.

iii) If (X, d) is a compact metric space, then $(P_{cl}(X), H_d)$ is a compact metric space.

iv) If (X, d) is a separable metric space, then $(P_{cp}(X), H_d)$ is a separable metric space.

v) If (X, d) is a ε-chainable metric space, then $(P_{cp}(X), H_d)$ is also an ε-chainable metric space.

Proof. i) We will prove that each Cauchy sequence in $(P_{cl}(X), H_d)$ is convergent. Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(P_{cl}(X), H_d)$. Let us consider the set A defined as follows:

$$A = \bigcap_{n=1}^{\infty} \left(\overline{\bigcup_{m=n}^{\infty} A_m} \right).$$

We have two steps in the proof:

1) $A \neq \emptyset$.

In this respect, consider $\varepsilon > 0$. Then for each $k \in \mathbb{N}$ there is $N_k \in \mathbb{N}$ such that for all $n, m \geq N_k$ we have $H(A_n, A_m) < \frac{\varepsilon}{2^{k+1}}$. Let $(n_k)_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers such that $n_k \geq N_k$. Let $x_0 \in A_{n_0}$. Let us construct inductively a sequence $(x_k)_{k\in\mathbb{N}}$ having the following properties:

$\alpha)$ $x_k \in A_{n_k}$, for each $k \in \mathbb{N}$

$\beta)$ $d(x_k, x_{k+1}) < \frac{\varepsilon}{2^{k+1}}$, for each$k \in \mathbb{N}$.

Suppose that we have $x_0, x_1, \ldots, x_k$ satisfying $\alpha)$ and $\beta)$ and we will generate x_{k+1} in the following way.

We have:

$$D(x_k, A_{n_{k+1}}) \leq H(A_{n_k}, A_{n_{k+1}}) < \frac{\varepsilon}{2^{k+1}}.$$

It follows that there exists $x_{k+1} \in A_{n_{k+1}}$ such that $d(x_k, x_{k+1}) < \frac{\varepsilon}{2^{k+1}}$.

Hence, we have proved that there exist a sequence $(x_k)_{k\in\mathbb{N}}$ satisfying $\alpha)$ and $\beta)$.

From $\beta)$ we get that $(x_k)_{k\in\mathbb{N}}$ is Cauchy in (X, d). Because (X, d) is complete it follows that there exists $x \in X$ such that $x = \lim_{k\to\infty} x_k$. We need to show now that $x \in A$. Since $(n_k)_{k\in\mathbb{N}}$ is an increasing sequence it follows that for $n \in \mathbb{N}^*$ there exists $k_n \in \mathbb{N}^*$ such that $n_{k_n} \geq n$. Then $x_k \in \bigcup_{m\geq n} A_m$, for $k \geq k_n$, $n \in \mathbb{N}^*$ implies that $x \in \overline{\bigcup_{m\geq n} A_m}$, $n \in \mathbb{N}^*$. Hence $x \in A$.

2) In the second step of the proof, we will establish that $H(A_n, A) \to 0$ as $n \to \infty$.

The following inequalities hold:

$$d(x_k, x_{k+p}) \leq d(x_k, x_{k+1}) + \cdots + d(x_{k+p-1}, x_{k+p}) <$$

$$< \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+2}} + \cdots + \frac{\varepsilon}{2^{k+p}} < \varepsilon \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^k} + \ldots \right) =$$

$$= \varepsilon \frac{1}{1 - \frac{1}{2}} = 2\varepsilon, \text{ for all } p \in \mathbb{N}^*.$$

If in $d(x_k, x_{k+p}) < 2\varepsilon$ we are letting $p \to \infty$ we obtain $d(x_k, x) < 2\varepsilon$, for each $k \in \mathbb{N}$. In particular $d(x_0, x) < 2\varepsilon$. So, for each $n_0 \in \mathbb{N}$, $n_0 \geq N_0$ and for $x_0 \in A_{n_0}$ there exists $x \in A$ such that $d(x_0, x) \leq 2\varepsilon$, which imply

$$\rho(A_{n_0}, A) \leq 2\varepsilon, \text{ for all } n_0 \geq N_0 \qquad (1).$$

On the other side, because the sequence $(A_n)_{n\in\mathbb{N}}$ is Cauchy, it follows that there exists $N_\varepsilon \in \mathbb{N}$ such that for $m, n \geq N_\varepsilon$ we have $H(A_n, A_m) < \varepsilon$. Let $x \in A$ be arbitrarily. Then $x \in \overline{\bigcup_{m=n}^{\infty} A_m}$, for $n \in \mathbb{N}^*$, which implies that there exist $n_0 \in \mathbb{N}$, $n_0 \geq N_\varepsilon$ and $y \in A_{n_0}$ such that $d(x, y) < \varepsilon$. Hence, there exists $m \in \mathbb{N}$, $m \geq N_\varepsilon$ and there is $y \in A_m$ such that $d(x, y) < \varepsilon$.

Then, for $n \in \mathbb{N}^*$, with $n \geq N_\varepsilon$ we have:

$$D(x, A_n) \leq d(x, y) + D(y, A_n) \leq d(x, y) + H(A_m, A_n) < \varepsilon + \varepsilon = 2\varepsilon.$$

So,

$$\rho(A, A_n) < 2\varepsilon, \text{ for each } n \in \mathbb{N} \text{ with } n \geq N_\varepsilon. \qquad (2)$$

From (1) and (2) and choosing $n_\varepsilon := \max\{N_0, N_\varepsilon\}$ it follows that $H(A_n, A) < 2\varepsilon$, for each $n \geq n_\varepsilon$. Hence $H(A_n, A) \to 0$ as $n \to \infty$.

v) (X, d) being an ε-chainable metric space (where $\varepsilon > 0$) it follows, by definition, that for all $x, y \in X$ there exists a finite subset (the so-called

ε-net) of X, let say $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_{k-1}, x_k) < \varepsilon$, for all $k = 1, 2, \ldots, n$.

Let $y \in X$ arbitrary and $Y = \{y\}$. Obviously, $Y \in P_{cp}(X)$. Because the ε-chainability property is transitive, it is sufficient to prove that for all $A \in P_{cp}(X)$ there exist an ε-net in $P_{cp}(X)$ linking Y with A. We have two steps in our proof:

a) Let suppose first that A is a finite set, let say $A = \{a_1, a_2, ..., a_n\}$ We will use the induction method after the number of elements of A. If $n = 1$ then $A = \{a\}$ and the conclusion follows from the ε-chainability of (X, d). Let suppose now that the conclusion holds for each subsets of X consisting of at most n elements. Let A be a subset of X with $n+1$ points, $A = \{x_1, x_2, \ldots, x_{n+1}\}$. Using the ε-chainability of the space (X, d) it follows that there exist an ε-net in X, namely $x_1 = u_0, u_1, \ldots, u_m = x_2$ linking the points x_1 and x_2. We obtain that the following finite set: A, $\{u_1, x_2, \ldots, x_{n+1}\}, \ldots,$ $\{u_{m-1}, x_2, \ldots, x_{n+1},$ $\{x_2, \ldots, x_{n+1}\}$ is an ε-net in $P_{cp}(X)$ from A to $B := \{x_2, \ldots, x_{n+1}\}$. But, from the hypothesis B is ε-chainable with Y, and hence A is ε-chainable with Y in $P_{cp}(X)$.

b) Let consider now $A \in P_{cp}(X)$ be arbitrary.

A being compact, there exists a finite family of nonempty compact subsets of A, namely $\{A_k\}_{k=1}^n$, having $diam(A_k) < \varepsilon$ such that $A = \bigcup_{k=1}^{n} A_k$. For each $k = 1, 2, \ldots n$ we can choose $x_k \in A_k$ and define $C = \{x_1, \ldots, x_n\}$. Then for all $z \in A$ there exists $k \in \{1, 2, \ldots, n\}$ such that $D(z, C) \leq \delta(A_k)$. We obtain:

$$H(A, C) = \max\left\{\sup_{z \in A} D(z, C), sup_{y \in C} D(y, A)\right\} =$$

$$= \sup_{z \in A} D(z, C) \leq \max_{i \leq k \leq n} \delta(A_k) < \varepsilon,$$

meaning that A is ε-chainable by C in $P_{cp}(X)$. Using the conclusion a) of this proof, we get that C is ε-chainable by Y in $P_{cp}(X)$ and so we have proved that A is ε-chainable by Y in $P_{cp}(X)$. $\square$

v) (X, d) being an ε-chainable metric space (where $\varepsilon > 0$) it follows, by definition, that for all $x, y \in X$ there exists a finite subset (the so-called ε-net) of X, let say $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_{k-1}, x_k) < \varepsilon$, for all $k = 1, 2, \ldots, n$.

Let $y \in X$ arbitrary and $Y = \{y\}$. Obviously, $Y \in P_{cp}(X)$. Because the ε-chainability property is transitive, it is sufficient to prove that for all $A \in P_{cp}(X)$ there exist an ε-net in $P_{cp}(X)$ linking Y with A. We have two steps in our proof:

a) Let suppose first that A is a finite set, let say $A = \{a_1, a_2, ..., a_n\}$ We will use the induction method after the number of elements of A. If $n = 1$ then $A = \{a\}$ and the conclusion follows from the ε-chainability of (X, d). Let suppose now that the conclusion holds for each subsets of X consisting of at most n elements. Let A be a subset of X with $n+1$ points, $A = \{x_1, x_2, \ldots, x_{n+1}\}$. Using the ε-chainability of the space (X, d) it follows that there exist an ε-net in X, namely $x_1 = u_0, u_1, \ldots, u_m = x_2$ linking the points x_1 and x_2. We obtain that the following finite set: A, $\{u_1, x_2, \ldots, x_{n+1}\}, \ldots,$ $\{u_{m-1}, x_2, \ldots, x_{n+1},$ $\{x_2, \ldots, x_{n+1}\}$ is an ε-net in $P_{cp}(X)$ from A to $B := \{x_2, \ldots, x_{n+1}\}$. But, from the hypothesis B is ε-chainable with Y, and hence A is ε-chainable with Y in $P_{cp}(X)$.

b) Let consider now $A \in P_{cp}(X)$ be arbitrary.

A being compact, there exists a finite family of nonempty compact subsets of A, namely $\{A_k\}_{k=1}^n$, having $diam(A_k) < \varepsilon$ such that $A = \bigcup_{k=1}^n A_k$. For each $k = 1, 2, \ldots n$ we can choose $x_k \in A_k$ and define $C = \{x_1, \ldots, x_n\}$. Then for all $z \in A$ there exists $k \in \{1, 2, \ldots, n\}$ such that $D(z, C) \leq \delta(A_k)$. We obtain:

$$H(A, C) = \max \left\{ \sup_{z \in A} D(z, C), sup_{y \in C} D(y, A) \right\} =$$

$$= \sup_{z \in A} D(z, C) \leq \max_{i \leq k \leq n} \delta(A_k) < \varepsilon,$$

meaning that A is ε-chainable by C in $P_{cp}(X)$. Using the conclusion a) of this proof, we get that C is ε-chainable by Y in $P_{cp}(X)$ and so we have proved that A is ε-chainable by Y in $P_{cp}(X)$. $\square$

1.2 Basic concepts for multi-valued operators

In this section, we describe some basic concepts and results for multi-valued operators.

Let X and Y two nonempty sets. A multi-valued operator (or a multi-function) from X into Y is a correspondence which associates to each element $x \in X$ a subset $F(x)$ of Y. We will denote this correspondence by the symbol: $F : X \to \mathcal{P}(Y)$ or occasionally by: $F : X \multimap Y$. Throughout this book we denote single-valued operators by small letters and multi-valued operators by capital letters.

Multi-valued operators arises in various branches of pure and applied mathematics, as we can see from the following examples:

i) The metric projection multi-function. Let (X, d) be a metric space and $Y \in P(X)$. Then the metric projection on Y is the multi-function $P_Y : X \to \mathcal{P}(Y)$ defined by:

$$P_Y(x) = \{y \in Y |\ D(x, Y) = d(x, y)\}.$$

If X is a Hilbert space, for example, then P_Y becomes a single-valued operator.

ii) Implicit differential equations. Consider the implicit differential equation:

$$f(t, x, x') = 0, x(0) = x^0.$$

This problem may be reduced to a multi-valued initial value problem:

$$x'(t) \in F(t, x(t)), x(0) = x^0$$

involving the multi-valued operator $F(t,x)=\{v|f(t,x,v)=0\}$.

iii) Differential inequalities. The differential inequality:

$$\|x'(t)-g(t,x)\|\leq f(t,x), x(0)=x^0$$

may be recast into the form:

$$x'(t)\in F(t,x(t)), x(0)=x^0$$

with $F(t,x)=\tilde{B}(g(t,x),f(t,x))$.

iv) Control theory. If $f:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ determines the dynamics of a control system having the equations of motion given by:

$$x'(t)=f(t,x(t),u(t)), x(0)=x^0,$$

where the control function u may be chosen as any measurable function from $U(t,x(t))$. (denote by $U:\mathbb{R}\times\mathbb{R}^n\to P(\mathbb{R}^m)$ the feedback multi-function), then the description of this system can be presented in a differential inclusion form:

$$x'(t)\in F(t,x(t)), x(0)=x^0,$$

where $F(t,x)=\{f(t,x(t),u(t))|\ u\in U(t,u(t)\}$.

v) Variational inequalities. Let E be a Banach space, $f:E\to\mathbb{R}^n$ be a differentiable function and $C\in P_{cv}(E)$. Consider the following problem (called a (differential) variational inequality):

Find $x_0\in C$ such that $(\nabla f(x_0),x-x_0)\geq 0$, for each $x\in C$.

If we denote by $N_C(x_0)$ the normal cone for the set C at x_0 (i.e. $N_C(x_0)=\{w\in\mathbb{R}^n|(w,x_0-x)\geq 0,\forall\ x\in C\}$), then the above problem can be written as follows:

Find $x_0\in X$ such that $0\in F(x_0)$, where $F(x_0)=\nabla f(x_0)+N_C(x_0)$ is a multi-valued operator.

More generally, if $K := \mathbb{R}^n_+$ is the convex cone of nonnegative vectors in $\mathbb{R}^n$ and $g : K \to \mathbb{R}^n$, then a variational inequality means the following problem:

Find $x_0 \in K$ such that $(g(x_0), x - x_0) \geq 0$, for each $x \in K$

or, equivalently

Find $x_0 \in K$ such that $0 \in F(x_0)$, where $F(x_0) = g(x_0) + N_K(x_0)$.

vi) Mathematical economies. Let us consider now the Arrow-Debreu model of an economy. Let $\mathbb{R}^n$ be the commodity space. A vector $x \in \mathbb{R}^n$ specifies a list of quantities of each commodity. A price p is also an element of $\mathbb{R}^n$, because p lists the value of an unit of each commodity. The main "actors" in a economy are the consumers. We assume that there is a given finite number of consumers. If M is the income of the consumer, then his budget set is $B = \{x \in X | p \cdot x \leq M\}$, where X denotes the consumption set (i.e. the set of all admissible consumption vectors of the consumer). The problem faced by a consumer is to choose a consumption vector or a set of them from the budget set. In order to do this, the consumer must have some criterion for choosing. Let us denote by U the preferences multi-valued operator for our consumer: $U : X \to \mathcal{P}(X)$, $U(x) = \{y \in X |$ y is strictly prefered to x$\}$.

An element $x^* \in X$ is an optimal preference for the consumer if $U(x^*) = \emptyset$. This is the so-called consumer's problem.

Another important question from mathematical economies is the equilibrium price problem. The set of sums of demand vectors minus sums of supply vectors is, by definition, the excess-demand multi-function, denoted by $E(p)$. A Walrasian equilibrium price problem means the following:

$$\text{find a price } p^* \in \mathbb{R}^n \text{ such that } 0 \in E(p^*).$$

Let us recall now some basic notions in the analysis of multi-valued operators.

Definition 1.2.1. Let X, Y be two nonempty sets. For the multi-valued operator $F : X \to \mathcal{P}(Y)$ we define:

i) the effective domain: $Dom\, F := \{x \in X \mid F(x) \neq \emptyset\}$

ii) the graphic: $Graf\, F := \{(x, y) \in X \times Y \mid y \in F(x)\}$

iii) the range: $F(X) := \bigcup\limits_{x \in X} F(x)$

iv) the image of the set $A \in P(X)$: $F(A) := \bigcup\limits_{x \in A} F(x)$

v) the inverse image of the set $B \in P(Y)$:

$$F^{-}(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}$$

vi) the strict inverse image of the set $B \in P(Y)$:

$$F^{+}(B) := \{x \in Dom\, F \mid F(x) \subset B\}.$$

vii) the inverse multi-valued operator, denoted $F^{-1} : Y \to \mathcal{P}(X)$ and defined by $F^{-1}(y) := \{x \in X \mid y \in F(x)\}$. The set $F^{-1}(y)$ is called the fibre of F at the point y.

Remark 1.2.2. We consider, by convention: $F^{-}(\emptyset) = \emptyset$ and $F^{+}(\emptyset) = \emptyset$.

Definition 1.2.3. Let $F, G : X \to \mathcal{P}(Y)$ be multi-valued operators. Then:

i) If $\otimes$ defines a certain operation between sets, then we will use the same symbol $\otimes$ for the corresponding operation between multi-functions, namely: $F \otimes G : X \to \mathcal{P}(Y)$, $(F \otimes G)(x) := F(x) \otimes G(x)$, $\forall\, x \in X$. (where $\otimes$ could be $\cap$, $\cup$, $+$, etc.)

iii) If $\eta : \mathcal{P}(Y) \to \mathcal{P}(Y)$, then we define $\eta(F) : X \to \mathcal{P}(Y)$ by $\eta(F)(x) := \eta(F(x))$, for all $x \in X$. In such way, we are able to define in topological spaces, for example, $\overline{F} : X \to \mathcal{P}(Y)$, $\overline{F}(x) = \overline{F(x)}$, for all $x \in X$ or $conv\, F : X \to \mathcal{P}(Y)$, $(conv\, F)(x) := conv(F(x))$, for all $x \in X$ in linear spaces, etc.

Definition 1.2.4. Let X, Y, Z be nonempty sets and $F : X \multimap Y$, $G : Y \multimap Z$ be multi-valued operators. The composite of G and F is the multi-valued operator $H = G \circ F$, defined by the relation $H : X \multimap Z$, $H(x) := \bigcup_{y \in F(x)} G(y)$.

If X is a nonempty set, then $Y \in P(X)$ is said to be invariant with respect to a multi-valued operator $F : X \to P(X)$ if $F(Y) \subset Y$. The family of all invariant subsets of F will be denoted by $I(F)$. Also, if $f : X \to \mathbb{R}$, then Z_f denotes the set of all zero point of f, i. e. $Z_f = \{x \in X | f(x) = 0\}$.

Definition 1.2.5. Let (X, d), (Y, d') be metric spaces and $F : X \to P(Y)$. Then, F is called:

i) a-Lipschitz if $a \geq 0$ and $H(F(x_1), F(x_2)) \leq ad(x_1, x_2)$, for all elements $x_1, x_2 \in X$.

ii) a-contraction if it is a-Lipschitz, with $a < 1$.

iii) contractive if $H(F(x_1), F(x_2)) < d(x_1, x_2)$, for all $x_1, x_2 \in X,\ x_1 \neq x_2$.

Lemma 1.2.6. *Let (X, d), (Y, d') and (Z, d'') be metric spaces. Then:*

i) If $F : X \to P_{b,cl}(Y)$ is a-Lipschitz and $G : X \to P_{b,cl}(Y)$ is b-Lipschitz, then $F \cup G$ is $\max\{a, b\}$*-Lipschitz.*

ii) If $F : X \to P_{cp}(Y)$ is a-Lipschitz and $G : Y \to P_{cp}(Z)$ is b-Lipschitz, then $G \circ F$ is ab-Lipschitz.

Lemma 1.2.7. *Let X be a Banach space and $F : X \to P_{b,cl}(X)$ be a-Lipschitz. Then $\overline{conv}\, F : X \to P_{b,cl}(X)$ defined by $(\overline{conv}\, F)(x) = \overline{conv}(F(x))$, for all $x \in X$ is a-Lipschitz. Moreover, if $F : X \to P_{cp}(X)$ then $\overline{conv}\, F : X \to P_{cp}(X)$.*

Let us remark now that, if (X, d) is a metric space and Y is a Banach space, then a multi-function $F : X \to \mathcal{P}(Y)$ is said to be α -Lipschitz on the set $K \in P(X)$ if $\alpha \geq 0$ and

$$F(x_1) \subseteq F(x_2) + \alpha d(x_1, x_2)\widetilde{B}(0;1), \text{ for all } x_1, x_2 \in K.$$

It is quite obviously that, if there exists $a > 0$ such that F is a-Lipschitz in the sense of Definition 1.2.5., then F is α-Lipschitz in the above mentioned sense with any $\alpha > a$ and also reversely.

1.3 Continuity of multi-valued operators

Let us consider, for the beginning, the notion of upper semi-continuity of a multi-function.

Definition 1.3.1. Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$. Then F is said to be upper semi-continuous in $x_0 \in X$ (briefly u.s.c.) if and only if for each open subset U of Y with $F(x_0) \subset U$ there exists an open neighborhood V of x_0 such that for all $x \in V$ we have $F(x) \subset U$.

F is u.s.c. on X if it is u.s.c. in each $x_0 \in X$.

Remark 1.3.2. If $x_0 \in X$ has the property $F(x_0) = \emptyset$ then F is u.s.c. in x_0 if and only if there exists a neighborhood V of x_0 such that $F(V) = \emptyset$.

Remark 1.3.3. If X, Y are metric spaces, then $F : X \to P(Y)$ is u.s.c. in $x_0 \in X$ if and only if for all $U \subset Y$ open, with $F(x_0) \subset U$ there exists $\eta > 0$ such that for all $x \in B(x_0; \eta)$ we have $F(x) \subset U$.

Definition 1.3.4. Let (X, d), (Y, d') be metric spaces and $F : X \to P(Y)$. Then F is called H-upper semi-continuous in $x_0 \in X$ (briefly H-u.s.c.) if and only if for all $\varepsilon > 0$ there exists $\eta > 0$ such that for all $x \in B(x_0; \eta)$ we have $F(x) \subset V(F(x_0); \varepsilon)$.

F is H-u.s.c. on X if it is H-u.s.c. in each $x_0 \in X$.

Remark 1.3.5. If $F : X \to P_{b,cl}(Y)$ then F is H-u.s.c. in $x_0 \in X$ if and only if for all $\varepsilon > 0$ there exists $\eta > 0$ such that for all $x \in B(x_0; \eta)$ we have $\rho_{d'}(F(x), F(x_0)) \leq \varepsilon$.

The connection between Definition 1.3.1 and Definition 1.3.4 is given by:

Lemma 1.3.6. *Let $(X, d), (Y, d')$ be metric spaces and $F : X \to P(Y)$. If F is u.s.c. in $x_0 \in X$ then F is H-u.s.c. in $x_0 \in X$.*

For a reverse implication, we have:

Lemma 1.3.7. *Let $(X, d), (Y, d')$ be metric spaces. If $F : X \to P_{cp}(Y)$ is H-u.s.c. in $x_0 \in X$ then F is u.s.c. in $x_0 \in X$.*

Remark 1.3.8. $F : X \to P_{b,cl}(X)$ is H-u.s.c. in $x_0 \in X$ if and only if for each sequence $(x_n)_{n\in\mathbb{N}^*} \subset X$ such that $\lim\limits_{n\to\infty} x_n = x_0$ we have $\lim\limits_{n\to\infty} \rho(F(x_n), F(x_0)) = 0$.

For Hausdorff topological spaces, we have the following characterization of global upper semi-continuity:

Theorem 1.3.9. *Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$. The following assertions are equivalent:*

i) F is u.s.c. on X

ii) $F^+(V) = \{x \in X |\ F(x) \subset V\}$ is open, for each open set $V \subset Y$.

iii) $F^-(W) = \{x \in X |\ F(x) \cap W \neq \emptyset\}$ is closed, for each closed set $W \subset Y$.

Lemma 1.3.10. *a) Let X, Y, Z be Hausdorff topological spaces and $F : X \to P(Y)$, $G : Y \to P(Z)$ be u.s.c. on X respectively on Y. Then $G \circ F : X \to P(Z)$ is u.s.c. on X.*

b) If X, Y are Hausdorff topological spaces and $F : X \to P_{cl}(Y)$ is u.s.c. on X, then $Graf\,F$ is a closed set in $X \times Y$.

Lemma 1.3.11. *Let $(X, d), (Y, d')$ be metric spaces, $f : X \to Y$ be a continuous operator and $F : X \to P_{b,cl}(Y)$ be a multi-valued operator*

H-u.s.c. on X. then the functional $p : X \to \mathbb{R}_+$, defined by $p(x) := D(f(x), F(x))$, for all $x \in X$ is lower semi-continuous (briefly l.s.c.) on X.

Proof. Let $x \in X$ be a fixed point and $(x_n)_{n\in\mathbb{N}} \subset X$ convergent to x. It follows that for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $d(f(x), f(x_n)) < \frac{\varepsilon}{2}$, for all $n \geq N_\varepsilon$. From the H-u.s.c. of F in x we have that $\rho(F(x_n), F(x)) < \frac{\varepsilon}{2}$, for all $n \geq N_\varepsilon$. Hence, for each $n \geq N_\varepsilon$ we have: $p(x) = D(f(x), F(x)) \leq d(f(x), f(x_n)) + D(f(x_n), F(x_n)) + \rho(F(x_n), F(x)) < \varepsilon + p(x_n)$. In conclusion, $p(x) \leq \liminf_{n\to\infty} p(x_n) + \varepsilon$, for all $\varepsilon > 0$. It follows $p(x) \leq \liminf_{n\to\infty} p(x_n)$ proving that p is l.s.c. in x. $\square$

Lemma 1.3.12. *Let (X, d) be a metric space, Y be a Banach space and $F : X \to P_{cp}(Y)$ be u.s.c. on X. Then, the multi-valued operator $\overline{conv}\, F : X \to P(Y)$ is u.s.c. on X.*

Proof. From Mazur's theorem (see Dugundji [83]) $\overline{conv}\, F(x)$ is compact, for all $x \in X$ and hence $\overline{conv}\, F$ has compact values. Using Lemma 1.3.7. it is sufficient to prove that $\overline{conv}\, F$ is H-u.s.c. on X. Let $x \in X$ be an arbitrary point and $(x_n)_{n\in\mathbb{N}} \subset X$ which converges to x. From

$$\rho(\overline{conv}\, F(x_n), \overline{conv}\, F(x)) \leq \rho(F(x_n), F(x)), \text{ for all } \ n \in \mathbb{N}^*$$

and using the hypothesis that F is H-u.s.c. on X we got the desired conclusion.$\square$

Lemma 1.3.13. *Let X, Y be Hausdorff topological spaces and $F : X \to P_{cp}(Y)$ be u.s.c. on X. Then, for each compact subset K of X, $F(K)$ is a compact set in Y.*

Lemma 1.3.14. *a) Let X, Y be Hausdorff topological spaces, $F_i : X \to P_{cp}(Y)$ be u.s.c. on X for each $i \in I$ such that $\bigcap_{i\in I} F_i(x) \neq \emptyset$ for* *each $x \in X$ and $H_j : X \to P_{cp}(Y)$, be u.s.c. for each $j \in \{1, 2, \ldots, n\}$. Then:*

i) $F := \bigcap_{i \in I} F_i$ is u.s.c. on X and has compact values.

ii) $H := \bigcup_{j=1}^{n} H_j$ is u.s.c. on X and has compact values.

b) If Y is a normed spaces and $F_1, F_2 : X \to P_{cp}(Y)$ are u.s.c. then, $T : X \to P_{cp}(Y)$, $T = F_1 + F_2$ is u.s.c. on X.

Another continuity notion for a multi-function is defined as follows:

Definition 1.3.15. Let $(X, d), (Y, d')$ be metric spaces and $F : X \to P(Y)$. Then F is said to be closed in $x_0 \in X$ if and only if for all $(x_n)_{n \in \mathbb{N}^*} \subset X$ such that $\lim_{n \to \infty} x_n = x_0$ and for all $(y_n)_{n \in \mathbb{N}^*} \subset Y$, with $y_n \in F(x_n)$, for all $n \in \mathbb{N}^*$ and $\lim_{n \to \infty} y_n = y_0$ we have $y_0 \in F(x_0)$.

F is closed on X if it is closed in each point $x_0 \in X$.

Remark 1.3.16. An equivalent definition is the following: $F : X \to P(Y)$ is said to be closed in $x_0 \in X$ if and only if for each $y_0 \notin F(x_0)$ there exist a neighborhood V of x_0 and a neighborhood U of y_0 such that for all $x \in V$ it follows that $F(x) \cap U = \emptyset$.

Lemma 1.3.17. *Let $(X, d), (Y, d')$ be metric spaces and $F : X \to P(Y)$ closed on X. Then:*

i) $F(x) \in P_{cl}(Y)$, for all $x \in X$

ii) $Graf\ F$ is a closed set with respect to the Pompeiu-Hausdorff topology from $X \times Y$. Moreover, the condition ii) implies that F is closed on X.

Lemma 1.3.18. *Let X, Y be Hausdorff topological spaces, $F_i : X \to P(Y)$, $i \in I$ be closed on X such that $\bigcap_{i \in I} F_i(x) \neq \emptyset$ for each $x \in X$ and $H_j : X \to P(Y)$, $j \in \{1, \ldots, n\}$ be closed on X. Then:*

i) $F := \bigcap_{i \in I} F_i$ is closed on X.

ii) $H := \bigcup_{j=1}^{n} H_j$ *is closed on* X.

The relation between an upper semi-continuous and a closed multi-valued operator are given by the following results:

Lemma 1.3.19. *Let* $(X, d), (Y, d')$ *be metric spaces and* $F : X \to P_{b,cl}(Y)$ *be* H*-u.s.c. on* X*. Then* F *is closed on* X.

Proof. Let $x \in X$ and $((x_n, y_n))_{n\in\mathbb{N}} \subset X \times Y$ such that $(x_n, y_n) \to (x, y)$ as $n \to \infty$ with $y_n \in F(x_n)$, for all $n \in \mathbb{N}$. F is H-u.s.c. in x and hence $\lim_{n\to\infty} \rho(F(x_n), F(x)) = 0$. On the other side, $D(y, F(x)) \leq d(y, y_n) + D(y_n, F(x_n)) + \rho(F(x_n), F(x))$, for all $n \in \mathbb{N}$. If we take $n \to \infty$ it follows that $D(y, F(x)) \leq 0$ and so $y \in \overline{F(x)} = F(x)$. $\square$

For a reverse proposition, we have:

Theorem 1.3.20. *Let* $(X, d), (Y, d')$ *be metric spaces,* $F_1 : X \to P(Y)$ *closed and* $F_2 : X \to P_{cp}(Y)$ *u.s.c.. Suppose that* $F_1(x) \cap F_2(x) \neq \emptyset$ *for each* $x \in X$*. Then, the multi-valued operator* $F = F_1 \cap F_2$ *is u.s.c. and it has compact values.*

Corollary 1.3.21. *Let* (X, d) *be a metric space,* (Y, d') *be a compact metric space and* $F : X \to P(Y)$ *closed on* X*. Then* F *is u.s.c. on* X *and it has compact values.*

Definition 1.3.22. Let X, Y be topological spaces. A multi-function $F : X \to P(Y)$ is said to be compact if its range $F(X)$ is relatively compact in Y.

Lemma 1.3.23. *Let* X, Y *be metric spaces and* $F : X \to P_{cp}(Y)$ *be a closed and compact multi-function. Then* F *is u.s.c.*

Lemma 1.3.24. *Let* X, Y *be metric spaces and* $F : X \to P_{cl}(Y)$ *be a closed multi-function. Then for each compact subset* K *of* X *its image* $F(K)$ *is closed in* Y.

Let us consider now the concept of lower semi-continuous multi-function.

Definition 1.3.25. Let X, Y be Hausdorff topological spaces and $F : X \to \mathcal{P}(\mathcal{Y})$. Then, F is said to be lower semi-continuous (briefly l.s.c.) in $x_0 \in X$ if and only if for each open subset $U \subset Y$ with $F(x_0) \cap U \neq \emptyset$ there exists an open neighborhood V of x_0 such that $F(x) \cap U = \emptyset$, for all $x \in V$.

F is l.s.c. on X if it is l.s.c. in each $x_0 \in X$.

Remark 1.3.26. If $(X, d), (Y, d')$ are metric spaces and $F : X \to P(Y)$, then F is l.s.c. in $x_0 \in X$ if and only if for all $(x_n)_{n\in\mathbb{N}^*} \subset X$ such that $\lim\limits_{n\to\infty} x_n = x_0$ and for all $y_0 \in F(x_0)$ there exists a sequence $(y_n)_{n\in\mathbb{N}^*} \subset Y$ such that $y_n \in F(x_n)$, for all $n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} y_n = y_0$.

Another lower semi-continuity notion is given by:

Definition 1.3.27. Let (X, d) and (Y, d') be metric spaces and $F : X \to P(Y)$. Then, F is called H-lower semi-continuous (briefly H-l.s.c.) in $x_0 \in X$ if and only if for each $\varepsilon > 0$ there exists $\eta > 0$ such that $F(x_0) \subset V(F(x); \varepsilon)$, for all $x \in B(x_0; \eta)$.

F is H-l.s.c. on X if it is l.s.c. in each point $x_0 \in X$.

Remark 1.3.28. $F : X \to P_{b,cl}(Y)$ is H-l.s.c. in $x_0 \in X$ if and only if for each $\varepsilon > 0$ there exists $\eta > 0$ such that $\rho_{d'}(F(x_0), F(x)) \leq \varepsilon$, for all $x \in B(x_0; \eta)$.

Lemma 1.3.29. *Let (X, d), (Y, d') be metric spaces and $F : X \to P(Y)$ be H-l.s.c. in $x_0 \in X$. Then F is l.s.c. in $x_0 \in X$.*

Regarding the reverse implication we have:

Lemma 1.3.30. *Let (X, d), (Y, d') be metric spaces and $F : X \to P_{cp}(Y)$ be l.s.c. in $x_0 \in X$. then F is H-l.s.c. in $x_0 \in X$.*

A characterization result for l.s.c. multi-functions is:

Theorem 1.3.31. *Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$. Then, the following assertions are equivalent:*

i) F is l.s.c. on X

ii) $F^+(V) := \{x \in X \mid F(x) \subset V\}$ is closed, for each closed set $V \subset Y$.

iii) $F^-(W) := \{x \in X \mid F(x) \cap W \neq \emptyset\}$ is open, for each open set $W \subset Y$.

Lemma 1.3.32. *Let (X, d) be a metric space, Y be a Banach space and $F : X \to P(Y)$ be l.s.c.. Then, the multi-valued operators $\operatorname{conv} F$ and $\overline{\operatorname{conv}} F$ are l.s.c..*

Lemma 1.3.33. *Let X, Y, Z be Hausdorff topological spaces. Then:*

i) If $F : X \to P(Y)$ and $G : Y \to P(Z)$ are l.s.c. on X respectively on Y then $G \circ F : X \to P(Z)$ is l.s.c. on X.

ii) If $F_i : X \to P(Y)$, are l.s.c. on X, for each $i \in I$, then $F := \bigcup\limits_{i \in I} F_i$ is l.s.c. on X.

An useful result is:

Lemma 1.3.34. *Let (X, d), (Y, d') be metric spaces. If $F_1 : X \to P(Y)$ is l.s.c. and $F_2 : X \to P(Y)$ has open graph, such that $F_1(x) \cap F_2(x) \neq \emptyset$ for each $x \in X$, then the multi-valued operator $F_1 \cap F_2$ is l.s.c..*

Definition 1.3.35. Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$. Then F is said to be continuous in $x_0 \in X$ if and only if it is l.s.c. and u.s.c. in $x_0 \in X$.

Definition 1.3.36. Let (X, d), (Y, d') be metric spaces and $F : X \to P(Y)$. Then F is called H-continuous in $x_0 \in X$ (briefly H-c.) if and only if it is H-l.s.c. and H-u.s.c. in $x_0 \in X$.

Remark 1.3.37. If $(X, d), (Y, d')$ are metric spaces, then $F : X \to$

$P_{b,cl}(Y)$ is H-c. in $x_0 \in X$ if and only if for each $\varepsilon > 0$ there exists $\eta > 0$ such that $x \in B(x_0; \eta)$ implies $H_{d'}(F(x), F(x_0)) < \varepsilon$.

Theorem 1.3.38. *Let (X, d) and (Y, d') be metric spaces. Then $F : X \to P_{cp}(Y)$ is continuous on X if and only if F is H-c. on X.*

The relations between H-continuity and lower semi-continuity is given in:

Lemma 1.3.39. *Let $(X, d), (Y, d')$ be metric spaces and $F : X \to P_{b,cl}(Y)$ be H-c. on X. Then F is l.s.c. on X.*

Further on, we will present some properties of multi-valued Lipschitz-type operators.

Lemma 1.3.40. *Let (X, d) be a metric space and $F : X \to P_{b,cl}(X)$ be a-Lipschitz. Then:*

a) F is closed on X

b) F is H-l.s.c. on X

c) F is H-u.s.c. on X

Proof. a) Let $(x_n, y_n)_{n\in\mathbb{N}} \subset X \times Y$ such that $(x_n, y_n) \to (x, y)$, when $n \to \infty$ and $y_n \in F(x_n)$, for all $n \in \mathbb{N}$. It follows that $D(y, F(x)) \leq d(y, y_n) + D(y_n, F(x)) \leq d(y, y_n) + H(F(x_n), F(x)) \leq d(y, y_n) + ad(x_n, x)$, for all $n \in \mathbb{N}$. Let us consider $n \to \infty$ and we obtain $D(y, F(x)) \leq 0$, proving that $y \in \overline{F(x)} = F(x)$.

b) Let $x \in X$ such that $x_n \to x$. We have: $\rho(F(x), F(x_n)) \leq H(F(x), F(x_n)) \leq ad(x, x_n) \to 0$. In conclusion, F is H-l.s.c. on X.

c) Using the relation: $\rho(F(x_n), F(x)) \leq H(F(x_n), F(x)) \leq ad(x, x_n) \to 0$, the conclusion follows as before. □

Lemma 1.3.41. Let (X, d) be a metric space and $F : X \to P_{cp}(X)$ be contractive. Then F is u.s.c. on X.

Proof. Let $H \subset Y$ be a closed set. We will prove that $F^-(H)$ is closed in X. Let $x \in \overline{F^-(H)} \setminus F^-(H)$ and $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \to x$, when $n \to \infty$, $x_n \neq x$, for all $n \in \mathbb{N}$ and $x_n \in F^-(H)$, for all $n \in \mathbb{N}$. It follows $F(x_n) \cap H \neq \emptyset$, for all $n \in \mathbb{N}$. Let $y_n \in F(x_n) \cap H$, $n \in \mathbb{N}$. Then $D(y_n, F(x)) \leq H(F(x_n), F(x)) < d(x_n, x)$. If $n \to \infty$ we got that $\lim_{n\to\infty} D(y_n, F(x)) = 0$. But $D(y_n, F(x)) = \inf_{y\in F(x)} d(y_n, y) = d(y_n, y'_n)$ (using the compactness of the set $F(x)$). When $n \to \infty$ we have $d(y_n, x'_n) \to 0$, $n \to \infty$. Because $(x'_n)_{n\in\mathbb{N}} \subset F(x)$ we obtain that there exists a subsequence $(x'_{n_k})_{k\in\mathbb{N}}$ which converges to an element $\widetilde{x} \in F(x)$. Then:

$$d(y_{n_k}, \widetilde{x}) \leq d(y_{n_k}, x'_{n_k}) + d(x'_{n_k}, \widetilde{x}) \to 0 \text{ când } k \to \infty$$

Hence, $y_{n_k} \to \widetilde{x} \in F(x)$, as $n \to \infty$. Because, $(y_{n_k})_{k\in\mathbb{N}} \subset H$ and H is closed, we got that $\widetilde{x} \in H$. So $F(x) \cap H \neq \emptyset$, which implies $x \in F^-(H)$, a contradiction. In conclusion, $\overline{F^-(H)} = F^-(H)$ and hence $F^-(H)$ is closed in X. $\square$

1.4 Measurability of multi-valued operators

Let $(T, \mathcal{A})$ be a measurable space and S be a family of subsets of T.

Definition 1.4.1. The σ-algebra generated by S is the intersection of all σ-algebras containing S.

Remark 1.4.2. If T is a topological space, then the Borel σ-algebra, denoted by $\mathcal{B}(T)$, is the σ-algebra generated by the family of all open sets from T.

Remark 1.4.3. If $T = \mathbb{R}^n$ or $T \subset \mathbb{R}^n$ then the σ-algebra $\mathcal{A}$ is the family $\mathcal{L}(T)$ of all measurable Lebesgue subsets of T.

Remark 1.4.4. If $(T_1, \mathcal{A}_1), (T_2, \mathcal{A}_2)$ are measurable spaces, then the σ-algebra generated by the family of sets of the form $A_1 \times A_2$, with

$A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$ and which will be denoted by $\mathcal{A}_1 \otimes \mathcal{A}_2$ is called the product σ-algebra of $\mathcal{A}_1$ with $\mathcal{A}_2$. The measurable space $(T_1 \times T_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ is said to be the product space of $(T_1, \mathcal{A}_1)$ and $(T_2, \mathcal{A}_2)$. In particular, if T_1, T_2 are topological spaces, then $\mathcal{B}(T_1) \otimes \mathcal{B}(T_2) = \mathcal{B}(T_1 \times T_2)$.

Definition 1.4.5. Let $(T, \mathcal{A})$ be a measurable space. Then, the function $\mu : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ is said to be a positive measure if for each sequence of disjoint sets $A_n \in \mathcal{A}$, $n \in \mathbb{N}^*$ we have: $\mu\left(\bigcup_{n \in \mathbb{N}^*} A_n\right) = \sum_{n \in \mathbb{N}^*} \mu(A_n)$.

Definition 1.4.6. Let $(T, \mathcal{A})$ be a measurable space. A positive measure μ is called σ-finite, if T can be represented as a countable reunion of measurable sets having finite measures.

Definition 1.4.7. Let $(T, \mathcal{A})$ be a measurable space and μ a positive measure. Then, the σ-algebra $\mathcal{A}$ is called μ-complete if for each $A \in \mathcal{A}$ with $\mu(A) = 0$ and for each $A_1 \subset A$, we have that $A_1 \in \mathcal{A}$.

Remark 1.4.8. If $T \subset \mathbb{R}^n$ is open (or closed), then $\mathcal{L}(T)$ is complete with respect to the Lebesgue measure. Moreover, the Lebesgue measure is σ-finite.

Definition 1.4.9. $(T, \mathcal{A}, \mu)$ is said to be a complete space with σ-finite measure if μ is a positive and σ-finite measure, while $\mathcal{A}$ is μ-complete.

Definition 1.4.10. Let $(T, \mathcal{A})$ be a measurable space, (X, d) be a separable metric space. Then $f : T \to X$ is called $\mathcal{A}$-measurable (or measurable) if and only if $f^{-1}(A) \in \mathcal{A}$, for each $A \in P_{op}(X)$ (or each $A \in P_{cl}(X)$.

Definition 1.4.11. Let $(T, \mathcal{A})$ be a measurable space, (X, d) be a separable metric space and $F : T \to P(X)$. Then F is called weak measurable (respectively measurable, respectively β-measurable) if and only if $F^{-}(E) := \{t \in T|\ F(t) \cap E \neq \emptyset\} \in \mathcal{A}$, for each $E \subset X$ open (respectively closed, respectively Borel).

Remark 1.4.12. The β-measurability implies the measurability, which implies the weak measurability of a multi-function.

Some equivalences between these concepts are included in the following lemma:

Lemma 1.4.13. *Let $(T, \mathcal{A})$ be a measurable space, (X, d) be a complete and separable metric space and $F : T \to P_{cp}(X)$. Then:*

i) F is weak measurable if and only if F is measurable.

ii) F is measurable if and only if the single-valued operator $F : (T, \mathcal{A}) \to (P_{cp}(X), H_d)$ is measurable.

Definition 1.4.14. Let $(T, \mathcal{A})$ be a measurable space and X, Y be metric spaces. Then $\varphi : T \times X \to Y$ is said to be a Carathéodory mapping if and only if:

i) for all $x \in X$, $\varphi(\cdot, x)$ is measurable.

ii) for all $t \in T$, $\varphi(t, \cdot)$ is continuous.

Lemma 1.4.15. *Let X and Y be complete and separable metric spaces, $(T, \mathcal{A})$ be a measurable space and $\varphi : T \times X \to Y$ be a Carathéodory mapping. Then:*

a) for each measurable function $f : T \to X$ we have that $t \mapsto \varphi(t, f(t))$ is measurable.

b) φ is $\mathcal{A} \otimes \mathcal{B}$ measurable.

In this framework, a very important theorem belong to Kuratowski and Ryll Nardzewski. It is an existence result of a measurable selection for a weak measurable multi-function. Let us recall that if X, Y are two nonempty sets and $F : X \to P(Y)$ is a multi-function, then a single-valued operator $f : X \to Y$ is said to be a selection of F if and only if $f(x) \in F(x)$, for each $x \in X$.

Theorem 1.4.16. *Let $(T, \mathcal{A})$ be a measurable space, (X, d) be a complete separable metric space and $F : T \to P_{cl}(X)$ be weak measurable. Then there exists $f : T \to X$ a measurable selection for F.*

Proof. Let $\{x_1, x_2, \dots\}$ be a countable and dense subset of X. Let $B_n(i) := \left\{x \in X |\ d(x, x_i) \leq \frac{1}{n}\right\}$, for $i, n \in \mathbb{N}$. We will define inductively a sequence of measurable multi-functions $(F_n)_{n\in\mathbb{N}}$ such that $\bigcap_{n=1}^{\infty} F_n$ will be the desired selection.

Let $F_0 = F$ and $F_{n+1}(t) = F_n(t) \cap B_{n+1}(I_n(t))$, where $I_n(t) := \min\{i \in \mathbb{N} |\ F(t) \cap B_{n+1}(i) \neq \emptyset\}$, for all $n \in \mathbb{N}$. For each $t \in T$ the sequence $(F_n(t))_{n\in\mathbb{N}} \subset P_{b,cl}(X)$ is decreasing and $\delta(F_n(t)) \to 0$, as $n \to \infty$. Using Cantor's theorem we obtain that $\bigcap_{n=1}^{\infty} F_n(t)$ consist in exactly one point. Let us define now $f(t) = \bigcap_{n=1}^{\infty} F_n(t)$, for all $t \in T$. Obviously, f is a selection for F. Let also prove that f is measurable. We shall prove first that each F_n are measurable, i.e. $\{t \in T |\ F_n(t) \cap E \neq \emptyset\} \in \mathcal{A}$, for each closed subset E of X. From the hypothesis, we have that $F_0 = F$ is measurable. Let suppose that F_n is measurable. Then we obtain:

$$\{t \in T |\ F_{n+1}(t) \cap E \neq \emptyset\} = \{t \in T |\ F_n(t) \cap B_{n+1}(I_n(t)) \cap E \neq \emptyset\} =$$

$$= \bigcap_{n=1}^{\infty} [\{t \in T |\ F_n(t) \cap B_{n+1}(i) \cap E \neq \emptyset\} \cap \{t \in T |\ I_n(t) = i\}].$$

But the final set is in $\mathcal{A}$ (taking account that $\{t \in T |\ I_n(t) = i\} = \bigcap_{j=1}^{i-1} [\{t \in T |\ F_n(t) \cap B_{n+1}(i) = \emptyset\} \cap \{t \in T |\ F_n(t) \cap B_{n+1}(i) \neq \emptyset\}] \in \mathcal{A}$). Hence the induction is finished.

Because X is complete, for each closed subset E of X we have

$$f^{-1}(E) = \bigcap_{n=0}^{\infty} \{t \in T |\ F_n(t) \cap E \neq \emptyset\}$$

and in conclusion $f^{-1}(E) \in \mathcal{A}$, proving that f is measurable. □

The following characterization theorem describe the main properties of measurable multi-functions:

Theorem 1.4.17. *Let $(T, \mathcal{A}, \mu)$ be a complete space with a σ-finite measure, (X, d) be a complete and separable metric space and $F : T \to P_{cl}(X)$. Then the following assertions are equivalent:*

i) F is weak measurable

ii) $Graf\, F \in \mathcal{A} \otimes \mathcal{B}(X)$

iii) $F^{-}(A) \in \mathcal{A}$, for all $A \in P_{cl}(X)$

iv) $F^{-}(A) \in \mathcal{A}$, for each Borel subset of X

v) for each $x \in X$ the mapping $D : T \to \mathbb{R}$ defined by $t \mapsto D(x, F(t))$ is measurable

vi) There exists a sequence of measurable selections $\{f_n\}_{n\in\mathbb{N}^*}$ of F such that for each $t \in T$, $F(t) = \overline{\bigcup_{n\geq 1} f_n(t)}$. (Castaing representation of F)

Lemma 1.4.18. *Let $(T, \mathcal{A}, \mu)$ be a complete space with a σ-finite measure, (X, d) be a measurable Banach space and $F : T \to P_{cl}(X)$. Then the multi-valued operator $\overline{conv}\, F : T \multimap X$ is measurable. Moreover, if $I \subset \mathbb{R}$ is compact and $F : I \to P_{cp}(\mathbb{R}^n)$ is measurable then $conv\, F : I \multimap \mathbb{R}^n$ is measurable.*

Lemma 1.4.19. *a) Let T be a metric space such that $(T, \mathcal{A}, \mu)$ is complete with a σ-finite measure while $\mathcal{A}$ contain all open sets from T. Let (X, d) be a complete and separable metric space and $F : T \to P_{cl}(X)$. If F is u.s.c. on T (or l.s.c. on T) then F is weak measurable.*

b) Let X be a complete and separable metric space, Y be a metric space, T a measurable space, $f : T \times X \to Y$ a Carathéodory mapping, $F : T \to P_{cp}(X)$ be measurable and $g : T \to Y$ a measurable mapping such that $g(t) \in f(t, F(t))$, for all $t \in T$. Then there exists $h : T \to X$

a measurable selection of F, such that $g(t) = f(t, h(t))$, for all $t \in T$. (Fillipov implicit function lemma)

c) Let I be a compact interval of the real axis, X be a complete and separable metric space, $\varphi : I \to X$ a measurable mapping and $F : I \times X \to P_{cl}(\mathbb{R}^n)$ a multi-valued operator satisfying the conditions:

i) $F(\cdot, x)$ is measurable, for all $x \in X$

ii) $F(t, \cdot)$ is H-c, for all $t \in I$.

Then:

1) F is measurable

2) the multi-function G defined by $G(t) = F(t, \varphi(t))$, for all $t \in I$ is measurable too.

3) if $H : I \to P_{cp}(\mathbb{R}^n)$ is measurable then the multifunction $P : I \to P(\mathbb{R}^n)$, by $P(t) := F(t, H(t))$, for each $t \in I$ is measurable. Moreover, same conclusion is true when the conditions i) and ii) are replaced by the u.s.c. or l.s.c. of the multi-valued operator F.

Lemma 1.4.20. *Let I be a compact interval of the real axis, Y a separable Banach space and $F_1, F_2 : I \to P_{cp}(Y)$ weak measurable multi-functions. Then:*

a) the deviation functional denoted $d^* : I \to \mathbb{R}_+$, and defined by $d^*(t) := \rho(F_1(t), F_2(t)$, for each $t \in I$ is measurable.

b) the functional $h : I \to \mathbb{R}_+$, defined by $h(t) := H(F_1(t), F_2(t))$, for each $t \in I$ is measurable.

Definition 1.4.21. Let $(T, \mathcal{A}, \mu)$ be a complete space with a σ-finite measure, X be a separable Banach space and $F : T \to P_{cl}(X)$ a measurable multi-function. Let us denote by $L^1(T, X)$ the set of all measurable and Bochner integrable mappings from T to X. Then $\mathcal{S}_F$ will denote the set of all integrable selections of F, i.e.:

$$\mathcal{S}_F := \{f \in L^1(T, X) | \ f(t) \in F(t) \text{ a.p.t. } t \in T\}.$$

Definition 1.4.22. Let $(T, \mathcal{A}, \mu)$ be a complete space with a σ-finite measure, X be a separable Banach space and $F : T \to P_{cl}(X)$ a measurable multi-function. Then F is said to be integrably bounded if there exists a real valued function $m : T \to \mathbb{R}_+$ such that $m \in L^1(T, \mathbb{R})$ and $F(t) \subset B_X(0, m(t))$, a.e. $t \in T$.

Remark 1.4.23. If $F : T \to P_{cp}(X)$ is integrably bounded then $\|F(t)\| \leq m(t)$ a.e. $t \in T$.

Remark 1.4.24. If $F : T \to P_{cl}(X)$ is integrably bounded then for each measurable selection $f : T \to X$ of F we have $f \in \mathcal{S}_F$. Moreover, if $F : T \to P_{cp}(X)$ is measurable and integrably bounded, then $\mathcal{S}_F \neq \emptyset$.

Definition 1.4.25. Let $F : T \to P_{cl}(X)$ be a measurable multi-function. Then F is said to be integrable in Aumann' sense on T if and only if $\mathcal{S}_F \neq \emptyset$. In this case, the multi-valued integral of F is:

$$\int_T F(t)d\mu := \left\{ \int_T f(t)d\mu |\ f \in \mathcal{S}_F \right\}.$$

Let us report now several properties of the multi-valued integral for the following particular case: $T = I$, $I \subseteq \mathbb{R}$ compact and $X = \mathbb{R}^n$.

Lemma 1.4.26. Let $K \in P_{cl,cv}(\mathbb{R}^n)$ and $t_1, t_2 \in I$, $t_1 < t_2$. Then $\int_{t_1}^{t_2} K dt = (t_2 - t_1)K$.

Definition 1.4.27. A set $K \subset L^1(I, \mathbb{R}^n)$ is said to be decomposable if and only if for all $u, v \in K$ and for each measurable subset E of I we have that $\chi_E u + \chi_{I \setminus E} v \in K$.

Let us remark that, if $F : I \to P_{cp}(\mathbb{R}^n)$ is measurable and integrably bounded then $\mathcal{S}_F$ is decomposable.

Lemma 1.4.28. Let $K \subset L^1(I, \mathbb{R}^n)$ be a decomposable set. Then the set

$$J(K) := \left\{ \int_I f(t)dt |\ f \in K \right\}$$

is convex in $\mathbb{R}^n$.

Proof. Let us suppose that $K \neq \emptyset$ and let $z_1, z_2 \in J(K)$ and $\lambda \in]0,1[$. Then there exist $f_1, f_2 \in K$ such that

$$z_1 = \int_I f_1(t)dt \quad \text{and} \quad z_2 = \int_I f_2(t)dt.$$

Let $\alpha(I)$ be the family of all Lebesgue measurable sets of I and denote

$$\gamma(E) = \left(\int_I f_1(t)dt, \int_I f_2(t)dt \right), \text{ for all } E \in \mathcal{A}(I).$$

From Lyapunov's convexity theorem (see for example [100])we obtain that $\gamma(\alpha(I))$ is a convex compact subset of $\mathbb{R}^{2n}$. Because $(0,0)$ and (z_1, z_2) belong to $\gamma(\alpha(I))$ it follows that $(\lambda z_1, \lambda z_2) \in \gamma(\alpha(I))$. Hence, there exists $F \in \alpha(I)$ such that $(\lambda z_1, \lambda z_2) = \gamma(F)$. Define $f = \chi_F f_1 + \chi_{I \setminus F} f_2$. Using the decomposability property of the set K we obtain that $f \in K$ and so $\int_I f(t)dt \in J(K)$. But $\int_I f(t)dt = \lambda z_1 + (1-\lambda) z_2$ and in conclusion $\lambda z_1 + (1-\lambda) z_2 \in J(K)$. $\square$

Theorem 1.4.29. *Let $F : I \to P_{cp}(\mathbb{R}^n)$ be a measurable and integrably bounded multi-function. Then*

$$\int_I F(t)dt = \int_I conv\, F(t)dt$$

and both are non-empty, convex, compact subset of $\mathbb{R}^n$.

In the general case of complete spaces with a σ-finite measure we have the following theorems.

Definition 1.4.30. Let $(T, \mathcal{A}, \mu)$ be a complete space with a σ-finite measure. Then the set $A \in \mathcal{A}$ is said to be an atom with respect to μ if and only if $\mu(A) > 0$ and for each $A_1 \subset A$ measurable we have that $\mu(A_1)$ is equal to 0 or $\mu(A)$. By definition, a measure μ is called non-atomic if $\mathcal{A}$ does not contain atoms.

Remark 1.4.31. The Lebesgue measure is non-atomic, while the Dirac measure, for example, is atomic.

Theorem 1.4.32. *Let $(T, \mathcal{A}, \mu)$ be a complete space with a σ-finite and non-atomic measure and $F : T \to P_{cl}(\mathbb{R}^n)$ be measurable. Then the following assertions hold:*

i) $\displaystyle\int_T F d\mu$ *is convex in* $\mathbb{R}^n$

ii) If F is integrably bounded, then $\displaystyle\int_T F d\mu$ *is non-empty and compact in* $\mathbb{R}^n$.

For the case of an arbitrary Banach space the following result belong to Hiai and Umegaki.

Theorem 1.4.33. *Let $(T, \mathcal{A}, \mu)$ be a complete space with a σ-finite and non-atomic measure, X separable Banach space and $F : T \to P_{cl}(X)$ be measurable. Then:*

i) $\overline{\int_T F(t)d\mu}$ *is convex and* $\overline{\int_T F(t)d\mu} = \overline{conv}\left(\int_T F(t)d\mu\right)$.

ii) If F is integrably bounded, then $\overline{\int_T F(t)d\mu} = \int_T \overline{conv}\, F(t)d\mu$.

iii) If X is reflexive and $F : T \to P_{cl,cv}(X)$ is integrably bounded, then $\displaystyle\int_T F(t)d\mu$ *is closed in X.*

Bibliographical comments.

The notions and results given in this chapter can be found in books and papers on multi-valued analysis such as: Aubin-Cellina [15], Aubin-Frankowska [16], Beer [30], Berge [32], Cernea [56], Deimling [80], Hu-Papageorgiou [105], Kamenskii-Obuhovskii-Zecca [118], Kirk-Sims (eds.) [124], Kisielewicz [127], M. Mureşan [152], Petruşel [197], I. A. Rus [223] and Xu [267].

Chapter 2

Operatorial inclusions

One of the connections between the "multi-valued analysis" and the "single-valued analysis" is given by the notion of selection. The first purpose of this chapter is to present continuous selection theorems for l.s.c. and u.s.c. multi-function with convex values. Then, the case of multi-functions with decomposable values is considered. Second, we will discuss the fixed point and the coincidence point theory for multi-valued operators. In this respect, we will report first the basic theory of the fixed points for multi-functions. Then, we will focus our interest on the main properties of the fixed point set of some multi-valued generalized contractions. In the third section, single-valued and multi-valued Caristi type operators are considered. Then, the connection between Meir-Keeler type operators and fractals is discussed. Coincidence theorems is the subject of the next paragraph. Finally, Krasnoselskii type theorems for multi-functions and the topological dimension of the fixed point set for several classes of multi-valued operators are the main topics of the last sections. Some applications to integral and differential inclusions are also presented.

2.1 Continuous selection theorems

In what follows, we will consider the basic selection theorems for l.s.c. and u.s.c. multi-functions.

Definition 2.1.1. Let X, Y be nonempty sets and $F : X \to P(Y)$. Then the single-valued operator $f : X \to Y$ is called a selection of F if and only if $f(x) \in F(x)$, for each $x \in X$.

Recall that, if X is a topological space, then a set $K \subset X$ is called compact if every open covering of K admits a finite subcovering. Moreover, if $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ are two coverings of X, then $(U_i)_{i\in I}$ is said to be a refinement of $(V_j)_{j\in J}$ if, for every $i \in I$ there exists $j \in J$ such that $U_i \subset V_j$.

An open covering $(V_j)_{j\in J}$ of a topological space X is called locally finite if for every $x \in X$ there exists a neigherhood $V \in \mathcal{V}(x)$ such that

$$card\{i \in I | V_i \cap V \neq \emptyset\}$$

is finite.

A space X is called paracompact if every open covering of it has a locally finite refinement.

For example, every compact set is paracompact and every metrizable space is paracompact. In particular, every metric space is paracompact.

For every topological space X and for $f : X \to \mathbb{R}$ the set $supp(f)$ is defined by

$$supp(f) := \overline{\{x \in X | f(x) \neq 0\}}.$$

Let X be a topological space and $(U_i)_{i\in I}$ be an open covering of X. Then, a continuous partition of unity subordinate to $(U_i)_{i\in I}$ means a family of continuous functions $\alpha_i : X \to [0, 1]$ such that:

(i) $supp(\alpha_i) \subset U_i$, for each $i \in I$;

(ii) $(supp(\alpha_i))_{i\in I}$ is a closed locally finite covering of X;

(iii) $\sum_{i\in I} \alpha_i(t) = 1$.

Remark. Let X be a paracompact space and $(U_i)_{i\in I}$ be an open covering of X. Then:

(a) $(U_i)_{i\in I}$ admits an open locally finite refinement $(V_i)_{i\in J}$ with $\overline{V_i} \subset U_i$, for every $i \in I$;

(b) $(U_i)_{i\in I}$ has a subordinate partition of unity.

In particular, if X is compact, then the open locally finite refinement is actually a finite one. As a consequence, the subordinate partition of unity consists in a finite number of maps.

Let us consider now the selection theorem of Browder.

Theorem 2.1.2. (Browder' selection theorem) *Let X and Y be Hausdorff topological vectorial space and $K \in P_{cp}(X)$. Let $F : K \to P_{cv}(Y)$ be a multi-valued operator such that $F^{-1}(y)$ is open, for each $y \in Y$. Then there exists a continuous selection f of F.*

Proof. Because $(F^{-1}(y))_{y\in Y}$ is an open covering of K, there exists a finite refinement of it, denoted by $(F^{-1}(y_i))_{i=\overline{1,n}}$. Let $(\alpha_i)_{i\in\overline{1,n}}$ be the continuous partition of unity corresponding to this finite covering. We define $f : K \to Y$ by the following relation: $f(x) = \sum_{i=1}^{n} \alpha_i(x)y_i$. Then f is continuous and each time when $\alpha_i(x) > 0$ it follows $y_i \in F(x)$. But for each $x \in X$, the set $F(x)$ is convex, and hence we obtain that $f(x) \in F(x)$, for all $x \in X$. $\square$

A very famous result is the so-called Michael' selection theorem. We start by proving the following auxiliary result:

Lemma 2.1.3. *Let (X, d) be a metric space, Y a Banach space and $F : X \to P_{cv}(Y)$ be l.s.c. on X. Then, for each $\varepsilon > 0$ there exists $f_\varepsilon : X \to Y$ a continuous function such that for all $x \in X$, we have: $f_\varepsilon(x) \in V(F(x); \varepsilon)$.*

Proof. Because F is l.s.c. we associate to each $x \in X$ and to each $y_x \in F(x)$ an open neighborhood U_x of x such that $F(x') \cap B(y_x; \varepsilon) \neq \emptyset$, for all $x' \in U_x$. X is a paracompact space and so there exists a locally finite refinement $\{U'_x\}_{x\in X}$ of $\{U_x\}_{x\in X}$. Let us recall that $\{\Omega_i\}_{i\in I}$ is a locally finite covering of X if for each $x \in X$ there exists V a neighborhood of x satisfying $\Omega_i \cap V \neq \emptyset$, for all $i = \overline{1,k}$. Moreover, to each locally finite covering it is possible to associate a continuous partition of unity, let say $\{\pi_x\}_{x\in X}$. We define: $f_\varepsilon(t) = \sum_{x\in X} \pi_x(t)y_x$. Then f_ε is continuous, being, locally, a finite sum of continuous functions. Moreover, if $\pi_x(t) > 0$, for $t \in U'_x \subset U_x$ then $y_x \in V(F(t), \varepsilon)$ implies that $f_\varepsilon(t) \in V(F(t), \varepsilon)$. $\square$

Theorem 2.1.4. (Michael' selection theorem) *Let (X, d) be a metric space, Y be a Banach space and $F : X \to P_{cl,cv}(Y)$ be l.s.c. on X. Then there exists $f : X \to Y$ a continuous selection of F.*

Proof. Let us define inductively a sequence of continuous functions $u_n : X \to Y$, $n = 1, 2, \dots$ satisfying the following assertions:

i) for all $x \in X$, $D(u_n(x), F(x)) < \dfrac{1}{2^n}$, for each $n \in \mathbb{N}^*$

ii) for all $x \in X$, $\|u_n(x) - u_{n-1}(x)\| < \dfrac{1}{2^{n-2}}$, for each $n = 2, 3, \dots$

1. Case $n = 1$. The conclusion follows from Lemma 2.1.3 with $\varepsilon = \dfrac{1}{2}$.

2. Case $n = n + 1$. Let us suppose that we have defined the mappings $u_1, \dots, u_n$ and we will construct the map u_{n+1} such that i) and ii) hold. For this purpose, we consider the multi-valued operator F_{n+1} given by:

$$F_{n+1}(x) = F(x) \cap B\left(u_n(x); \frac{1}{2^n}\right), \text{ for each } x \in X.$$

From i) we obtain that $F_{n+1}(x) \neq \emptyset$, for all $x \in X$. Moreover $F_{n+1}(x)$ is convex, for all $x \in X$. Using Lemma 1.3.34., we have that F_{n+1} is l.s.c.. From Lemma 2.1.3., applied for F_{n+1} (with $\epsilon := \frac{1}{2^{n+1}}$), we obtain the existence of a continuous function $u_{n+1} : X \to Y$ such that: $D(u_{n+1}(x), F_{n+1}(x)) < \frac{1}{2^{n+1}}$, for each $x \in X$. It follows that $D(u_{n+1}(x), F(x)) < \frac{1}{2^{n+1}}$. On the same time, by the above relation we have:

$$u_{n+1}(x) \in V^0\left(F_{n+1}(x), \frac{1}{2^{n+1}}\right).$$

Then
$$u_{n+1}(x) \in B(u_n(x); \frac{1}{2^n} + \frac{1}{2^{n+1}}).$$
Thus
$$\|u_{n+1}(x) - u_n(x)\| < \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{3}{2^{n+1}} < \frac{1}{2^{n-1}}.$$
This completes the induction.

Further on, from ii) we obtain that $(u_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence convergent to a continuous function $u : X \to Y$. From i) and the fact that $F(x)$ are closed for each $x \in X$, we obtain that $u(x) \in F(x)$, for all $x \in X$. Hence, u is the desired continuous selection and the proof is complete. □

Corollary 2.1.5. *i) Let (X, d) be a metric space, Y a Banach space and $F : X \to P_{cl,cv}(Y)$ be l.s.c. on X. Let $Z \subset X$ be a nonempty set and $\varphi : Z \to Y$ a continuous selection of $F|_Z$. Then φ admits an extension to a continuous selection of F. In particular, we have that for each $y_0 \in F(x_0)$, with $x_0 \in X$ arbitrary, there exists a continuous selection φ of F such that $\varphi(x_0) = y_0$.*

ii) Let X be a metric space, Y be a Banach space, $F : X \to P_{cl,cv}(Y)$ be l.s.c. on X and $G : X \to P(Y)$ with open graph. If $F(x) \cap G(x) \neq \emptyset$, for all $x \in X$, then $F \cap G$ has a continuous selection.

For u.s.c. multi-functions we have the following approximate selection theorem given by Cellina [15]:

Theorem 2.1.6. (Cellina's approximate selection theorem) *Let (X, d) be a metric space, Y be a Banach space and $F : X \to P_{cv}(Y)$ be u.s.c. on X. Then for each $\varepsilon > 0$ there exists $f_\varepsilon : X \to Y$ locally Lipschitz such that:*

a) $f_\varepsilon(X) \in conv\, F(X)$,

b) $Graf\ f_\varepsilon \subset V(Graf\ F, \varepsilon)$.

The concept of locally selectionable multi-function characterize the multi-valued operators having "exact" continuous selections. More precisely, we define:

Definition 2.1.7. Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$. Then F is called locally selectionable at $x_0 \in X$ if for each $y_0 \in F(x_0)$ there exist an open neighborhood V of x_0 and a continuous mapping $f : V \to Y$ such that $f(x_0) = y_0$ and $f(x) \in F(x)$, for all $x \in X$. F is said to be locally selectionable if it is locally selectionable at every $x_0 \in X$.

Remark 2.1.8. Any locally selectionable multi-function is l.s.c.

Some examples of locally selectionable multi-functions are:

Lemma 2.1.9. *Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$ such that $F^{-1}(y)$ is open for each $y \in Y$. Then F is locally selectionable.*

We note that a similar result hold for multi-functions with open graph. (It is easy to see that if the graph of F is open then $F^{-1}(y)$ is open for each $y \in X$.)

Lemma 2.1.10. *Let X, Y be Hausdorff topological spaces and $F, G : X \to P(Y)$ such that $F(x) \cap G(x) \neq \emptyset$, for each $x \in X$. If F is locally selectionable and G has open graph then the multi-valued operator $F \cap G$ is locally selectionable.*

A global continuous selection theorem for a locally selectionable multi-function is:

Theorem 2.1.11. (Aubin-Cellina [15])*Let X be a paracompact space and Y a Hausdorff topological vector space. Then any locally selectionable multi-function $F : X \to P_{cv}(Y)$ has a continuous selection.*

Proof We associate with each $y \in X$ an element $z \in F(x)$ and a continuous selection $f_y : V \to Y$ such that $f_y(x) \in F(x)$ and $f(y) = z$. Since the space X is paracompact there exists a continuous partition of unity $(a_y)_{y \in X}$ associated with the open covering of X given by $V(y), y \in X$. Denote by $I(x)$ the non-empty finite set of points $y \in X$ having the property that $a_y(x) > 0$. Let us define the function $f : X \to Y$ by

$$f(x) = \sum_{y \in X} a_y(x) f_y(x) = \sum_{y \in I(x)} a_y(x) f_y(x).$$

Obviously, f is continuous as a finite sum of continuous functions and because $F(x)$ is convex, the convex combination $f(x)$ is also in $F(x)$. $\square$

A very interesting selection result for a continuous multi-function with not necessarily convex values is the following:

Theorem 2.1.12. (Strother [251]) *Let $F : [0,1] \to P([0,1])$ be a continuous multi-valued operator. Then there exists a continuous selection of F.*

Proof. Let us define $f : [0,1] \to [0,1]$, by $f(x) := inf\{y | y \in F(x)\}$. We will prove that f is a continuous selection of F. Let $x^{'} \in [0,1]$ be arbitrary and $r > 0$ be a real positive number. Denote by V_{2r} an open interval of length $2r$ with center $f(x^{'})$. Obviously, V_r is also an open set containing $f(x^{'})$. Using the l.s.c. of F there exists an open set U_1 containing x_0 such that $F(x) \cap V_r \neq \emptyset$, for each $x \in U_1$. Hence $x \in U_1$ implies that $inf\{y | y \in F(x)\} = f(x) \geq f(x^{'}) - r$. On the other side, consider $V = \{y | y < r + f(x^{'})\}$. The set V is open and it contains $F(x^{'})$. From the u.s.c. of F there exists an open set U_2 containing $x^{'}$ such that $F(x) \subset V$, for each $x \in U_2$. Then for each $x \in U_2$ we have that $f(x) = inf\{y | y \in F(x)\} \leq f(x^{'}) + r$.

Let consider now $U := U_1 \cap U_2$. Then for each $x \in U$ we obtain that $|f(x) - f(x^{'})| \leq r$ and therefore $f(x) \in V_{2r}$, proving that f is continuous in $x^{'}$. $\square$.

Let us consider now the problem of the existence of a Lipschitz selection for a multi-function.

Definition 2.1.13. Let $F : \mathbb{R}^n \to P_{cp}(\tilde{B}(0;R)))$ be a H-c. multi-function and let $S = \tilde{B}(y^0;b) \subset \mathbb{R}^n$. Let q be any finite collection of points $x_1, x_2, ..., x_{k+1}$ in S such that $\sum_{p=1}^{k} |x_{p+1} - x_p| \le b$ and Q denote the set of all such collections. Let $V(F, S, q) := \sum_{i=1}^{k} H(F(x_{i+1}), F(x_i))$ and $V(F, S) := sup\{V(F, S, q) | q \in Q\}$. If $V(F, S) < \infty$, then we say that F has bounded variation in S.

Moreover, if $F : [0, T] \to P_{cp}(\tilde{B}(0; R)))$ then, by definition, the variation of F on the subinterval $[t - q, t]$, where $q > 0$, denoted by $V_{t-q}^{t}(F)$ is defined as follows: let R be a partition of $[t - q, t]$ (i.e. $t - q = t_0, t_1 < ... < t_{k+1} = t$) and let $\mathcal{R}$ be the set of all such partitions. Then $V_{t-q}^{t}(F, R) := \sum_{p=1}^{k} H(F(t_{p+1}), F(t_p))$ and $V_{t-q}^{t}(F) := sup\{V_{t-q}^{t}(F, R) | R \in \mathcal{R}\}$.

Theorem 2.1.14. (Hermes [98], [99]) *Let $T > 0$ and $F : [0, T] \to P_{cp}(\tilde{B}(0; R)))$. Then:*

i) If F is H-c and has bounded variation in $[0, T]$, then F admits a continuous selection.

ii) If F is a-Lipschitz, then there exists an a-Lipschitz selection of F.

Proof. For each positive integer k, consider the points $0, \frac{T}{k}, \frac{2T}{k}, ..., T$. Choose $x_0^k \in F(0), x_1^q \in F(\frac{T}{k})$ such that $|x_0^k - x_1^k| = D(x_0^k, F(\frac{T}{k}))$ and then inductively $x_j^k \in F(\frac{jT}{k})$ such that $|x_{j-1}^k - x_j^k| = D(x_{j-1}^k, F(\frac{jT}{k}))$. Define $f^k : [0, T] \to \mathcal{R}$ be the polygonal arc joining the points $x_j^k, j \in \{0, 1, .., k\}$. Then:

i) For each $t \in [0, T]$ and each k there exists an integer $j = j(k)$ such that $|t - \frac{jT}{k}|, \frac{T}{k}$. We can assume, without any loss of generality, that $t \in [\frac{(j-1)T}{k}, \frac{jT}{k}]$. Then $D(f^k(t), F(t)) \le |f^k(t) - f^k(\frac{jT}{k})| + D(f^k(\frac{jT}{k}), F(t)) \le H(F(\frac{(j-1)T}{k}), F(\frac{jT}{k})) + H(F(\frac{jT}{k}), F(t))$.

ii) For each t and s from $[0, T]$ and each k, let j, l be integers such that: $|t - \frac{jT}{k}| < \frac{T}{k}$ and $|s - \frac{lT}{k}| < \frac{T}{k}$. We have: $|f^k(t) - f^k(s)| \le$

$$|f^k(t) - f^k(\tfrac{jT}{k})| + \sum_{r=j}^{l-1} |f^k(\frac{(r+1)T}{k}) - f^k(\frac{rT}{k})| + |f^k(\frac{lT}{k}) - f^k(s)| \leq$$

$$H(F(t), F(\frac{jT}{k})| + \sum_{r=j}^{l-1} H(F(\frac{(r+1)T}{k}), F(\frac{rT}{k})) + H(F(s), F(\frac{lT}{k})).$$

Now, we are able to prove a). Let us first remark that the sequence $(f^k)_{k\in bbN^*}$ is equicontinuous. Indeed, for any $\varepsilon > 0$ choose k^* sufficiently large such that if $k \leq k^*$ and $|t_1 - t_2|, \frac{T}{k^*}$ we have $H(t_1), F(t_2)) < \frac{\varepsilon}{3}$. Next, since F is of bounded variation, we obtain that $V_0^t(F)$ is continuous as a function of t on $[0, T]$ and hence uniformly continuous. We can choose $\delta > 0$ such that $V_a^b(F) < \frac{\varepsilon}{3}$, for $|a - b| < \delta$. Since $|\frac{jT}{k} - \frac{lT}{k}| \leq |t - s| + \frac{2T}{k}$ if $k.\frac{4T}{\delta}$ and $|t - s| < \frac{\delta}{2}$, we obtain $V_{\frac{jT}{k}}^{\frac{lT}{k}} < \frac{\varepsilon}{3}$. Then, from ii) we have for $k \geq max(\frac{4T}{j}, k^*)$ and $|t - s| < \delta$ that $|f^k(t) - f^k(s)| < \varepsilon$ and equicontinuity is shown. The sequence (f^k) being bounded, it has an uniformly convergent subsequence converging to $f \in C[0, T]$. let $t \in [0, T]$ and $j(k)$ be an integer such that $|t - \frac{j(k)T}{k}| < \frac{T}{k}$. Using i) and the fact that the images $F(t)$ are closed, we obtain by taking $k \to +\infty$ $f(t) \in F(t)$.

For b), let us assume in ii) that $t < \frac{jT}{k} < ... < \frac{lT}{k} < s$. From the Lipschitz condition, relation ii) becomes: $|f^k(t) - f^k(s)| \leq a[(\frac{jT}{k} - t) + \sum_{p=j}^{l-1}(\frac{(p+1)T}{k} - \frac{pT}{k}) + (s - \frac{lT}{k})] = a|s - t|$. Thus $(f^k)_{k\in bbN^*}$ is equicontinuous, bounded and has a subsequence converging uniformly to $f \in C[0, T]$ and $|f(t) - f(s)| \leq a|t - s|$. From i) we conclude again that $f(t) \in F(t)$, for each $t \in [0, T]$. $\square$

For more general spaces, the Steiner point approach generate a Lipschitz selection as follows:

Theorem 2.1.15. *Let X be a metric space and $F : X \to P_{cp,cv}(\mathbb{R}^n)$ be a-Lipschitz. Then F admits a b-Lipschitz selection with $b = ak(n)$ and $k(n) = \frac{n!!}{(n-1)!!}$ if n is odd and $k(n) = \frac{n!!}{\pi(n-1)!!}$ if n is even.*

Finally, let us remark that the problem of existence of a Lipschitz selection for a Lipschitz multi-function was settled by Yost (see for example Hu-Papageorgiou [105]) as follows:

Theorem 2.1.16. (Yost) *Let X be a metric space and Y be a Banach space. Then every a-Lipschitz multi-function $F : X \to P_{b,cl,cv}(Y)$ admits a Lipschitz selection if and only if Y is finite dimensional.*

A extension of the concept of selection is given by Deguire-Lassonde as follows:

Definition 2.1.17. Let X be a topological space and $(Y_i)_{i\in I}$ an arbitrary family of topological spaces. The family of continuous functions $\{f_i : X \to Y_i\}_{i\in I}$ is called a selecting family for the family $\{F_i : X \to \mathcal{P}(Y_i)\}_{i\in I}$ of multi-functions if for each $x \in X$ there exists $i \in I$ such that $f_i(x) \in F_i(x)$.

One easily observe that the notion of selecting family reduces to the concept of continuous selection when I has only one element.

Definition 2.1.18. Let X be a topological space, $(E_i)_{i\in I}$ be an arbitrary family of Hausdorff topological vector spaces and $Y_i \in P_{cv}(E_i)$, for all $i \in I$. Then the family $\{F_i : X \to \mathcal{P}(Y_i)\}_{i\in I}$ of multi-functions is said to be a Ky Fan family if the following are verified:

i) $F_i(x)$ is convex for each $x \in X$ and each $i \in I$.

ii) $F^{-1}(y_i)$ is open for each $y_i \in Y_i$ and each $i \in I$.

iii) for each $x \in X$ there exists $i \in I$ such that $F_i(x) \neq \emptyset$.

In this setting, an important result is:

Theorem 2.1.19. (Deguire-Lassonde [79]) *Let X be a paracompact space, $(E_i)_{i\in I}$ be an arbitrary family of Hausdorff topological vector spaces and $Y_i \in P_{cv}(E_i)$, for all $i \in I$. Then any Ky Fan family of multi-valued operators $\{F_i : X \to \mathcal{P}(Y_i)\}_{i\in I}$ admits a selecting family $\{f_i : X \to Y_i\}_{i\in I}$.*

Proof. From the definition of the Ky Fan family of multi-functions, we have that the system $(DomF_i(x))_{i\in I}$ is an open covering of X. Using the paracompactness of the space X it follows the existence of a closed refinement $(U_i)_{i\in I}$ such that $U_i \subset Dom(F_i)$, for each $i \in I$. Let us define, for each $i \in I$ the multi-valued operator $G_i : X \to Y_i$,by the relation:

$$G_i(x) = \begin{cases} F_i(x), & \text{if } x \in U_i \\ Y_i, & \text{if } x \notin Y_i \end{cases}$$

Then, for each $i \in I$, G_i has nonempty and closed values and the sets $F_i^{-1}(y)$ are open for each $y \in Y_i$. From Browder selection theorem, we have the existence of a continuous selection $f_i : X \to Y_i$ of F_i, for each $i \in I$. Because for each $x \in X$ there exists $i \in I$ such that $x \in U_i$ implies $f_i(x) \in G_i(x) = F_i(x)$, we obtain that $\{f_i : X \to Y_i | i \in I\}$ is a selecting family for $\{F_i : X \to \mathcal{P}(Y_i)\}_{i\in I}$. The proof is complete. □

Using a similar argument (via Michael' selection theorem), we have:

Theorem 2.1.20. (Deguire-Lassonde [79]) *Let X be a paracompact space, $(E_i)_{i\in I}$ be an arbitrary family of Hausdorff topological vector spaces and $Y_i \in P_{cv}(E_i)$, for all $i \in I$. Then any family of l.s.c. multi-valued operators $\{F_i : X \to \mathcal{P}(Y_i)\}_{i\in I}$ having the property that for each $x \in X$ there is $i \in I$ with $F_i(x) \neq \emptyset$ admits a selecting family $\{f_i : X \to Y_i\}_{i\in I}$.*

Bibliographical comments. Basic continuous selections theorems can be found in many books on multi-valued analysis such as: Aubin [14], Aubin-Cellina [15], Aubin-Frankowska [16], Border [36], Deimling [80], Gorniewicz [92], Hu-Papageorgiou [105], Kamenskii-Obuhovskii-Zecca [118], Kisielewicz [127], Repovs-Simeonov [214] Tolstonogov [257] and Yuan [270]. Theorem 2.1.12. belong to Strother [251], meanwhile results regarding the existence of Lipschitz selections for multi-functions maybe found in Hermes [98] and [99]. The notion of selecting family and the corresponding results were given by Deguire and Lassonde in [78] and [79].

2.2 Selection theorems. The decomposable case

Throughout this section $(T, \mathcal{A}, \mu)$ is a complete σ-finite non-atomic measure space and E is a Banach space. Let $L^1(T, E)$ be the Banach space of all measurable functions $u : T \to E$ which are Bochner μ-integrable. We recall that a set $K \subset L^1(T, E)$ is said to be decomposable if for all $u, v \in K$ and each $A \in \mathcal{A}$:

$$u\chi_A + v\chi_{T\setminus A} \in K,$$

where χ_A stands for the characteristic function of the set A.

This notion is, somehow, similar to convexity, but there exist also major differences. However, in several cases the decomposability condition is a good substitute for convexity. The purpose of this section is to present some results in the field of multi-valued analysis related to this topic: convexity replaced by decomposability.

A decomposable set has been considered for the first time in the field of multi-valued analysis by Antosiewicz and Cellina [12] in connection with the problem of the existence of a continuous selection for a continuous multifunction with not necessarily convex values.

There are several results in the analysis of multi-valued operators where in the assumptions, convexity can be replaced by decomposability. Some of these theorems will be considered in what follows.

First theorem is a "decomposable" version of the Michael's selection theorem for l.s.c. multi-functions with convex values.

Let consider, without proofs, two technical auxiliary results.

Lemma 2.2.1. *Let* $(g_n)_{n\in\mathbb{N}} \subseteq L^1(T, E)$, *with* $g_0 = 1$. *Then there exists* $S : \mathbb{R}^+ \times [0, 1] \to \mathcal{A}$, *such that for all* $\tau, \tau_1, \tau_2 \in \mathbb{R}^+$ *and all* $\lambda, \lambda_1, \lambda_2 \in [0, 1]$ *we have:*

a)$S(\tau, \lambda_1) \subseteq S(\tau, \lambda_2)$, *if* $\lambda_1 \leq \lambda_2$

b)$\mu(S(\tau_1, \lambda_1)\Delta S(\tau_2, \lambda_2)) \leq |\lambda_1 - \lambda_2| + 2|\tau_1 - \tau_2|$

(c) $\int_{S(\tau,\lambda)} g_n d\mu = \lambda \int_S g_n d_\mu$, *for all* $n \leq \tau$.

Lemma 2.2.2. *Let* X *be a separable metric space,* E *be a separable Banach space and let* $F : X \to P_{cl,dec}(L^1(T,E))$ *be a l.s.c. multi-valued operator. Then for every* $\varepsilon > 0$ *there exist* $f_\varepsilon : X \to L^1(T,E)$ *and a continuous mapping* $g_\varepsilon : X \to L^1(T,E)$ *such that the multi-function* $F_\varepsilon(x) := \{u \in F(x) | \parallel u(t) - f_\varepsilon(x)(t) \parallel < g_\varepsilon(x)(t), a.e.\}$ *has nonempty values and* $\|g_\varepsilon(x)\| < \varepsilon$ *for each* $x \in X$*.*

Now we present the decomposable version of Michael' selection theorem:

Theorem 2.2.3. (Fryszkowski [89], Bressan-Colombo [40]) *Let* (X,d) *be a separable metric space,* E *a separable Banach space and let* $F : X \to P_{cl,dec}(L^1(T,E))$ *be a l.s.c. multi-valued operator. Then* F *has a continuous selection.*

Proof. Using Lemma 2.2.2., by induction we will generate two sequences of continuous functions $(f_n : X \to L^1(T,E))$ and $(g_n : X \to L^1(T,E))$ and a sequence of l.s.c. multi-functions $F_n : X \to P_{cl}(L^1(T,E))$ such that:

i) $\|g_n(x)\| \leq 2^{-n}$

ii) $\|f_n(x)(t) - f_{n-1}(x)(t)\| \leq g_n(x) + g_{n-1}(x)\|$, a.e. and for each $n \geq 2$

iii) $F_n(x) := \{u \in F(x) | \parallel u(t) - f_n(x)(t) \parallel < g_n(x)(t), a.e.\} \neq \emptyset$, for each $x \in X$

Indeed, for the first step of the induction let us consider f_1 and g_1 be defined by Lemma 2.2.2. with F and $\varepsilon = 2^{-1}$. Suppose that f_n, g_n, F_n have been defined satisfying the conditions i)-iii). Using again Lemma 2.2.2. for $\overline{F_{n-1}(x)}$ and $\varepsilon = 2^{-n}$ we complete the induction. Because $(f_n(x))_{n\in\mathbb{N}}$ is uniformly Cauchy in $L^1(T,E)$, we have that $f_n(x)$ converges to $f(x)$, with f a continuous function from X to $L^1(T,E)$. Also since $D(f_n(x), F_n(x)) < 2^{-n}$, for each $n \geq 1$, and F has closed values we conclude that f is the desired continuous selection of F. $\square$

For the u.s.c. case we have:

Theorem 2.2.4. (Bressan-Colombo [40]) *Let (X, d) be a separable metric space and let $F : X \to \mathcal{P}_{dec}(L^1(T, E))$ be a H-u.s.c. multi-valued operator. If either X or $L^1(T, E)$ is separable, then for each $\varepsilon > 0$ there is a continuous function $f_\varepsilon : X \to L^1(T, E)$ such that $Graph\, f_\varepsilon \subseteq V(Graph\, F, \varepsilon)$ and $f_\varepsilon(X) \in \mathcal{P}_{dec}(F(X))$.*

Remark 2.2.5. As we have seen, the main tool for the decomposable case is to consider instead of convex combinations some continuous interpolations between different elements of a decomposable set. More precisely, consider an increasing family $\{A_\lambda \mid \lambda \in [0, 1]\}$ (where $A_\lambda \in \mathcal{A}$ with $\mu(A_\lambda) = \lambda\mu(T)$, for every $\lambda \in [0, 1]$) and let $u_1, ..., u_n$ be elements of a decomposable set $K \subset L^1(T, E)$. Let λ_i be p nonnegative numbers such that $\sum\limits_{i=1}^{p} \lambda_i = 1$. Setting $\eta_0 = 0$ and $\eta_i = \sum\limits_{j=1}^{i} \lambda_i$, $(i \in \{1, 2, ..., p\}$ then the decomposable combination $\sum\limits_{i=1}^{p} u_i \cdot \chi_{A_{\eta_i} \setminus A_{\eta_{i-1}}}$ lies inside K. For the compact case the construction below is given by Fryszkowski in [89] and the extension for the paracompact case appear in Bressan-Colombo in [40]. (They consider continuous combinations of an infinite family of functions, taking advantage of the fact that at any given time only a finite number of u_i enter in a decomposable combination).

Let us prove now an auxiliary result, concerning the existence of continuous selections for a locally selectionable multi-function with decomposable values.

Lemma 2.2.6. *Let (X, d) be a separable metric space, $(T, \mathcal{A}, \mu)$ be a complete σ-finite and non-atomic measure space and E be a Banach space. Let $F : X \to P_{dec}(L^1(T, E))$ be a locally selectionable multi-valued operator. Then F has a continuous selection.*

Proof. We associate to any $y \in X$ and $z \in F(y)$ an open neighborhood $N(y)$ and a local continuous selection $f_y : N(y) \to L^1(T, E)$, satisfying $f_y(y) = z$ and $f_y(x) \in F(x)$ when $x \in N(y)$. We denote by $\{V_n\}_{n\in\mathbb{N}^*}$ a countable locally finite open refinement of the open covering $\{N(y)|\ y \in X\}$ and by $\{\psi_n\}_{n\in\mathbb{N}^*}$ a continuous partition of unity associated to $\{V_n\}_{n\in\mathbb{N}^*}$.

Then, for each $n \in \mathbb{N}^*$ there exist $y_n \in X$ such that $V_n \subset N(y_n)$ and a continuous function $f_{y_n} : N(y_n) \to L^1(T, E)$ with $f_{y_n}(y_n) = z_n$, $f_{y_n}(x) \in F(x)$, for all $x \in N(y_n)$. We define $\lambda_0(x) = 0$ and $\lambda_n(x) = \sum\limits_{m\leq n} \psi_m(x)$, $n \in \mathbb{N}^*$. Let $g_{m,n} \in L^1(T, \mathbb{R}_+)$ be the function defined by $g_{m,n}(t) = \|z_n(t) - z_m(t)\|$, for each $m, n \geq 1$.

We arrange these functions into a sequence $\{g_k\}_{k\in\mathbb{N}^*}$.

Consider the function $\tau(x) = \sum\limits_{m,n\geq 1} \psi_m(x)\psi_n(x)$. From Lemma 2.2.1., there exists a family $\{T(\tau, \lambda)\}$ of measurable subsets of T such that:

(a) $T(\tau, \lambda_1) \subseteq T(\tau, \lambda_2)$, if $\lambda_1 \leq \lambda_2$

(b) $\mu(T(\tau_1, \lambda_1)\Delta T(\tau_2, \lambda_2)) \leq |\lambda_1 - \lambda_2| + 2|\tau_1 - \tau_2|$

(c) $\int_{T(\tau,\lambda)} g_n d\mu = \lambda \int_T g_n d_\mu$, $\forall\ n \leq \tau_0$ for all $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, and all $\tau_0, \tau_1, \tau_2 \geq 0$.

Define $f_n(x) = f_{y_n}(x)$ and $\chi_n(x) = \chi_{T(\tau(x),\lambda_n(x))\setminus T(\tau(x),\lambda_{n-1}(x))}$ for each $n \in \mathbb{N}^*$.

Let us consider the single-valued operator $f : X \to L^1(T, E)$, defined by $f(x) = \sum_{n\geq 1} f_n(x)\chi_n(x)$, $x \in X$.Then, f is continuous because the functions τ and λ_n are continuous, the characteristic function of the set $T(\tau, \lambda)$ varies continuously in $L^1(T, E)$ with respect to the parameters τ and λ and because the summation defining f is locally finite. On the other hand, from the properties of the sets $T(\tau, \lambda)$ (see Remark 2.2.5. and Hu-Papageorgiou [105], 241-244pp. for more details) and because F has decomposable values, it follows that f is a selection of F. $\square$

The following result is similar to Corollary 2.1.4.(ii).

Theorem 2.2.7. *Let (X, d) be a separable metric space, E a separable Banach space, $F : X \to P_{cl,dec}(L^1(T, E))$ be a l.s.c. multi-valued operator* *and $G : X \to P_{dec}(L^1(T,E))$ be with open graph. If $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$ then there exists a continuous selection of $F \cap G$.*

Proof. Let $x_0 \in X$ and for each $y_0 \in F(x_0)$ we define the multifunction

$$F_0(x) = \begin{cases} \{y_0\}, & \text{if } x = x_0 \\ F(x), & \text{if } x \neq x_0. \end{cases}$$

Obviously $F_0 : X \to \mathcal{P}_{cl,dec}(L^1(T,E))$ is l.s.c. From Theorem 2.2.3. there exists a continuous selection f of F_0, i.e. $f_0(x_0) = y_0$ and $f_0(x) \in F(x)$, for each $x \in X$, $x \neq x_0$. Using Lemma 2.1.10. it follows that $F \cap G$ is locally selectionable at x_0 and has decomposable values. From Lemma 2.2.6. the conclusion follows. □

An important result is the following Browder-type selection theorem:

Theorem 2.2.8. *Let E be a Banach space such that $L^1(T,E)$ is separable. Let K be a nonempty, paracompact, decomposable subset of $L^1(T,E)$ and let $F : K \to P_{dec}(K)$ be a multi-valued operator with open fibres. Then F has a continuous selection.*

Proof. For each $y \in K$, $F^{-1}(y)$ is an open subset of K. Since K is paracompact it follows that the open covering $\{F^{-1}(y)\}_{y\in K}$ admits a locally finite open refinement, let say $K = \bigcup\limits_{j\in J} F^{-1}(y_j)$, with $y_j \in K$. Let $\{\psi_j\}_{j\in J}$ be a continuous partition of unity subordinate to $\{F^{-1}(y_j)\}_{j\in J}$. Using the same construction as in the proof of Lemma 2.2.6., one can construct a continuous function $f : K \to K$, $f(x) = \sum\limits_{j\in J} f_j(x)\chi_j(x)$, where $f_j(x) \in F(x)$ for each $x \in K$. This function is a continuous selection for F. □

The following results are decomposable versions of Deguire-Lassonde theorems. (see Theorem 2.1.19 and Theorem 2.1.20.)

Theorem 2.2.9. *Let E be a Banach space such that $L^1(T,E)$ is separable. Let I be an arbitrary set of indices, $\{K_i | i \in I\}$ be a family* *of nonempty, decomposable subsets of* $L^1(T, E)$ *and* X *a paracompact space. Let us suppose that the family* $\{F_i : X \to \mathcal{P}_{dec}(K_i) | i \in I\}$ *is of Ky Fan-type. Then there exists a selecting family for* $\{F_i\}_{i \in I}$*.*

Proof. Let $\{U_i\}_{i \in I}$ be the open covering of the paracompact space X given by $U_i = \{x \in X |\ F_i(x) \neq \emptyset\}$ for each $i \in I$. It follows that there exists a locally finite open cover $\{W_i\}_{i \in I}$ such that $\overline{W_i} \subset U_i$ for $i \in I$. Let $V_i = \overline{W_i}$. For each $i \in I$ let us consider the multi-valued operator $G_i : X \to \mathcal{P}(K_i)$, defined by the relation

$$G_i(x) = \begin{cases} F_i(x), & \text{if } x \in V_i \\ K_i, & \text{if } x \notin V_i. \end{cases}$$

Then G_i is a multifunction with nonempty and decomposable values having open fibres (indeed, $G_i^{-1}(y) = F_i^{-1}(y) \cup (X \backslash V_i))$, for each $i \in I$.

Using Theorem 2.2.8. we have that there exists $f_i : X \to K_i$ continuous selection for G_i $(i \in I)$, for each $i \in I$. It follows that for each $x \in X$ there exists $i \in I$ such that $x \in V_i$ and hence $f_i(x) \in G_i(x) = F_i(x)$, proving that $\{f_i\}_{i \in I}$ is a selecting family for $\{F_i\}_{i \in I}$. $\square$

Using a similar argument we have:

Theorem 2.2.10. *Let* E *be a separable Banach space and* X *a separable metric space. Let* I *be an arbitrary set of indices,* $\{K_i | i \in I\}$ *be a family of nonempty, closed, decomposable subsets of* $L^1(T, E)$*. Let* $\{F_i : X \to P_{cl,dec}(K_i) | i \in I\}$ *be a family of l.s.c. multi-valued operators such that for each* $x \in X$ *there is* $i \in I$ *such that* $F_i(x) \neq \emptyset$*. Then* $\{F_i\}_{i \in I}$ *has a selecting family.*

Proof. There are only minor modifications of the above arguments. More precisely, the proof runs exactly as in the previous theorem, but instead of using Theorem 2.2.8., the conclusion follows from Theorem 2.2.3. $\square$

Bibliographical comments. Mainly, this section is based on the works of Fryszkowski [89], Bressan-Colombo [40], Petruşel-Muntean [201] and Petruşel-Moţ [198]. Further results can be found in Aubin-Cellina [15], Bressan-Colombo-Fryszkowski [41], Browder [44], Cellina-Colombo [55], Deguire [78], Deguire-Lassonde [79], Fryskowski [90], Hiai-Umegaki [100], Hu-Papageorgiou [105], Kisielewicz [127], Marano [138], Olech [158], Petruşel [187].

2.3 Basic fixed point theorems

The aim of this section is to report some basic theorems of the fixed point theory for multi-functions.

Let us recall first some basic notations and concepts.

Definition 2.3.1. Let X be a metric space. If $F : X \to P(X)$ is a multi-valued operator and $x_0 \in X$ is an arbitrary point, then the sequence $(x_n)_{n\in\mathbb{N}}$ is, by definition, the successive approximations sequence of F starting from x_0 if and only if $x_k \in F(x_{k-1})$, for all $k \in \mathbb{N}^*$. Let us remark that in the theory of dynamical systems, the sequence of successive approximations is called the motion of the system F at x_0 or a dynamic process of F starting at x_0. The set $T(x_0) := \{x_n : x_{n+1} \in F(x_n), n \in \mathbb{N}\}$ is called the trajectory of this motion and the space X is the phase space.

Definition 2.3.2. Let (X, d) be a generalized metric space and let $F : X \to P_{cl}(X)$ be a multi-valued operator. Then F is said to be:

i) a-contraction if and only if $a \in [0, 1[$ and $H(F(x_1), F(x_2)) \leq ad(x_1, x_2),$ for all $x_1, x_2 \in X$ with $d(x_1, x_2) < \infty$.

ii) (ε, a)-contraction if and only if $\varepsilon > 0,\ a \in [0, 1[$ and $H(F(x_1), F(x_2)) \leq ad(x_1, x_2),$ for all $x_1, x_2 \in X$ with $d(x_1, x_2) < \varepsilon$.

Remark 2.3.3. Obviously, each multi-valued a-contraction is an (ε, a)-contraction.

Theorem 2.3.4. (Covitz-Nadler [71]) *Let (X, d) be a generalized com-*

plete metric space. Let $x_0 \in X$ arbitrary and $F : X \to P_{cl}(X)$ be a multi-valued (ε, a)-contraction. Then the following alternative holds:

(1) for each sequence of successive approximations of F starting from x_0 we have $d(x_{i-1}, x_i) \geq \varepsilon$, for all $i \in \mathbb{N}^*$

or

(2) there exists a sequence of successive approximations of F starting from x_0 which converges to a fixed point of F.

Corollary 2.3.5. *Let (X, d) be a generalized complete metric space and $x_0 \in X$ be arbitrary. If $F : X \to P_{cl}(X)$ is a multi-valued a-contraction, then the following alternative holds:*

(1) for each sequence of successive approximations of F starting from x_0 we have $d(x_{i-1}, x_i) = \infty$, for all $i \in \mathbb{N}^*$

or

(2) there exists a sequence of successive approximations of F starting from x_0 which converges to a fixed point of F.

The following result is known in the literature as Covitz-Nadler theorem (see [71]):

Theorem 2.3.6. (Covitz-Nadler [71]) *Let (X, d) be a complete metric space and $x_0 \in X$ be arbitrary. If $F : X \to P_{cl}(X)$ is a multi-valued a-contraction, then there exists a sequence of successive approximations of F starting from x_0 which converges to a fixed point of F.*

As regards to the strict fixed points set of a multi-valued a-contraction, we have the following result of I. A. Rus ([223]):

Theorem 2.3.7. (Rus [223]) *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multi-valued a-contraction. If $SFixF \neq \emptyset$ then $FixF = SFixF = \{x^*\}$.*

Definition 2.3.8. Let (X, d) be a metric space and $F : X \to P_{cl}(X)$ be a multi-valued operator. If there exists $a, b, c \in \mathbb{R}_+$, with $a + b + c < 1$

such that for all $x_1, x_2 \in X$ we have:

$$H(F(x_1), F(x_2)) \leq ad(x_1, x_2) + bD(x_1, F(x_1)) + cD(x_2, F(x_2))$$

then F is called a Reich type multi-valued operator.

Reich's fixed point theorem (see [213]) is an extension of the Covitz-Nadler principle:

Theorem 2.3.9. (Reich [213]) *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a Reich type multi-valued operator. Then $FixF \neq \emptyset$.*

If the multi-valued operator is contractive and the space is compact, then we have the following result:

Theorem 2.3.10. (Smithson [249]) *Let (X, d) be a compact metric space and $F : X \to P_{cl}(X)$ be a contractive multi-valued operator. Then $FixF \neq \emptyset$.*

Another generalization of the Covitz-Nadler principle is:

Theorem 2.3.11. (Mizoguchi-Takahashi (see [147]) *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ a multi-function such that $H(F(x), F(y)) \leq k(d(x, y))d(x, y)$, for each $x, y \in X$ with $x \neq y$, where $k :]0, \infty[\to [0, 1[$ satisfies $lim_{r \to t^+} k(r) < 1$, for every $t \in [0, \infty[$. Then $FixF \neq \emptyset$.*

For the case of multi-functions from a closed ball of a metric space X into X, Frigon and Granas (see [88]) proved the following extension of Covitz-Nadler principle:

Theorem 2.3.12. (Frigon and Granas [88]) *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $F : \widetilde{B}(x_0; r) \to P_{cl}(X)$ be an a-contraction such that $D(x_0, F(x_0)) < (1 - a)r$. Then $FixF \neq \emptyset$.*

Using the above theorem, Frigon and Granas have proved some continuation results for multi-functions on complete metric spaces.

Definition 2.3.13 If X, Y are metric spaces and $F_t : X \to P_{cl}(Y)$ is a family of multi-functions depending on a parameter $t \in [0,1]$ then, by definition, $(F_t)_{t\in[0,1]}$ is said to be a family of k-contractions if:

i) F_t is a k-contraction, for each $t \in [0,1]$.

ii) $H(F_t(x), F_s(x)) \leq |\phi(t) - \phi(s)|$, for each $t, s \in [0,1]$ and each $x \in X$, where $\phi : [0,1] \to \mathbb{R}$ is a continuous and strictly increasing function.

If (X, d) is a complete metric space and U is an open connected subset of X, then we will denote by $K(\overline{U}, X)$ the set of all k-contractions $F : \overline{U} \to P_{cl}(X)$. Also, denote by $\mathcal{K}_0(\overline{U}, X) = \{F \in \mathcal{K}(\overline{U}, X) | x \notin F(x), \text{ for each } x \in \partial U\}$.

Definition 2.3.14. $F \in \mathcal{K}_0(\overline{U}, X)$ is called essential if and only if $FixF \neq \emptyset$. Otherwise F is said to be inessential.

Definition 2.3.15. A family of k-contractions $(F_t)_{t\in[0,1]}$ is called a homotopy of contractions if and only if $F_t \in \mathcal{K}_0(\overline{U}, X)$, for each $t \in [0,1]$. The multi-functions S and T are said to be homotopic if there exists a homotopy of contractions $(F_t)_{t\in[0,1]}$ such that $F_0 = S$ and $F_1 = T$.

The topological transversality theorem read as follows:

Theorem 2.3.16. (Frigon-Granas [88]) *Let $S, T \in \mathcal{K}_0(\overline{U}, X)$ two homotopic multi-functions. Then S is essential if and only if T is essential.*

The non-linear alternative for multi-valued contractions was proved by Frigon and Granas:

Theorem 2.3.17. (Frigon-Granas [88]) *Let X be a Banach space and $U \in P_{op}(X)$ such that $0 \in U$. If $T : \overline{U} \to P_{cl}(X)$ is a multi-valued k-contraction such that $T(\overline{U})$ is bounded, then either:*

i) there exists $x \in \overline{U}$ such that $x \in T(x)$.

or

ii) there exists $y \in \partial U$ and $\lambda \in]0,1[$ such that $y \in \lambda T(y)$.

Let us present now the Leray-Schauder principle for multi-valued contractions:

Theorem 2.3.18. (Frigon-Granas [88]) *Let X be a Banach space and $T : X \to P_{cl}(X)$ such that for each $r > 0$ the multi-function $T|_{\widetilde{B}(0,r)}$ is a k-contraction. Denote by $\mathcal{E}_T := \{x \in X | x \in \lambda T(x),$ for some $\lambda \in]0,1[\}$. Then at least one of the following assertions hold:*

i) $\mathcal{E}_T$ is unbounded

ii) $FixT \neq \emptyset$.

Corollary 2.3.19. *Let X be a Banach space and $T : \overline{U} \to P_{cl}(X)$ be a k-contractions such that for each $x \in \partial U$ at least one of the following assertions hold:*

i) $\|T(x)\| \leq \|x\|$

ii) $\|T(x)\| \leq D(x, T(x))$

iii) $\|T(x)\| \leq (D(x, T(x))^2 + \|x\|^2)^{\frac{1}{2}}$

iv) $\|T(x)\| \leq max(\|x\|, D(x, T(x)))$

Then $FixT \neq \emptyset$

In case F is a nonexpansive (i.e. 1-Lipschitz) multi-function, we have:

Theorem 2.3.20. (Lim [135]) *Let X be an uniformly convex Banach space $Y \in P_{b,cl,cv}(X)$ and $F : Y \to P_{cp}(Y)$ be nonexpansive. Then $FixF \neq \emptyset$.*

Definition 2.3.21. Let X be a real Banach space, $Y \in P_{cl}(X)$ and $x \in Y$. We let:

$$T_Y(x) = \left\{ y \in X | \lim_{h \to 0_+} \inf D(x + hy, Y) h^{-1} = 0 \right\}$$

$$\tilde{I}_Y(x) := x + T_Y(x)$$

$$I_Y(x) = \{x + \lambda(y - x) | \ \lambda \geq 0, \ y \in Y\}, \quad \text{for} \quad Y \in P_{cl,cv}(X).$$

The set $I_Y(x)$ is called the inward set at x. Notice that $\tilde{I}_Y(x) = I_Y(x)$ for convex subset Y of X.

Definition 2.3.22. Let X be a real Banach space, $Y \in P_{cl}(X)$ and the mappings $f : Y \to X$ and $F : Y \to P(X)$. Then:

i) f is called weakly inward if $f(x) \in \tilde{I}_Y(x)$, for each $x \in Y$

ii) F is called weakly inward if $F(x) \subset \tilde{I}_Y(x)$, for each $x \in Y$

iii) F is called inward if $F(x) \cap \tilde{I}_Y(x) \neq \emptyset$, for each $x \in Y$

For weakly inward multi-valued contractions we have the following recent result of T. -C. Lim ([134]):

Theorem 2.3.23. (Lim [134]) *Let X be a Banach space and Y be a nonempty closed subset of X. Assume that $F : Y \to P_{cl}(X)$ is a weakly inward multi-valued contraction. Then F has a fixed point in Y.*

Let us consider now some basic topological fixed point principles.

For the beginning, we define the notion of Kakutani-type multi-function:

Definition 2.3.24. Let X, Y be two vector topological spaces. Then $F : X \to P(Y)$ is said to be a Kakutani-type multi-function if and only if:

i) $F(x) \in P_{cp,cv}(Y)$, for all $x \in X$

ii) F is u.s.c. on X.

Definition 2.3.25. Let X be a vector topological space and $Y \in P(X)$. Then, by definition, Y has the Kakutani fixed point property (briefly K.f.p.p.) if and only if each Kakutani-type multi-function $F : Y \to P(Y)$ has at least a fixed point in Y.

The most famous topological fixed point result is the Kakutani-Fan theorem (see [117]):

Theorem 2.3.26. (Kakutani-Fan [117]) *Any compact convex subset K of a Banach space has the K.f.p.p.*

For the infinite dimensional case we also have the following result (see for example Kirk-Sims [124]) of Bohnenblust-Karlin:

Theorem 2.3.27. (Bohnenblust-Karlin) *Let X be a Banach space and $Y \in P_{b,cl,cv}(X)$. The any u.s.c. multi-function $F : Y \to P_{cl,cv}(Y)$ with relatively compact range has at least a fixed point in Y.*

As consequence of the Kakutani-Fan result, Browder and Fan proved:

Theorem 2.3.28. (Browder-Fan [44]) *Let X be a Hausdorff vector topological space and K be a nonempty compact and convex subset of X. Let $F : K \to P_{cv}(K)$ be a multi-valued operator with open fibres. Then $FixF \neq \emptyset$.*

Another generalization of the Kakutani-Fan fixed point principle has been proved by Himmelberg as follows:

Theorem 2.3.29. (Himmelberg [103]) *Let X be a convex subset of a locally convex Hausdorff topological vector space and Y be a nonempty compact subset of X. Let $F : X \to P_{cl,cv}(Y)$ be an u.s.c. multi-function. Then there exists a point $\overline{x} \in Y$ such that $\overline{x} \in F(\overline{x})$.*

Recently, X. Wu (see [264]) proved a fixed point theorem for lower semi-continuous multivalued operators in locally convex Hausdorff topological vector spaces. This theorem is the lower semi-continuous version of Himmelberg's fixed point theorem.

Theorem 2.3.30. (Wu [264]) *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space, Y a nonempty compact metrizable subset of X and $F : X \to P_{cl,cv}(Y)$ a l.s.c. multi-function. Then the exists a point $\overline{x} \in Y$ such that $\overline{x} \in F(\overline{x})$.*

We recall now the definitions of Kuratowski and Hausdorff noncompactness measures α_K, respectively α_H:

Definition 2.3.31. Let X be a metric space and S a bounded subset of X. We set:

$\alpha_K(S) := \inf\{\varepsilon > 0|$ there exists $m \in \mathbb{N}^*$ such that $S = \bigcup_{i\leq m} S_i,\ S_i \in P(X),\ diam(S_i) \leq \varepsilon\}$.

Definition 2.3.32. Let X be a metric space and S a bounded subset of X. Then
$\alpha_H(S) := \inf\{\varepsilon > 0 |$ there exist $m \in \mathbb{N}^*$ and $x_i \in X$ such that $S = \bigcup_{i \leq m} B(x_i, \varepsilon)\}$.

Definition 2.3.33. Let (X, d) be a metric space. A multi-valued operator $F : X \to P_{cl}(X)$ is called:

i) γ-condensing if and only if $\gamma(F(A)) < \gamma(A))$, for each $A \in P_b(X)$, with $\gamma(A) > 0$.

ii) (γ, a)-contraction if and only if $a \in [0, 1[$ and $\gamma(F(A)) \leq a\gamma(A)$, for each $A \in P_b(X)$.

(where γ is α_K or α_H. Moreover, γ could be an abstract measure of noncompactness, see for example Ayerbe Toledano, Dominguez Benavides, López Acedo [23]).

The following results can be found, for example, in Deimling [80] and [81].

Theorem 2.3.34. *Let X be a Banach space and $Y \in P_{b,cl,cv}(X)$. Let $F : Y \to P_{cl,cv}(X)$ be u.s.c. , γ-condensing and inward. Then $FixF \neq \emptyset$.*

As a consequence of the degree theory for multi-functions one can prove:

Theorem 2.3.35. *Let X be a Banach space, $Y \in P_b(X)$ and $F : \overline{Y} \to P_{cl,cv}(X)$ an u.s.c. and (γ, a)-contraction multi-function. Suppose that one of the following conditions holds:*

i) Y is open and there exists $x_0 \in Y$ such that $x_0 + \lambda(x - x_0) \notin F(x)$, for each $x \in \partial Y$ and each $\lambda > 1$

ii) Y is closed, convex and $F(Y) \subset Y$

Then $FixF \neq \emptyset$.

Finally, let us consider some fixed point principles for single-valued operators.

Definition 2.3.36. Let (X, d) be a metric space and $f : X \to X$ a single-valued operator. Then:

i) f is a Meir-Keeler type operator if and only if for each $\eta > 0$ there exists $\delta > 0$ such that $x, y \in X$, $\eta \le d(x, y) < \eta + \delta \;\Rightarrow\; d(f(x), f(y)) < \eta$.

ii) f is a ε-locally Meir-Keeler type operator (where $\varepsilon > 0$) if and only if for each $0 < \eta < \varepsilon$ there is $\delta > 0$ such that $x, y \in X$, $\eta \le d(x, y) < \eta + \delta \;\Rightarrow\; d(f(x), f(y)) < \eta$.

Let us remark that each Meir-Keeler type operator is contractive, i.e. $d(f(x), f(y)) < d(x, y)$, for each $x, y \in X$ with $x \neq y$.

Theorem 2.3.37. (Meir-Keeler [144]) *Let (X, d) be a complete metric space and f a mapping from X into itself. If f is a Meir-Keeler-type operator then f has a unique fixed point, i.e. $Fixf = \{x^*\}$. Moreover for any $x \in X$, $\lim\limits_{n\to\infty} f^n(x) = x^*$.*

Theorem 2.3.38. (Xu [267]) *Let (X, d) be a complete ε-chainable metric space and $f : X \to X$ be a ε-locally Meir-Keeler type operator. Then f has at least a fixed point in X.*

For single-valued operators satisfying to a Boyd-Wong type condition we have:

Theorem 2.3.39. (Boyd-Wong [39] and H. K. Xu [267]) *Let (X, d) be a complete metric space, $\varepsilon > 0$ and $f : X \to X$ be a single-valued operator such that:*

$d(f(x), f(y)) \le k(d(x, y))d(x, y)$, for all $x, y \in X$ with $0 < d(x, y) < \varepsilon$, where $k :]0, \infty[\to]0, 1[$ is a real function with the property:

$$(P)\quad \begin{cases} \text{For each } 0 < t < \varepsilon \text{ there exist } e(t) > 0 \text{ and } s(t) < 1 \\ \text{such that } k(r) \le s(t) \text{ provided } t \le r < t + e(t) \end{cases}$$

Then $Fixf \neq \emptyset$.

Combining a metrical fixed point result (namely, the Banach contraction principle) with a topological ones (Schauder's theorem), Krasnoselskii (see for example [128])proved the following fixed point principle for the sum of two single-valued operators:

Theorem 2.3.40. (Krasnoselskii [128]) *Let X be a Banach space, $Y \in P_{cl,cv}(X)$ and consider $f : Y \to X$, $g : Y \to X$ two single-valued operators. If the following conditions are satisfied:*

i) $f(y) + g(y) \in Y$, for each $y \in Y$

ii) f is a-contraction

iii) g is continuous and has relatively compact range

then $Fix(f + g) \neq \emptyset$.

Bibliographical comments. Basic fixed point theorems for multi-function can be found in several sources, such as: Agarwal-Meehan-O'Regan [1], Border [36], Covitz-Nadler [71], Deimling [80], [81], Frigon-Granas [88], Hu-Papageorgiou [105], M. Kamenskii-Obuhovskii-Zecca [118], Kirk-Sims [124], I. A. Rus [223], Smithson [249], X. Wu [264], Z. Wu [265], Yuan [270].

2.4 The fixed point set

The purpose of this section is to present several properties of the fixed point set for some multi-valued generalized contractions.

Throughout this section, the symbol $\mathcal{M}$ indicates the family of all metric spaces. Let $X \in \mathcal{M}$.

The following notions appear in Rus- Petruşel-Sîntămărian (see [228] and [229]).

Definition 2.4.1. Let (X, d) be a metric space and $T : X \to P(X)$ a multi-valued operator. By definition, T is a multi-valued weakly Picard (briefly MWP) operator if and only if for all $x \in X$ and all $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

i) $x_0 = x$, $x_1 = y$

ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$

iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of the multi-valued operator T.

Let us remark that a sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the conditions (i) and (ii) in the previous definition is, by definition, a sequence of successive approximations of T, starting from (x, y).

We can illustrate this notions by several examples.

Example 2.4.2. (Covitz-Nadler [71]) Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multi-valued a-contraction. Then T is a MWP operator.

Example 2.4.3. (Reich [213]) Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multi-valued Reich-type operator. Then T is a MWP operator.

Example 2.4.4. (I. A. Rus [224]) Let (X, d) be a complete metric space. A multi-valued operator $T : X \to P_{cl}(X)$ is said to be a multi-valued Rus-type graphic-contraction if Graf(T) is closed and the following condition is satisfied: there exist $\alpha, \beta \in \mathbb{R}_+$, $\alpha + \beta < 1$ such that: $H(T(x), T(y)) \le \alpha d(x, y) + \beta D(y, T(y))$, for every $x \in X$ and every $y \in T(x)$

Then T is a MWP operator.

Example 2.4.5. (Petruşel [182]) Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. The multi-valued operator T is called a Frigon-Granas type operator if $T : \tilde{B}(x_0; r) \to P_{cl}(X)$ and satisfies the following assertion:

i) there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$, $\alpha + \beta + \gamma < 1$ such that:

$$H(T(x), T(y)) \le \alpha d(x, y) + \beta D(x, T(x)) + \gamma D(y, T(y)), \text{ for all } x, y \in \tilde{B}(x_0; r)$$

If T is a Frigon-Granas type operator such that:

ii) $\delta(x_0, T(x_0)) < [1 - (\alpha + \beta + \gamma)](1 - \gamma)^{-1} r$,

then T is a MWP operator.

Let us recall that in 1985, T. -C. Lim (see [132]) proved that if T_1 and T_2 are multi-valued contractions on a complete metric space X with a same contraction constant $\alpha < 1$ and if $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$, then the data dependence phenomenon for the fixed point set holds, i.e.

$$H(FixT_1, FixT_2) \leq \eta\{1 - a\}^{-1}.$$

We will show now that the data dependence problem for the fixed point set for some generalized multi-valued contractions has also a positive answer.

Definition 2.4.6. Let (X, d) be a metric space and $T : X \to P(X)$ a MWP operator. Then we define the multi-valued operator $T^{\infty} : Graf(T) \to P(FixT)$ by the formula:

$T^{\infty}(x, y) := \{z \in FixT |$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\}$.

An important abstract concept in this approach is the following:

Definition 2.4.7. Let (X, d) be a metric space and $T : X \to P(X)$ a MWP operator. Then T is a c-multi-valued weakly Picard operator (briefly c-MWP operator) if there is a selection t^{∞} of T^{∞} such that: $d(x, t^{\infty}(x, y)) \leq cd(x, y)$, for all $(x, y) \in Graf(T)$.

Further on we shall present several examples of c-MWP operators.

Example 2.4.8. A multi-valued α-contraction on a complete metric space is a c-MWP operator with $c = (1 - \alpha)^{-1}$.

Example 2.4.9. A multi-valued Reich type operator on a complete metric space is a c-MWP operator with $c = [1 - (\alpha + \beta + \gamma)]^{-1}(1 - \gamma)$.

Example 2.4.10. A multi-valued Rus-type graphic contraction on a complete metric space is a c-MWP operator with $c = (1-\beta)[1-(\alpha+\beta)]^{-1}$.

Example 2.4.11. A multi-valued Frigon-Granas type operator $T : \tilde{B}(x_0; r) \to P_{cl}(X)$ satisfying the condition $\delta(x_0, T(x_0)) < [1 - (\alpha + \beta + \gamma)](1 - \gamma)^{-1} r$ is a c-MWP operator.

An important abstract result of is the following:

Theorem 2.4.12. *Let (X, d) be a metric space and $T_1, T_2 : X \to P(X)$. We suppose that:*

i) T_i is a c_i-MWP operator for $i \in \{1, 2\}$

ii) there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$.

Then $H(FixT_1, FixT_2) \leq \eta \max\{c_1, c_2\}$.

Proof. Let $t_i : X \to X$ be a selection of T_i for $i \in \{1, 2\}$. Let us remark that

$$H(FixF_1, FixT_2) \leq \max\left\{\sup_{x \in FixT_2} d(x, t_1^\infty(x, t_1(x))), \sup_{x \in FixT_2} d(x, t_2^\infty(x, t_2(x)))\right\}$$

Let $q > 1$. Then we can choose t_i $(i \in \{1, 2\})$ such that

$$d(x, t_1^\infty(x, t_1(x))) \leq c_1 q H(T_2(x), T_1(x)), \text{ for all } x \in FixT_2$$

and

$$d(x, t_2^\infty(x, t_2(x)) \leq c_2 q H(T_1(x), T_2(x)), \text{ for all } x \in FixT_1.$$

Thus we have $H(FixT_1, FixT_2) \leq q\eta \max\{c_1, c_2\}$. Letting $q \searrow 1$, the proof is complete. $\square$

Remark 2.4.13. As consequences of this abstract principle, we deduce that the data dependence phenomenon regarding the fixed points set for some generalized multi-valued contractions (such as Reich-type operators, Rus-type graphic contractions, Frigon-Granas type operators) holds.

Contrary to the single-valued case, if $T : X \to P_{cl}(X)$ is a multi-valued contraction on a complete metric space, then $FixT$ is not necessarily a singleton and hence it is of interest to study the topological properties of it.

Let us recall that a metric space X is called an absolute retract for metric spaces (briefly $X \in AR(\mathcal{M})$) if, for any $Y \in \mathcal{M}$ and any $Y_0 \in P_{cl}(X)$, every continuous function $f_0 : Y_0 \to X$ has a continuous extension over Y, that is $f : Y \to X$. Obviously, any absolute retract is arcwise connected. In this setting, B. Ricceri (see [215]), stated the following important theorem:

Theorem 2.4.14. (Ricceri) *Let E be a Banach space and let X be a nonempty, closed, convex subset of E. Suppose $T : X \to P_{cl,cv}(X)$ is a multi-valued contraction. Then $FixT$ is an absolute retract for metric spaces.*

A decomposable version of this result was proved by Bressan-Cellina-Fryszkowski (see [41]):

Theorem 2.4.15. (Bressan-Cellina-Fryszkowski) *Let $F : L^1(T, E) \to P_{b,cl,dec}(L^1(T, E))$ be a multi-valued a-contraction. Then $FixF$ is an absolute retract for metric spaces.*

We establish the following result on the structure of the fixed point set for a multi-valued Reich type operator with convex values.

Theorem 2.4.16. *Let E be a Banach space, $X \in P_{clc,cv}(E)$ and $T : X \to P_{cl,cv}(X)$ be a l.s.c. multi-valued Reich-type operator. Then $FixT \in AR(\mathcal{M})$.*

Proof. Let us remark first that $FixT \in P_{cl}(X)$. (see for example Reich [213]) Let K be a paracompact topological space, $A \in P_{cl}(K)$ and $\psi : A \to FixT$ a continuous mapping. Using Theorem 2 from B. Ricceri [215] (taking $G(t) = X$, for each $t \in K$) it follows the existence of a continuous function $\varphi_0 : K \to X$ such that $\varphi_0|_A = \psi$. We next consider $q \in]1, (\alpha + \beta + \gamma)^{-1}[$. We claim that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of continuous functions from K to X with the following properties:

(i) $\varphi_n|_A = \psi$

(ii) $\varphi_n(t) \in T(\varphi_{n-1}(t))$, for all $t \in K$

(iii) $\|\varphi_n(t)-\varphi_{n-1}(t)\| \leq [(\alpha+\beta+\gamma)q]^{n-1}\|\varphi_1(t)-\varphi_0(t)\|$, for all $t \in K$.

To see this, we proceed by induction on n. Clearly, for each $t \in A$ we have that $\psi(t) \in T(\varphi_0(t))$. On the other hand, the multi-function $t \mapsto T(\varphi_0(t))$ is l.s.c. on K with closed, convex values and hence using again Theorem 2 in [215] it follows that there is a continuous function $\varphi_1 : K \to X$ such that $\varphi_1|_A = \psi$ and $\varphi_1(t) \in T(\varphi_0(t))$, for all $t \in K$. Hence, the conditions (i), (ii), (iii) are true for φ_1. Suppose now we have constructed p continuous functions $\varphi_1, \varphi_2, \ldots, \varphi_p$ from K to X in such a way that (i), (ii), (iii) are true for $n \in \{1, 2, \ldots, p\}$. Using the Reich type contraction condition for T, we have

$$D(\varphi_p(A), T(\varphi_p(t))) \leq H(T(\varphi_{p-1}(t)), T(\varphi_p(t))) \leq$$

$$\leq \alpha\|\varphi_{p-1}(t) - \varphi_p(t)\| + \beta D(\varphi_{p-1}(t), T(\varphi_{p-1}(t))) + \gamma D(\varphi_p(t), T(\varphi_p(t))) \leq$$

$$\leq \alpha\|\varphi_{p-1}(t) - \varphi_p(t)\| + \beta\|\varphi_{p-1}(t) - \varphi_p(t)\| + \gamma D(\varphi_p(t), T(\varphi_p(t)))$$

so that

$$D(\varphi_p(t), T(\varphi_p(t))) \leq (\alpha + \beta)(1 - \gamma)^{-1}\|\varphi_p(t) - \varphi_{p-1}(t)\| \leq$$

$$\leq (\alpha + \beta)(1 - \gamma)^{-1}[(\alpha + \beta + \gamma)q]^{p-1}\|\varphi_1(t) - \varphi_0(t)\|$$

$$< (\alpha + \beta + \gamma)^p q^{p-1}\|\varphi_1(t) - \varphi_0(t)\| < [(\alpha + \beta + \gamma)q]^p\|\varphi_1(t) - \varphi_0(t)\|.$$

We next define, for each $t \in K$

$$Q_p(t) = \begin{cases} B(\varphi_p(t), [(\alpha + \beta + \gamma)q]^p\|\varphi_1(t) - \varphi_0(t)\|), \text{ if } \varphi_1(t) \neq \varphi_0(t) \\ \{\varphi_p(t)\}, \text{ if } \varphi_1(t) = \varphi_0(t) \end{cases}$$

Obviously $T(\varphi_p(t)) \cap Q_p(t) \neq \emptyset$, for all $t \in K$. We can apply now (taking $G(t) = F(\varphi_p(t))$, $f(t) = \varphi_p(t)$ and the mapping $g(t) = [(\alpha + \beta + \gamma)q]^p\|\varphi_1(t) - \varphi_0(t)\|$, for all $t \in K$). Proposition 3 from Ricceri [215], we obtain that the multi-function $t \mapsto \overline{T(\varphi_p(t)) \cap Q_p(t)}$ is l.s.c. on K with nonempty, closed, convex values. Because of Theorem 2 in [215], this produces a continuous function $\varphi_{p+1} : K \to X$ such that $\varphi_{p+1}|_t = \psi$ and $\varphi_{p+1}(t) \in \overline{T(\varphi_p(H)) \cap Q_p(t)}$, for all $t \in T$. Thus the existence of the sequence $\{\varphi_n\}$ is established. Consider now the open covering of K defined by the formula: $(\{t \in K| \ \|\varphi_1(t) - \varphi_0(t)\| < \lambda\})_{\lambda>0}$. Moreover, because of (iii) and the fact that X is complete, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly on each of the following set $K_\lambda = \{t \in K| \ \|\varphi_1(t) - \varphi_0(t)\| < \lambda\}$ $(\lambda > 0)$. Let $\varphi : K \to X$ be the pointwise limit of $(\varphi_n)_{n\in\mathbb{N}}$. Obviously φ is continuous and $\varphi|_A = \psi$. Moreover, a simple computation ensures that : $\varphi(t) \in T(\varphi(t))$ for all $t \in K$ and this completes the proof. $\square$

Remark 2.4.17 If $\beta = \gamma = 0$ then the previous theorem coincides with B. Ricceri's result (Theorem 2.4.14. below).

Remark 2.4.18. Of course, it is also possible to formulate version of Theorem 2.4.16. for multi-valued Rus type graphic contraction. It is an open question if such a result holds for a Frigon-Granas type multi-function.

Several paper have been devoted to some extensions and generalizations of the previous results to a larger family of multi-valued contractions defined on arbitrary complete absolute retracts. For this purpose, an important abstract notion is:

Definition 2.4.19. Let (X, d) be a metric space, $F : X \to P_{cl}(X)$ be l.s.c. and $\mathcal{U} \subset \mathcal{X}$ be an arbitrary family of metric spaces. We say that F has the selection property with respect to $\mathcal{U}$ when for any $Y \in \mathcal{U}$, any pair of continuous functions $f : Y \to X$ and $r : Y \to]0, \infty[$ such that:

$$G(y) = \overline{F(f(y)) \cap B(f(y), r(y))} \neq \emptyset, \text{ for each } y \in Y$$

and any nonempty closed set $Z \subset Y$, every continuous selection g_0 of $G|_Z$ admits a continuous extension g over Y fulfilling $g(y) \in G(y)$, for all $y \in Y$. When $\mathcal{U} = \mathcal{X}$ we say that G has the selection property (briefly $G \in SP(X)$).

Some examples illustrating this notion are:

Example 2.4.20. Let X be a nonempty closed convex subset of a Banach space E and $F : X \to P_{cl,cv}(X)$ be l.s.c.. From Michael's selection theorem it follows that $F \in SP(X)$.

Example 2.4.21. Let X be a nonempty closed decomposable subset of $L^1(T, E)$ and let $F : X \to P_{cl,dec}(X)$ be l.s.c.. Gorniewicz and Marano proved (see [94]) that F has the selection property with respect to the family of all separable metric spaces.

Using this abstract setting the following results was proved in Gorniewicz-Marano [94]:

Theorem 2.4.22. (Gorniewicz-Marano) *Let X be a complete absolute abstract and let $F : X \to P_{cl}(X)$ be a multi-valued contraction. Suppose that $F \in SP(X)$. Then $FixF$ is a complete absolute retract.*

Remark 2.4.23. *Theorem 2.4.22. contains as particular cases both Theorem 2.4.14. and Theorem 2.4.15.*

An important result is the following generalization of Theorem 2.4.22.
:

Theorem 2.4.24. *Let X be a complete absolute retract and $F : X \to P_{cl}(X)$be a Reich type multi-function such that $F \in SP(X)$. Then the fixed point set $FixF$ is a complete absolute retract.*

Regarding to the compactness property of the fixed point set of a multi-valued contraction mapping, J. Saint Raymond (see [237]) established the following theorem:

Theorem 2.4.25. (Saint Raymond) *Let T be a multi-valued contraction from the complete metric space X to itself. If T takes compact values, the fixed point set $FixT$ is compact too.*

An extension of the previous result is:

Theorem 2.4.26. *Let (X,d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let us suppose that $T : \tilde{B}(x_0;r) \to P_{cp}(X)$ satisfies the following two conditions:*

i) there exist $\alpha, \beta \in \mathbb{R}_+$, $\alpha + 2\beta < 1$ such that

$$H(T(x),T(y)) \leq \alpha d(x,y) + \beta[D(x,T(x)) + D(y,T(y))],$$

for each $x, y \in \tilde{B}(x_0;r)$

ii) $D(x_0,T(x_0)) < [1-(\alpha+2\beta)](1-\gamma)^{-1}r$.

Then the fixed points set $FixT$ is compact.

Proof. From Reich's theorem [213] it follows that $FixT \in P_{cl}(\tilde{B}(x_0;r))$. Assume that $FixT$ is not compact. Because $FixT$ is complete, it cannot be precompact, so there exist $\delta > 0$ and a sequence $(x_i)_{i\in\mathbb{N}} \subset FixT$ such that $d(x_i,x_j) \geq \delta$, for each $i \neq j$. Put $\rho = \inf\{R|$ there exists $a \in \tilde{B}(x_0;r)$ such that $B(a,R)$ contains infinitely many x_i;s$\}$. Obviously $\rho \geq \dfrac{\delta}{2} > 0$. Let $\varepsilon > 0$ such that $\varepsilon < \dfrac{1-\alpha-2\beta}{1+\alpha}\rho$ and choose $a \in \tilde{B}(x_0;r)$ such that the set $J = \{i : \ x_i \in B(a,\rho+\varepsilon)\}$ is infinite.

For each $i \in J$, we have

$$D(x_i,T(a)) \leq H(T(x_i),T(a)) \leq \alpha d(x_i,a)+\beta_i[D(x_i,T(x_i))+D(a,T(a))] =$$

$$= \alpha d(x_i,a) + \beta D(a,T(a)) < \alpha(\rho+\varepsilon) + \beta d(a,y), \text{ for every } y \in T(a).$$

Then

$$D(x_i,T(a)) < \alpha(\rho+\varepsilon)+\beta[d(a,x_i)+d(x_i,y)] < \alpha(\rho+\varepsilon)+\beta(\rho+\varepsilon)+\beta d(x_i,y),$$

for every $y \in T(a)$. Taking $\inf\limits_{y\in T(a)}$ we get : $D(x_i,T(a)) \leq (\alpha+\beta)(\rho+\varepsilon)(1-\beta)^{-1}$, for each $i \in J$. So, we can choose some $y_i \in T(a)$ such that $d(x_i,y_i) \leq (\alpha+\beta)(\rho+\varepsilon)(1-\beta)^{-1}$, for each $i \in J$. By the compactness of $T(a)$ there exists $b \in T(a)$ such that the following set: $J' = \{i \in J| \ d(y_i,b) < \varepsilon\}$ is infinite. Then, for each $i \in J'$ we get $d(x_i,b) \leq d(x_i,y_i) + d(y_i,b) < (\alpha+\beta)(\rho+\varepsilon)(1-\beta)^{-1} + \varepsilon$

$= (\alpha+\beta)(1-\beta)^{-1}\rho + \varepsilon\left(1+(\alpha+\beta)(1-\beta)^{-1}\right) < \rho$. This contradicts the definition of ρ, because the set $B(b,R)$ contains infinitely many x_i's $\left(\text{where } R = (\alpha+\beta)\rho(1-\beta)^{-1} + \varepsilon\left(1+(\alpha+\beta)(1-\beta)^{-1}\right)\right)$. $\square$

The purpose of the last part of this section is to study the measurability of the fixed point set for some multi-valued generalized contractions.

Let (X,d) be a complete separable metric space and (Ω,Σ) be a measurable space. Recall also that a multi-valued operator $T:\Omega\times X\to\mathcal{P}(X)$ is said to be a random operator if, for any $x\in X$ $T(\cdot,x):\Omega\to P(X)$ is measurable. We will denote by $F(\omega)$ the fixed points set of $T(\omega,\cdot)$, i.e. $F(\omega) := \{x\in X|\ x\in T(\omega,x)\}$. A random fixed point of T is a measurable function $x:\Omega\to X$ such that $x(\omega)\in T(\omega,x(\omega))$, for all $\omega\in\Omega$, or equivalently, x is a measurable selection for F.

If $T:\Omega\times X\to P_{b,cl}(X)$ is a random contraction (that is, for each $x\in X$, $T(\cdot,x)$ is measurable and for each $\omega\in\Omega$ there exists a number $k(\omega)\in[0,1[$ such that

$$H(T(\omega,x),T(\omega,y)) \le k(\omega)d(x,y),\ \text{ for all } x,y\in X)$$

then Xu and Beg (see [18]) proved that the multi-function F is measurable and hence T admits a random fixed point.

Let us start with the following lemma.

Lemma 2.4.27. *Let (X,d) be a complete metric space and let $T: X\to P_{b,cl}(X)$ be a multi-valued Reich type operator. Then for each $p>0$ we have*

$$H(F_p, FixT) \le (1-\gamma)p[1-(\alpha+\beta+\gamma)]^{-1}$$

where $F_p := \{x\in X|\ D(x,T(x)) < p\}$.

Proof. Obviously $F_p \supseteq FixT$ and hence $H(F_p,FixT) = \rho(F_p,FixT)$. Let $x\in F_p$ and $\varepsilon>0$ be arbitrarily. We can choose $x_1\in T(x)$ such that $d(x,x_1) < (1+\varepsilon)p$.

Starting from $x_0 = x$ and x_1, we can construct a sequence $\{x_n\}$ of successive approximation of T starting from (x_0, x_1) such that

$$d(x_n, x^*) \leq L^n(q)(1 - L(q))^{-1} d(x_1, x_0), \text{ for each } n \geq 0$$

where $q \in \left]1, (\alpha + \beta + \gamma)^{-1}\right[$ is arbitrary, $L(q) = q(\alpha + \beta)(1 - q\gamma)^{-1}$ and $\lim\limits_{n\to\infty} x_n = x^* \in FixT$.

For $n = 0$ we obtain $d(x_0, x^*) \leq (1 - L(q))^{-1} d(x_1, x_0) \leq (1 + \varepsilon)p(1 - L(q))^{-1}$. Letting $\varepsilon \searrow 0$ and $q \searrow 1$ we have

$$d(x, x^*) \leq \frac{p}{1 - (\alpha + \beta)(1 - \gamma)^{-1}} = \frac{(1 - \gamma)p}{1 - (\alpha + \beta + \gamma)}.\square$$

Definition 2.4.28. Let (X, d) be a complete separable metric space, (Ω, Σ) is a measurable space. Then $T : \Omega \times X \to P_{b,cl}(X)$ is a random Reich type operator, if for each $x \in X$, $T(\cdot, x)$ is measurable and for each $\omega \in \Omega$ there exist $\alpha(\omega), \beta(\omega), \gamma(\omega) \in \mathbb{R}_+$ with $\alpha(\omega) + \beta(\omega) + \gamma(\omega) < 1$ such that

$$H(T(\omega, x), T(\omega, y)) \leq \alpha(\omega)d(x, y) + \beta(\omega)D(x, T(\omega, x)) + \gamma(\omega)D(y, T(\omega, y))$$

for each $x, y \in X$.

Main result of the last part of this section is:

Theorem 2.4.29. *Suppose that (X, d) is a complete separable metric space, (Ω, Σ) is a measurable space and $T : \Omega \times X \to P_{b,cl}(X)$ is a random continuous Reich type operator. Then the multi-function F of the fixed point set is measurable.*

Proof. By Reich's theorem [213] the set $F(\omega)$ is nonempty for every $\omega \in \Omega$. For each $n \geq 1$ we consider

$$F_n(\omega) = \overline{\left\{ x \in X |\ D(x, T(\omega, x)) < \frac{1}{n} \right\}}$$

Using Proposition 1.5 in Xu-Beg [268], each $F_n(\omega)$ is measurable and by Lemma 2.4.27. we have

$$H(F_n(\omega), F(\omega)) \leq \frac{1-\gamma}{1-(\alpha+\beta+\gamma)} \cdot \frac{1}{n} \to 0, \text{ as } n \to \infty.$$

So F is measurable. The proof is now complete. □

Theorem 2.4.30. *Suppose that* (X, d) *is a complete separable metric space,* (Ω, Σ) *is a measurable space and* $T : \Omega \times X \to P_{b,cl}(X)$ *is a random continuous Reich-type operator. Then the multi-function* T *has a random fixed point.*

Proof. The conclusion follows from Theorem 2.4.29. via Kuratowski and Ryll Nardzewski selection theorem. (see Theorem 1.4.16.) □

Bibliographical comments. The approach of this paragraph follows mainly Petruşel [174] and Rus-Petruşel-Sîntămărian [229]. There is an extensive literature on the subject of multi-valued generalized contractions. Excellent sources for the properties of the fixed point set are the following: Anisiu-Mark [7], Deimling [80], Gorniewicz-Marano-Slosarki [93], Gorniewicz-Marano [94], Kamenskii-Obuhovskii-Zecca [118], Lim [133], Marano [137], Markin [138], Naselli Ricceri and B. Ricceri [156], Ricceri [215], Rybinski [235], Saint Raymond [237], Schirmer [246], Wang [261], Xu-Beg [268].

2.5 Caristi type operators

Caristi's fixed point theorem states that each operator f from a complete metric space (X, d) into itself satisfying the condition:

there exists a lower semi-continuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that:

$$(2.5.1.) \qquad d(x, f(x)) + \varphi(f(x)) \leq \varphi(x), \text{ for each } x \in X,$$

has at least a fixed point $x^* \in X$, i. e. $x^* = f(x^*)$ (see Caristi [52]).

There are several extensions and generalizations of this important principle of the nonlinear analysis (see for example Jachymski [111], Ciric [62] etc.).

One of them, asserts that if (X, d) is a complete metric space, $x_0 \in X$, $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is lower semi-continuous and $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\int_0^\infty \frac{ds}{1+h(s)} = \infty$, then each single-valued operator f from X to itself satisfying the condition:

$$(2.5.2.)\ \text{ for each } x \in X, \frac{d(x, f(x))}{1 + h(d(x_0, x))} + \varphi(f(x)) \le \varphi(x),$$

has at least a fixed point. (see Zhong-Zhu-Zhao [274])

For the multi-valued case, if F is an operator of the complete metric space X into the space of all nonempty subsets of X and there exists a lower semi-continuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that:

$$(2.5.3.)\ \text{ for each } x \in X, \text{ there is } y \in F(x) \text{ so that } d(x, y) + \varphi(y) \le \varphi(x),$$

then the multi-valued map F has at least a fixed point $x^* \in X$, i. e. $x^* \in F(x^*)$. (see for example [147])

Moreover, if F satisfies the stronger condition:

$$(2.5.4.)\ \text{ for each } x \in X \text{ and each } y \in F(x) \text{ we have } d(x, y) + \varphi(y) \le \varphi(x),$$

then the multi-valued map F has at least a strict fixed point $x^* \in X$, i. e. $\{x^*\} = F(x^*)$. (see [18])

On the other hand, if F is a multi-valued operator with nonempty closed values and $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semi-continuous function such that the following condition holds:

$$(2.5.5.)\ \text{for each } x \in X, \inf \{\ d(x,y) + \varphi(y)\ :\ y \in F(x)\ \} \leq \varphi(x),$$

then F has at least a fixed point.(see [104])

In this framework, let us remark that if we replace condition (2.5.5.) by a weaker condition (see (2.5.6.) below), then the conjecture stated by J.-P. Penot in [171] as follows:

Let (X,d) be a complete metric space, $\varphi : X \to \mathbb{R}_+$ be a lower semi-continuous function and F be a multi-valued operator of X into the family of all nonempty closed subsets of X satisfying the following condition:

$$(2.5.6.)\ D(x, F(x)) + \inf \{\ \varphi(y)\ :\ y \in F(x)\ \} \leq \varphi(x),$$

then F has at least a fixed point.

is false. (see van Hot [104] for a counterexample).

It is easy to see that (2.5.4.) $\Rightarrow$ (2.5.3.) $\Rightarrow$ 2.5.5.) and (2.5.5.) $\Rightarrow$ (2.5.3.) provided that F has nonempty compact values.

The purpose of this section is to present several new results in connection with the above mentioned single-valued and multi-valued Caristi type operators in complete metric spaces.

Let (X,d) be a metric space and $F : X \to P(X)$ be a multi-valued map.

Definition 2.5.1. A function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is called:

(i) a weak entropy of F if the condition (2.5.3) holds.

(ii) an entropy of F if the condition (2.5.4.) holds.

Moreover, the map $F : X \to P(X)$ is said to be weakly dissipative if there exists a weak entropy of F and it is said to be dissipative if there is an entropy of it.

Definition 2.5.2. Let (X,d) be a metric space and $F : X \to P_{b,cl}(X)$ be a multi-valued operator. Then F is said to be a δ-Reich type operator if there exist $a, b, c \in \mathbb{R}_+$, with $a+b+c<1$ such that

$$\delta(F(x), F(y)) \leq a\, d(x,y) + b\, D(x, F(x)) + c\, D(y, F(y)), \text{ for each } x, y \in X.$$

Let us remark now, that if f is a (single-valued) a-contraction in a complete metric space X, then f satisfies condition (2.5.1.) with $\varphi(x) = (1-a)^{-1}\ d(x, f(x))$, for each $x \in X$, so that part of the Banach contraction principle which says about the existence of a fixed point can be obtained by Caristi's theorem. For the multi-valued case we have the following result:

Theorem 2.5.3. *Let (X, d) be a complete metric space and $F : X \to P_{cp}(X)$ be an a-contraction $(0 < a < 1)$. Then:*

(a) F satisfies the condition (2.5.5.) with $\varphi(x) = (1-a)^{-1}\ D(x, F(x))$, for each $x \in X$.

(b) If, in addition $F(x) \in P_{cp}(X)$, for each $x \in X$, then F is weakly dissipative with a weak entropy given by the formula $\varphi(x) = (1-a)^{-1}\ D(x, F(x))$, for each $x \in X$.

Proof. a) is Corollary 1 in [104] and b) follows immediately from a) and the conditions (2.5.3.) $\Leftrightarrow$ (2.5.5.). $\square$

Remark 2.5.4. It is an open question if a multi-valued a-contraction $(0 < a < 1)$ is dissipative.

First main result of this section is:

Theorem 2.5.5. *Let (X, d) be a metric space and $F : X \to P_{cl}(X)$ be a Reich type multi-valued map. Then there exists $f : X \to X$ a selection of F satisfying the Caristi type condition (2.5.1.).*

Proof. Let $\varepsilon > 0$ such that $a < \varepsilon < 1 - b - c$. We denote by $U_x = \{\ y \in F(x)\ :\ \varepsilon\ d(x, y) \leq (1 - b - c)\ D(x, F(x))\ \}$, for each $x \in X$. Obviously, for each $x \in X$, the set U_x is nonempty (otherwise, if $x \in X$

is not a fixed point of F and we suppose that for each $y \in F(x)$ we have $\varepsilon d(x, y) > (1 - b - c)\ D(x, F(x))$, then we reach the contradiction $\varepsilon D(x, F(x)) \geq (1 - b - c)\ D(x, F(x))$; if $x \in X$ is a fixed point of F, then clearly $U_x \neq \emptyset$).

We can choose a single-valued mapping $f : X \to X$ such that $f(x) \in U_x$, i. e. $f(x) \in F(x)$ and $\varepsilon\ d(x, f(x)) \leq (1 - b - c)\ D(x, F(x))$, for each $x \in X$.

Then we have successively: $D(f(x), F(f(x))) \leq H(F(x), F(f(x))) \leq a\ d(x, f(x)) + b\ D(x, F(x)) + c\ D(f(x), F(f(x)))$ and hence

$$(1 - c)\ D(f(x), F(f(x))) - b\ D(x, F(x)) \leq a\ d(x, f(x)).$$

In view of this we obtain:

$$d(x, f(x)) = (\varepsilon - a)^{-1}\ [\varepsilon\ d(x, f(x)) - a\ d(x, f(x))] \leq$$

$$\leq (\varepsilon - a)^{-1}\ [(1 - b - c)\ D(x, F(x)) - (1 - c)\ D(f(x), F(f(x))) + b\ D(x, F(x))] =$$

$$= (1 - c)/(\varepsilon - a)\ [D(x, F(x)) - D(f(x), F(f(x)))].$$

If we define $\varphi : X \to \mathbb{R}_+$ by $\varphi(x) = (1 - c)/(\varepsilon - a)\ D(x, F(x))$, for each $x \in X$, then it is easy to see that

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \text{ for each } x \in X.$$

Moreover,

$$|\varphi(x) - \varphi(y)| = (1 - c)/(\varepsilon - a)\ |D(x, F(x)) - D(y, F(y))| \leq$$

$$\leq (1 - c)/(\varepsilon - a)\ |d(x, y) + H(F(x), F(y))| \leq$$

$$\leq (1 - c)/(\varepsilon - a)\ [d(x, y) + a\ d(x, y) + b\ D(x, F(x)) + c\ D(y, F(y))] =$$

$= (1 - c)(1 + a)/(\varepsilon - a)\ d(x, y) + b(1 - c)/(\varepsilon - a)\ D(x, F(x)) + c(1 - c)/(\varepsilon - a)\ D(y, F(y))$, proving the fact that the selection f is a kind of single-valued Reich type operator. $\square$

Remark 2.5.6. If the multi-valued operator $F : X \to P_{cl}(X)$ is an upper semi-continuous Reich type operator, then φ is a lower semi-continuous entropy of f. (because the map $x \mapsto D(x, F(x))$ is lower semi-continuous.)

Remark 2.5.7. If in Theorem 2.5.5. we take $b = c = 0$, then we obtain a result of Jachymski, see [111]. Moreover, we get that a multi-valued a-contraction $(0 \le a < 1)$ is weakly dissipative.

Theorem 2.5.8. *Let (X, d) be a metric space and $F : X \to P(X)$ be a δ-Reich type operator. Then the multi-valued operator F is dissipative.*

Proof. Let $\varepsilon > 0$ such that $a < \varepsilon < 1-b-c$. Let $x \in X$ and $y \in F(x)$. It is not difficult to see that

$$\varepsilon \ d(x, y) \le (1 - b - c) \ \delta(x, F(x)).$$

Using the fact that $y \in F(x)$ and the condition from hypothesis we have

$$\delta(y, F(y)) \le \delta(F(x), F(y)) \le a \ d(x, y) + b \ \delta(x, F(x)) + c \ \delta(y, F(y)).$$

It follows that

$$-a \ d(x, y) \le b \ \delta(x, F(x)) - (1 - c) \ \delta(y, F(y)).$$

So, we have

$$d(x, y) = (\varepsilon - a)^{-1} \ [\varepsilon \ d(x, y) - a \ d(x, y)] \le$$

$$\le (\varepsilon - a)^{-1} \ [(1 - b - c) \ \delta(x, F(x)) + b \ \delta(x, F(x)) - (1 - c) \ \delta(y, F(y))] =$$

$$= (1 - c)/(\varepsilon - a) \ [\delta(x, F(x)) - \delta(y, F(y))].$$

We define $\varphi(x) : X \to \mathbb{R}_+$ as follows: $\varphi(x) = (1-c)/(\varepsilon-a) \ \delta(x, F(x))$, for each $x \in X$ and we get

$$d(x, y) + \varphi(y) \le \varphi(x), \text{ for each } x \in X \text{ and for all } y \in F(x),$$

i. e. the multi-valued operator F is dissipative. $\square$

The following result is an extension of Proposition 1 in van Hot [104].

Theorem 2.5.9. *Let* (X, d) *a complete metric space,* $x_0 \in X$ *be arbitrarily,* $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ *a lower semi-continuous function and* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *a continuous non-decreasing function such that* $\int_0^\infty \frac{ds}{1+h(s)} = \infty$. *Let* $F : X \to P_{cl}(X)$ *be a multi-valued operator such that:*

$$\inf\{\frac{d(x,y)}{1+h(d(x_0,x))} + \varphi(y) : y \in F(x)\} \leq \varphi(x), \textit{for each } x \in X.$$

Then F *has at least a fixed point.*

Proof. We shall prove that for each $x \in X$ there exists $f(x) \in F(x)$ such that:

$$\frac{d(x, f(x))}{1+h(d(x_0,x))} + 2\varphi(f(x)) \leq 2\varphi(x).$$

If $D(x, F(x)) = 0$ then $x \in F(x)$ and put $x = f(x)$.

If $D(x, F(x)) > 0$ then we get successively:

$$\frac{D(x, F(x)}{1+h(d(x_0,x)} + \inf\{\frac{d(x,y)}{1+h(d(x_0,x))} + 2\varphi(y) : y \in F(x)\}$$

$$\leq 2\inf\{\frac{d(x,y)}{1+h(d(x_0,x))} + \varphi(y) : y \in F(x)\} \leq 2\varphi(x), \text{for each } x \in X.$$

It follows that:

$$\inf\{\frac{d(x,y)}{1+h(d(x_0,x))} + 2\varphi(y) : y \in F(x)\} < 2\varphi(x)$$

and hence there exists $f(x) \in F(x)$ such that:

$$\frac{d(x, f(x))}{1+h(d(x_0,x))} + 2\varphi(f(x)) \leq 2\varphi(x).$$

If we define $\psi(t) = 2\varphi(t)$ we get that f satisfies the hypothesis of Lemma 1.2. in [274] and hence there exists $x^* \in X$ such that $x^* = f(x^*) \in F(x^*)$. $\square$

In what follows we shall discuss the data dependence of the fixed points set of multi-valued operators which satisfy the Caristi type condition (2.5.3) and the data dependence of the strict fixed points set of multi-valued operators which satisfy the Caristi type condition (2.5.4).

Theorem 2.5.10. *Let (X,d) be a complete metric space and $F_1, F_2 : X \to P(X)$ be two multi-valued operators. We suppose that:*

(i) there exist two lower semi-continuous functions $\varphi_1, \varphi_2 : X \to \mathbb{R}_+$ such that for all $x \in X$, there exists $y \in F_i(x)$ so that

$$d(x,y) \leq \varphi_i(x) - \varphi_i(y),\ i \in \{1,2\};$$

(ii) there exists $c_i \in \,]0,+\infty[$ such that

$$\varphi_i(x) \leq c_i\ d(x,y),\ \textit{for each } x \in X \textit{ and for all } y \in F_i(x),\ i \in \{1,2\};$$

(iii) there exists $\eta > 0$ such that

$$H(F_1(x), F_2(x)) \leq \eta,\ \textit{for all } x \in X.$$

Then

$$H(Fix(F_1), Fix(F_2)) \leq \eta\ \max\ \{\ c_1,\ c_2\ \}.$$

Proof. From the condition (i) we have that $Fix(F_i) \neq \emptyset$, $i \in \{1,2\}$. Let $\varepsilon \in \,]0,1[$ and $x_0 \in Fix(F_1)$. It follows, from Ekeland variational principle (see for example [81]), that there exists $x^* \in X$ such that

$$\varepsilon\ d(x_0, x^*) \leq \varphi_2(x_0) - \varphi_2(x^*)$$

and

$$\varphi_2(x^*) - \varphi_2(x) < \varepsilon\ d(x, x^*),\ \text{for each } x \in X \setminus \{x^*\}.$$

For $x^* \in X$, there exists $y \in F_2(x^*)$ so that

$$d(x^*, y) \leq \varphi_2(x^*) - \varphi_2(y).$$

If we suppose that $y \neq x^*$, then we reach the contradiction

$$d(x^*, y) \le \varphi_2(x^*) - \varphi_2(y) < \varepsilon\ d(y, x^*).$$

So $y = x^*$ and therefore $x^* \in F_2(x^*)$, i. e. $x^* \in Fix(F_2)$.

Let $q \in \mathbb{R},\ q > 1$. Then, there exists $x_1 \in F_2(x_0)$ such that

$$d(x_0, x_1) \le q\ H(F_1(x_0), F_2(x_0)).$$

Taking into account the conditions (ii) and (iii) we are able to write $\varepsilon\ d(x_0, x^*) \le \varphi_2(x_0) - \varphi_2(x^*) = \varphi_2(x_0) \le c_2\ d(x_0, x_1) \le c_2\ q\ H(F_1(x_0), F_2(x_0)) \le c_2\ q\ \eta$. Hence

$$d(x_0, x^*) \le \eta\ c_2\ q\ /\ \varepsilon.$$

Analogously, for all $y_0 \in Fix(F_2)$, there exists $y^* \in Fix(F_1)$ such that

$$d(y_0, y^*) \le \eta\ c_1\ q\ /\ \varepsilon.$$

Using the last two inequalities, we obtain

$$H(Fix(F_1), Fix(F_2)) \le \eta\ q\ \varepsilon^{-1}\ \max\ \{\ c_1,\ c_2\ \}.$$

From this, letting $q \searrow 1$ and $\varepsilon \nearrow 1$, the conclusion follows. $\square$

Remark 2.5.11. In the condition (ii) of the Theorem 2.5.10. it is sufficient to ask that $\varphi_i(x) = 0$, for all $x \in Fix(F_i)$ and the existence of $c_i \in\ ]0, +\infty[$ such that

$$\varphi_i(x) \le c_i\ d(x, y),$$

for each $x \in Fix(F_j)$ and for all $y \in F_i(x)$, $i, j \in \{1, 2\}$, $i \neq j$.

Theorem 2.5.12. *Let (X, d) be a complete metric space and $F : X \to P(X)$ be a multi-valued operator. We suppose that:*

(i) there exists $\varphi : X \to \mathbb{R}_+$ a lower semi-continuous function such that

$$d(x, y) \le \varphi(x) - \varphi(y),\ \textit{for each } x \in X \textit{ and for all } y \in F(x);$$

(ii)there exists $c \in\ ]0,+\infty[$, such that

$$\varphi(x) \leq c\ d(x,y),\ \textit{for each } x \in X \textit{ and for all } y \in F(x).$$

Then $Fix(F) = SFix(F) \neq \emptyset$.

Proof. From the condition (i) we have that $SFix(F) \neq \emptyset$. Let $x^* \in Fix(F)$ and $y \in F(x^*)$. It follows that

$$d(x^*,y) \leq \varphi(x^*) - \varphi(y) = -\varphi(y) \leq 0.$$

Hence $d(x^*,y) = 0$ and therefore $y = x^*$. So $F(x^*) = \{x^*\}$, i. e. $x^* \in SFix(F)$ and thus we are able to write that $Fix(F) \subseteq SFix(F)$. $\square$

Remark 2.5.13. In condition (ii) of Theorem 2.5.12. it is sufficient to ask that $\varphi(x) = 0$, for all $x \in Fix(F)$.

Example 2.5.14. Let $F : [0,1] \to P([0,1])$, $F(x) = [x/3, x/2]$, for each $x \in [0,1]$ and $\varphi : X \to \mathbb{R}_+$, $\varphi(x) = kx$, for each $x \in [0,1]$, where $k \in \mathbb{R},\ k \geq 1$. It is not difficult to see that $|x-y| \leq \varphi(x) - \varphi(y)$, for each $x \in [0,1]$ and for all $y \in F(x)$ and there exists $c = 2k > 0$ such that $\varphi(x) \leq c\ |x-y|$ for each $x \in [0,1]$ and for all $y \in F(x)$. From Theorem 2.5.12. we have $Fix(F) = SFix(F) \neq \emptyset$.

Theorem 2.5.15. *Let (X,d) be a complete metric space and $F_1, F_2 : X \to P(X)$ be two multi-valued operators. We suppose that:*

(i) there exist two lower semi-continuous functions $\varphi_1, \varphi_2 : X \to \mathbb{R}_+$ such that

$$d(x,y) \leq \varphi_i(x) - \varphi_i(y),\ \textit{for each } x \in X \textit{ and for all } y \in F_i(x),\ i \in \{1,2\};$$

(ii) there exists $c_i \in\ ]0,+\infty[$ such that

$$\varphi_i(x) \leq c_i\ d(x,y),\ \textit{for each } x \in X \textit{ and for all } y \in F_i(x),\ i \in \{1,2\};$$

(iii) there exists $\eta > 0$ such that

$$H(F_1(x), F_2(x)) \leq \eta,\ \textit{for all } x \in X.$$

Then

$$H(Fix(F_1), Fix(F_2)) = H(SFix(F_1), SFix(F_2)) \le \eta \ \max\{\ c_1,\ c_2\ \}.$$

Proof. From Theorem 2.5.12. we have $Fix(F_i) = SFix(F_i) \neq \emptyset$, $i \in \{1, 2\}$ and applying Theorem 2.5.10. the conclusion follows. □

Example 2.5.16. Let $F_1, F_2 : [0, 1] \to P([0, 1])$, $F_1(x) = [x/3, x/2]$, for each $x \in [0, 1]$ and $F_2(x) = [(x+1)/2, (x+2)/3]$, for each $x \in [0, 1]$. Let $\varphi_1, \varphi_2 : [0, 1] \to \mathbb{R}_+$, $\varphi_1(x) = x$, for each $x \in [0, 1]$ and $\varphi_2(x) = 1 - x$, for each $x \in [0, 1]$. By an easy calculation we get that $|x-y| \le \varphi_i(x) - \varphi_i(y)$, for each $x \in [0, 1]$ and for all $y \in F_i(x)$, $i \in \{1, 2\}$ and there exist $c_1 = 2$ and $c_2 = 2$ such that $\varphi_i(x) \le c_i\ |x - y|$, for each $x \in [0, 1]$ and for all $y \in F_i(x)$, $i \in \{1, 2\}$. Also, there exists $\eta = 2/3 > 0$ so that $H(F_1(x), F_2(x)) \le \eta$, for all $x \in [0, 1]$. Then, from Theorem 2.5.15. we have $H(Fix(F_1), Fix(F_2)) = H(SFix(F_1), SFix(F_2)) \le 4/3$.

Bibliographical comments. For the results of this section and more details see Petruşel-Sîntămărian [203]. Also, the works of Aubin-Siegel [18], Bae-Cho-Yeom [24], Caristi [52], Ciric [62], van Hot [104], Mizoguchi-Takahashi [147], Penot [172], Zhong-Zhu-Zhao [274] are important for the topic of single-valued and multi-valued Caristi operators.

2.6 Meir-Keeler type operators and fractals

It is well-known that, initiated by Mandelbrot and then developed by Barnsley, Hutchinson and Hata the mathematical study of self-similar sets is in connection with the mathematics of fractals. In few words, a self-similar set is a set consisting of retorts of itself. More precisely, let f_i, $i \in \{1, \ldots, m\}$ be continuous operators of X into itself. A nonempty compact set Y in X is, by definition, self-similar if it satisfies the condition $Y = \cup_{i=1}^{m} f_i(Y)$. Obviously, we may regard the above relation as a fixed point problem for an appropriate operator. More precisely, if $T : (P_{cp}(X), H) \to (P_{cp}(X), H)$ is defined by $T(Y) = \bigcup_{i=1}^{m} f_i(Y)$, then the self-similar sets in X are the fixed points of T. If $X = \mathbb{R}^n$, it is well known that a self-similar set is a global attractor with respect to the dynamics generated by T in the phase set $P_{cp}(X)$ and its Hausdorff dimension is not, in general, an integer. For this reason, Y is a fractal and $P_{cp}(X)$ is the space of fractals. Moreover, self-similar sets among the fractals form an important class, since many of them have computable Hausdorff dimensions. For example, if f_i are a-contractions for $i \in \{1, \ldots, m\}$ then the operator T is also an a-contraction and hence has a unique fixed point.

The purpose of this section is to present similar results for the case of single-valued and multi-valued Meir-Keeler type operators.

Definition 2.6.1. Let $f_i : X \to X$, $i \in \{1, \ldots, m\}$ be a finite family of continuous operators. Let us define $T_f : (P_{cp}(X), H) \to (P_{cp}(X), H)$ by $T_f(Y) = \cup_{i=1}^{m} f_i(Y)$. The operator T_f is the so-called Barnsley-Hutchinson operator or the fractal operator generated by the iterated function system $f = (f_1, f_2, ...f_m)$.

First result of this section is:

Theorem 2.6.2. *Let (X, d) be a complete metric space and $f_i : X \to X$, for $i \in \{1, 2, \ldots, m\}$ are Meir-Keeler type operators. Then the fractal operator $T_f : (P_{cp}(X), H) \to (P_{cp}(X), H)$ is a Meir-Keller type operator and hence $FixT_f = \{A^*\}$ and $(T_f^n(A))_{n\in\mathbb{N}}$ converges to A^*, for each $A \in P_{cp}(X)$*

Proof. We shall prove that for each $\eta > 0$ there is $\delta > 0$ such that the following implication holds

$$\eta \leq H(A, B) < \eta + \delta \text{ we have } H(T_f(A), T_f(B)) < \eta.$$

Let us consider $A, B \in P_{cp}(X)$ such that $\eta \leq H(A, B) < \eta + \delta$.

If $u \in T_f(A)$ then there exists $j \in \{1, \ldots, m\}$ and $x \in A$ such that $u = f_j(x)$.

For $x \in A$ we can choose $y \in B$ such that $d(x, y) \le H(A, B) < \eta + \delta$. We have the following alternative:

1) If $d(x, y) \ge \eta$ then $\eta \le d(x, y) < \eta+\delta$ implies $d(f_j(x), f_j(y)) < \eta$. Hence $D(u, T_f(B)) \le d(u, f_j(y)) < \eta$.

2) If $d(x, y) < \eta$ then from the definition of Meir-Keeler operator we have $d(f_j(x), f_j(y)) < d(x, y) < \eta$ and again the conclusion $D(u, T_f(B)) < \eta$.

Because $T_f(A)$ is compact we have that $\rho(T_f(A), T_f(B)) < \eta$.

Interchanging the roles of $T_f(A)$ and $T_f(B)$ we obtain $\rho(T_f(B), T_f(A)) < \eta$ and hence $H(T_f(A), T_f(B)) < \eta$, showing the fact that T_f is a Meir-Keeler-type operator. From Meir-Keeler fixed point result (see Theorem 2.3.37.), we obtain that there exists an unique $A^* \in P_{cp}(X)$ such that $T_f(A^*) = A^*$ and $(T_f^n(A))_{n\in\mathbb{N}}$ converges to A^*, for each $A \in P_{cp}(X)$. $\square$

Remark 2.6.3. By definition, the set A^* is called the attractor of the system $f = (f_1, f_2, ..., f_m)$. Hence, Theorem 2.6.2. is an existence result of an attractor.

Next we will prove a local version of the previous result:

Theorem 2.6.4. *Let (X, d) be a complete ϵ-chainable metric space and $f_i : X \to X$, for $i \in \{1, \ldots, m\}$ be ϵ-locally-Meir-Keeler type operators. Then the fractal operator T_f is an ϵ-locally-Meir-Keeler type operator, having at least a fixed point.*

Proof. Let us consider $0 < \eta < \epsilon$ and $\delta > 0$ such that $A, B \in P_{cp}(X)$ and $\eta \le H(A, B) < \eta+\delta$. We shall prove that $H(T_f(A), T_f(B)) < \eta$. For this purpose, let $u \in T_f(A)$ arbitrarily. Then there is $j \in \{1, \ldots, m\}$ and $x \in A$ such that $u = f_j(x)$. For $x \in A$ we can choose $y \in B$ such that $d(x, y) \le H(A, B) < \eta + \delta$.

If $d(x,y) \geq \eta$ then from the hypothesis we get $d(f_j(x), f_j(y)) < \eta$ and hence $D(u, T_f(B)) \leq d(f_j(x), f_j(y)) < \eta$.

If on the other hand $d(x,y) < \eta < \epsilon$ then $d(f_j(x), f_j(y)) < d(x,y)$ implies again that $D(u, T_f(B)) < \eta$.

As before we deduce that $H(T_f(A), T_f(B)) < \eta$ thus T_f is an ϵ-locally Meir-Keeler type operator. The existence of the fixed point for T_f is now an easy application of Theorem 2.3.38. □

Remark 2.6.5. Jachymski (see [112]), C. S. Wong (in [263]) and T. -C. Lim [134] proved that the Meir-Keeler type condition is equivalent to other conditions of this type:

(a) for any $\eta > 0$ there exists a $\delta > 0$ such that $x, y \in X$, $0 < d(x,y) < \eta + \delta$ we have $d(f(x), f(y)) < \eta$

(b) for any $\eta > 0$ there exists a $\delta > 0$ such that $x, y \in X$, $0 \leq d(x,y) < \eta + \delta$ we have $d(f(x), f(y)) < \eta$

(c) $\delta(\eta) > 0$, for each $\eta > 0$, where $\delta(\eta)$ denotes the modulus of uniform continuity of f.

(d) there exists a lower semi-continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(0) = 0$, $\psi(\epsilon) > 0$, for every $\epsilon > 0$ and $\psi(d(f(x), f(y))) \leq d(x,y)$, for every $x, y \in X$

(e) there exists a function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ with the properties $\lambda(0) = 0$, $\lambda(\epsilon) > 0$, for every $\epsilon > 0$ and for each $s > 0$ there exists $u > s$ with $\lambda(t) \leq s$, for each $t \in [s, u]$ such that $d(f(x), f(y)) \leq \lambda(d(x,y))$, for every $x, y \in X$, $x \neq y$.

Obviously, similar theorems for operators f_i, $i \in \{1, \ldots m\}$ satisfying the condition (a)-(e) can be proved.

Let us consider now the multi-valued case. In this respect, we need some definitions.

Definition 2.6.6. Let X be a metric space and $F_1, \ldots, F_m : X \to P_{cp}(X)$ be a finite family of upper semi-continuous multi-valued operators. We define the multi-fractal operator generated by $F = (F_1, \ldots, F_m)$

as follows: $T_F : P_{cp}(X) \to P_{cp}(X), \quad T_F(Y) = \cup_{i=1}^{m} F_i(Y)$.

Also, by definition, $A^* \in P_{cp}(X)$ is a multi-self-similar set if $A^* = T_F(A^*)$.

Let us consider now some generalized contraction conditions for a multi-valued operator on a metric space (X, d).

Definition 2.6.7. The multi-valued operator $F : X \to P(X)$ is said to be:

i) a Meir-Keeler type operator if:

for each $\eta > 0$ there is $\delta > 0$ such that $\eta \leq d(x, y) < \eta + \delta$ implies $H(F(x), F(y)) < \eta$.

ii) an ϵ-locally Meir-Keeler type operator (where $\epsilon > 0$) if:

for each $\eta \in]0, \epsilon[$ there is $\delta > 0$ such that $\eta \leq d(x, y) < \eta + \delta$ implies $H(F(x), F(y)) < \eta$.

It is easy to prove that a multi-valued Meir-Keeler operator is contractive and hence u.s.c. on X.

An existence and uniqueness result for a multi-self-similar set is:

Theorem 2.6.8. *Let (X, d) be a complete metric space and $F_i : X \to P_{cp}(X)$, $i \in \{1, \ldots, m\}$ be a finite family of multi-valued Meir-Keeler type operators. Then the multi-fractal operator $T_F : P_{cp}(X) \to P_{cp}(X)$ is a single-valued Meir-Keeler type operator and $FixT_F = \{A^*\}$.*

Proof. Let us suppose that for each $\eta > 0$ there exists $\delta > 0$ such that $\eta \leq d(x, y) < \eta + \delta$ implies $H(F_i(x), F_i(y)) < \eta$ for $i \in \{1, \ldots, m\}$.

Obviously, F_i is contractive and hence upper semi-continuous, for $i \in \{1, \ldots, m\}$. As consequence $T_F : P_{cp}(X) \to P_{cp}(X)$.

Let us consider $\eta > 0$ and $Y_1, Y_2 \in P_{cp}(X)$ such that $\eta \leq H(Y_1, Y_2) < \eta + \delta$. We will prove that $H(T_F(Y_1), T_F(Y_2)) < \eta$.

For this purpose, let $u \in T_F(Y_1)$ be arbitrary. Then there exist $k \in \{1, \ldots, m\}$ and $y_1 \in Y_1$ such that $u \in F_k(Y_1)$. For this $y_1 \in Y_1$ there is $y_2 \in Y_2$ such that $d(y_1, y_2) \leq H(Y_1, Y_2) < \eta + \delta$.

If $d(y_1, y_2) \geq \eta$, then from Meir-Keeler condition we get that

$H(F_k(y_1), F_k(y_2)) < \eta$. It follows that there is $v \in F_k(y_2)$ such that $d(u, v) < \eta$ and hence $D(u, T_F(Y_2)) \leq d(u, v) < \eta$.

On the other hand if $0 < d(y_1, y_2) < \eta$, using again Meir-Keeler condition we deduce that:

$$H(F_k(y_1), F_k(y_2)) < d(y_1, y_2) < \eta$$

and as before $D(u, T_F(Y_2)) < \eta$.

Because $T_F(Y_1)$ is a compact set, we have that $\rho(T_F(Y_1), T_F(Y_2)) < \eta$. Interchanging the roles of $T_F(Y_1)$ and $T_F(Y_2)$ we obtain $\rho(T_F(Y_2), T_F(Y_1)) < \eta$ and the conclusion $H(T_F(Y_1), T_F(Y_2)) < \eta$ follows.

So $T_F : P_{cp}(X) \to P_{cp}(X)$ is a Meir-Keeler type operator and by Theorem 2.3.37. has a unique fixed point, i.e. $A^* \in P_{cp}(X)$ such that $T_F(A^*) = A^*$. $\square$

The following abstract notion is giving by Rus (see [223] for example):

Definition 2.6.9. Let (X, d) be a metric space and $f : X \to X$ an operator. By definition, f is a Picard operator if for each $x \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by:

i) $x_0 = x$

ii) $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$

is convergent and its limit is the unique fixed point of f.

Corollary 2.6.10. *Let (X, d) be a complete metric space and $F_i : X \to P_{cp}(X)$, $i \in \{1, \dots, m\}$ be a finite family of multi-valued Meir-Keeler type operators. Then the multi-fractal operator T_F is a Picard operator.*

For multi-valued operators satisfying to some locally contractive type conditions, we have the following results:

Theorem 2.6.11. *Let (X, d) be a complete ϵ-chainable metric space (where $\epsilon > 0$) and $F_i : X \to P_{cp}(X)$, $i \in \{1, \dots, m\}$ be a finite family of multi-valued ϵ-locally Meir-Keeler type operators.*

Then the multi-fractal operator $T_F : P_{cp}(X) \to P_{cp}(X)$ is an single-valued ϵ-locally Meir-Keeler type operator, having a fixed point.

Proof. The proof runs exactly as in Theorem 2.6.8., but this time via Xu's Theorem 2.3.38. $\square$

Using an ϵ-locally Boyd-Wong type condition (see Theorem 2.3.39.) one can also prove:

Theorem 2.6.12. *Let (X, d) be a complete ϵ-chainable metric and let $F_i : X \to P_{cp}(X)$, $i \in \{1, \dots, m\}$ be multi-valued operators such that $H(F_i(x), F_i(y)) \leq k(d(x, y))d(x, y)$, for all $x, y \in X$, with $0 < d(x, y) < \epsilon$, where $k : (0, \infty) \to (0, 1)$ is a real function with the property:*

$$(P) \quad \begin{cases} \text{For each } 0 < t < \epsilon \text{ there exist } e(t) > 0 \text{ and } s(t) < 1 \\ \text{such that } k(r) \leq s(t) \text{ provided } t \leq r < t + e(t) \end{cases}$$

Then, the multi-fractal operator $T_F : P_{cp}(X) \to P_{cp}(X)$ satisfy the condition:

$$H(T_F(Y_1), T_F(Y_2)) \leq k(H(Y_1, Y_2))H(Y_1, Y_2),$$

for all $Y_1, Y_2 \in P_{cp}(X)$ with $0 < H(Y_1, Y_2) < \epsilon$ and has a fixed point.

Proof. Let $Y_1, Y_2 \in P_{cp}(X)$ such that $0 < H(Y_1, Y_2) < \epsilon$. Then

$$H(T_F(Y_1), T_F(Y_2)) \leq \max\{H(F_k(Y_1), F_k(Y_2)) |\ k \in \{1, \dots, m\}\} \leq$$

$$\leq k(H(Y_1, Y_2))H(Y_1, Y_2).$$

The conclusion follows now from Theorem 2.3.39. $\square$

Bibliographical comments. The theory of self-similar sets in connection with Meir-Keeler type operators can be found in Petruşel [183], [197] and Petruşel-Rus [204]. For the topics of iterated function systems, self-similarity and fractals we refer the works of Barnsley [28], Hutchinson [106], Jachymski [114], Máté [140], Rus [227], Yamaguti-Hata-Kigami [271]. Regarding the Meir-Keeler type operators the following papers should be mentioned: Meir-Keeler [144], Boyd-Wong [39], Lim [134], Kirk-Sims [124], Jachymski [112], [113].

2.7 Coincidence theorems

S. Sessa and G. Mehta (see [242]) established some general coincidence theorems for upper semi-continuous multi-functions using Himmelberg's fixed point principle.

The first aim of this section is to prove some coincidence theorems for lower semi-continuous multi-functions on locally convex Hausdorff topological vector spaces using, instead of Himmelberg's result, the new fixed point principle of X. Wu (see Theorem 2.3.30.). We also show that a lower semi-continuous version of the well-known Browder's coincidence theorem is an easy consequence of our main result.

Theorem 2.7.1. *Let X be a nonempty convex and paracompact subset of a locally convex Hausdorff topological vector space E, D a nonempty set of a topological vector space Y. If $S : D \to P(X)$ and $T : X \to P(D)$ are such that:*

(a) S is l.s.c.

(b) $S(y) \in P_{cl,cv}(X)$

(c) $Q(x) = conv\, T(x)$ is a subset of D

(d) $S(D) \subset C$, where C is a compact metrizable subset of X

(e) for each $x \in X$ there exists $y \in D$ such that $x \in int\, Q^{-1}(y)$.

Then there exist $\overline{x} \in X$ and $\overline{y} \in D$ such that $\overline{x} \in S(\overline{y})$ and $\overline{y} \in Q(\overline{x})$.

Proof. We denote by $U(y) = int\, Q^{-1}(y)$, for each $y \in D$. Then the family $(U(y))_{y\in D}$ is an open covering of the paracompact space X (see (e)). Let $(U(y_i))_{i\in I}$ be an open locally finite covering of X and $\{f_{y_i} | i \in I\}$ a partition of unity by continuous nonnegative real functions defined on X subordinate to the covering $(U(y_i))_{i\in I}$. We can define a continuous function $f : X \to D$ by $f(x) = \sum_{i\in I} f_{y_i}(x) y_i$ for each $x \in X$. If $f_{y_i}(x) \neq 0$ then $x \in supp f_{y_i} \subset U(y_i) \subset Q^{-1}(y_i)$, that is $y_i \in Q(x)$. Since $Q(x)$ is convex for each $x \in X$ by (c) and $f(x)$ is a convex combination of elements from $Q(x)$, it follows that $f(x) \in Q(x)$, for each $x \in X$. We consider now the multi-valued operator $W : X \to \mathcal{P}(X)$ by $W(x) = S(f(x))$, for each $x \in X$. Then W is l.s.c. since f is continuous and S is l.s.c. Moreover by (b) W has nonempty, closed, convex values and $W(X) \subset S(D) \subset C$. Since C is compact and metrizable, then using Theorem 2.3.30. we get that there exists $\overline{x} \in C$ such that $\overline{x} \in W(\overline{x})$. It follow that $\overline{x} \in S(f(\overline{x}))$ and hence $\overline{y} = f(\overline{x}) \in Q(\overline{x})$, proving the conclusion of this theorem. $\square$

If $E = Y$ and $T(x)$ is convex for each $x \in X$ then we get the following coincidence result, similar to Sessa's coincidence theorem for u.s.c. multifunctions (see [241]).

Corollary 2.7.2. *Let X be a nonempty convex and paracompact subset of a locally convex Hausdorff topological vector space E, D a nonempty set of E and $S : D \to P(X)$, $T : X \to P(D)$ two multi-valued operators satisfying the following assertions:*

a) S is l.s.c.

b) $S(y) \in P_{cl,cv}(X)$

c) $T(x) \in P_{cv}(D)$

d) $S(D) \subset C$, where C is a nonempty compact, metrizable subset of the space X

e) for each $x \in X$ there exists $y \in D$ such that $x \in int\, T^{-1}(y)$.

Then there exist $\overline{x} \in X$ and $\overline{y} \in D$ such that $\overline{x} \in S(\overline{y})$ and $\overline{y} \in T(\overline{x})$.

Remark 2.7.3. Condition (e) from Corollary 2.7.2. appears in Tarafdar [254] and it generalize the well-known Browder's condition:

(f) for each $y \in D$ the set $T^{-1}(y)$ is open in X.

Using condition (f) instead of (e) we deduce from Theorem 2.7.1. the following result:

Theorem 2.7.4. *Let X be a nonempty convex compact and metrizable subset of a locally convex Hausdorff topological vector space E, D*

a nonempty set of a topological vector space Y, and $S : D \to P(X)$, $T : X \to P(D)$ two multi-valued operators satisfying:

a) S is l.s.c.

b) $S(y) \in P_{cl,cv}(X)$, for each $y \in D$

c) $T(x) \in P_{cv}(D)$, for each $x \in X$

d) $T^{-1}(y)$ is open in X, for each $y \in D$.

Then there exist $\overline{x} \in X$ and $\overline{y} \in D$ such that $\overline{x} \in S(\overline{y})$ and $\overline{y} \in T(\overline{x})$.

As consequence of the previous result we get:

Theorem 2.7.5. *Let X be a nonempty convex compact and metrizable subset of a locally convex Hausdorff topological vector space E, D a nonempty subset of a topological vector space Y and $S, T : D \to P(X)$ be multi-functions such that:*

a) S is l.s.c.

b) $S(y) \in P_{cl,cv}(X)$ for each $y \in D$

c) $T^{-1}(x)$ is a nonempty convex subset of D for each $x \in X$

d) $T(y)$ is open in X for each $y \in D$.

Then there exists $\overline{y} \in D$ such that $S(y) \cap T(\overline{y}) \neq \emptyset$.

Proof. Let us define the multifunction $\widetilde{T} : X \to P(D)$ by $\widetilde{T}(x) = T^{-1}(x)$, for each $x \in D$. Then S and $\widetilde{T}$ satisfy all the hypothesis of Theorem 2.7.4. and hence there exist $\overline{x} \in X$ and $\overline{y} \in D$ such that $\overline{x} \in S(\overline{y})$ and $\overline{y} \in \widetilde{T}(\overline{x})$. From the definition of $\widetilde{T}$ we obtain $\overline{y} \in T^{-1}(\overline{x})$ and so $\overline{x} \in S(\overline{y}) \cap T(\overline{y})$. $\square$

Remark 2.7.6. Theorem 2.7.5. is a l.s.c. version of Browder's coincidence theorem (see [44]).

Bibliographical comments. The results given here extent to the l.s.c. multi-functions case some results from Sessa-Mehta (see [242]). Mainly, this section follow the paper Muntean-Petruşel [150]. For other results and interesting applications see: Ansari-Idzik-Yao [10], Buică [45], [46], Dugundji-Granas [84], Petruşel [190], [191], O'Regan [161], Rus [220], [223].

2.8 Fixed points and integral inclusions

First purpose of this section is to present some fixed point theorems for the sum of two multi-valued operators. Secondly, several applications to integral inclusions are given.

Let us start with some auxiliary results that will be used in the following proofs. (see Rybinski [235], Deimling [80], Petruşel [176] and [222])

Theorem 2.8.1. *Let X be a metric space and Y be a closed subset of a Banach space Z. Assume that the multi-valued operator $F : X \times Y \to P_{cl,cv}(Y)$ satisfies the following conditions:*

i) $H(F(x, y_1), F(x, y_2)) \leq L\|y_1 - y_2\|$, for each $(x, y_1), (x, y_2) \in X \times Y$;

ii) for every $y \in Y$, $F(\cdot, y)$ is lower semi-continuous (briefly l.s.c.) on the space X.

Then there exists a continuous mapping $f : X \times Y \to Y$ such that :

$$f(x, y) \in F(x, f(x, y)), \text{ for each } (x, y) \in X \times Y.$$

Theorem 2.8.2. *Let X be a Banach space and $F_1, F_2 : X \to P_{cp}(X)$ be two multi-valued operators, such that F_1 is an L-contraction and F_2 is compact. Then $F_1 + F_2$ is (α, L)-contraction.*

Theorem 2.8.3. *Let X be a Banach space and $Y \in P_{b,cl,cv}(X)$. If $F : Y \to P_{cp,cv}(Y)$ is an u.s.c. and (α, L)-contraction multi-valued operator then $FixF \neq \emptyset$.*

A first multi-valued version of the Krasnoselskii's fixed point principle is:

Theorem 2.8.4. *Let X be a Banach space, $Y \in P_{cl,cv}(X)$ and $A : Y \to P_{b,cl,cv}(X)$, $B : Y \to P_{cp,cv}(X)$ two multi-valued operators. If the following conditions are satisfied:*

(i) $A(y_1) + B(y_2) \subset Y$, for each $y_1, y_2 \in Y$

(ii) A is L-contraction

(iii) B is l.s.c. and $B(Y)$ is relatively compact

Then $Fix(A+B) \neq \emptyset$.

Proof. Let $C : Y \to \mathcal{P}(Y)$ be a multi-valued operator defined as follows:

a) For each $x \in Y$ consider the multi-valued operator $T_x : Y \to P_{cp,cv}(Y)$, $T_x(y) = A(y) + B(x)$. Since T_x is multi-valued L-contraction (indeed, on have:
$H(T_x(y_1), T_x(y_2)) = H(A(y_1)+B(x), A(y_2)+B(x)) \leq H(A(y_1), A(y_2)) \leq L\|y_1 - y_2\|$, for each $y_1, y_2 \in Y$), from Covitz-Nadler fixed point theorem it follows that for every $x \in Y$ the fixed point set for the multifunction T_x, $Fix\, T_x = \{y \in Y |\ y \in A(y) + B(x)\}$ is nonempty and closed.

b) From Theorem 2.8.1. it follows that there exists a continuous mapping $f : Y \times Y \to Y$ such that $f(x, y) \in A(f(x, y)) + B(x)$. Let us observe that the multi-valued operator $F : Y \times Y \to P_{cp,cv}(Y)$ defined by $F(x, y) = A(y) + B(x)$, for each $(x, y) \in Y \times Y$ satisfies the hypothesis of Theorem 2.8.1.

Let us define $C(x) = Fix\, T_x$, $C : Y \to P_{cl}(Y)$ and consider the single-valued operator $c : Y \to Y$ defined by $c(x) = f(x, x)$, for each $x \in Y$. Then c is a continuous mapping having the property that $c(x) = f(x, x) \in A(f(x, x)) + B(x) = A(c(x)) + B(x)$, for each $x \in Y$.

Now, we will prove that $c(Y)$ is relatively compact. For this purpose it is sufficient to show that $C(Y)$ is relatively compact. Let us observe that $C(Y)$ is totally bounded:

Indeed $B(Y)$ being relatively compact it is also totally bounded. So, there exists $Z = \{x_1, \ldots, x_n\} \subset Y$ such that $B(Y) \subset \{z_1, \ldots, z_n\} + B(0, (1-L)\epsilon) \subset \bigcup_{i=1}^{n} B(x_i) + B(0, (1-L)\epsilon)$ (where $z_i \in B(x_i)$, for each $i = 1, 2, \ldots, n$). It follows that, for each $x \in Y$, $B(x) \subset \bigcup_{i=1}^{n} B(x_i) + B(0, (1-L)\epsilon)$ and hence there exists an element $x_k \in Z$ such that

$\rho(B(x), B(x_k)) < (1-L)\epsilon$. Then:

$$\rho(C(x), C(x_k)) = \rho(Fix\, T_x, Fix\, T_{x_k}) \leq \frac{1}{1-L} \sup_{y \in Y} \rho(T_x(y), T_{x_k}(y)) =$$

$$\frac{1}{1-L} \sup_{y \in Y} \rho(A(y) + B(x), A(y) + B(x_k)) \leq \frac{1}{1-L} \sup_{y \in Y} \rho(B(x), B(x_k)) <$$

$$< \frac{1}{1-L}(1-L)\epsilon = \epsilon$$

It follows that for each $u \in C(x)$ there is $v_k \in C(x_k)$ such that $\|u - v_k\| < \epsilon$. Hence, for each $x \in Y$, $C(x) \subset Q + B(0, \epsilon)$, where $Q = \{v_1, \ldots, v_k, \ldots, v_n\}$, $v_i \in C(x_i)$, $i = 1, 2, \ldots, n$.

Since in a Banach space a totally bounded set is relatively compact the conclusion follows.

Finally, let us observe that the mapping $c : Y \to Y$ satisfies the assumptions of Schauder's fixed point theorem. Let $x^* \in Y$ be a fixed point for c. On have that $x^* = c(x^*) \in A(c(x^*)) + B(x^*) = A(x^*) + B(x^*)$. $\square$

Using the abstract measures of noncompactness technique another fixed point result for the sum of two multi-valued operators is the following:

Theorem 2.8.5. *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and $A, B : Y \to P_{cp,cv}(X)$ two multi-valued operators. If the following conditions are satisfied:*

(i) $A(y) + B(y) \subset Y$, for each $y \in Y$

(ii) A is L-contraction

(iii) B is u.s.c. and compact

Then $Fix(A + B) \neq \emptyset$.

Proof. Since A is L-contraction it follows that A is u.s.c. The multi-valued operator $T = A + B$ (i.e. $T(x) = A(x) + B(x)$, for each $x \in Y$) is (α, L)-contraction from Y into the space $P_{cp,cv}(Y)$. On the same time, T is u.s.c. The conclusion follows by Theorem 2.8.3. $\square$

Following a idea from T.A. Burton [48], let us observe that the condition i) in Theorem 2.8.4. can be relaxed as follows:

Theorem 2.8.6. *Let X be a Banach space, $Y \in P_{cl,cv}(X)$ and $A : X \to P_{b,cl,cv}(X)$, $B : Y \to P_{cp,cv}(X)$ two multi-valued operators. Suppose that:*

i) If $y \in A(y) + B(x)$, $x \in Y$ then $y \in Y$

ii) A is L-contraction

iii) B is l.s.c. and $B(Y)$ is relatively compact.

Then $Fix(A + B) \neq \emptyset$.

Proof. Let consider now the multi-valued operator $C : Y \to P(Y)$ as follows:

a) for each $x \in Y$ consider the multi-valued mapping $T_x : X \to P_{cp,cv}(X)$ defined by $T_x(y) = A(y) + B(x)$. As in the proof of Theorem 4.1 we have that T_x is L-contraction and hence for each $x \in Y$ the fixed points set $FixT_x = \{y \in X|\ y \in A(y) + B(x)\}$ is nonempty and closed. Moreover from i) we have that $FixT_x \subset Y$.

b) Using Theorem 2.8.1. one obtain a continuous function $f : Y \times X \to X$ such that $f(x, y) \in A(f(x, y)) + B(x)$. Let us define $C(x) = FixT_x$, for each $x \in Y$. From a) we have that $C : Y \to P_{cl}(Y)$. Let consider now the single-valued operator $c : Y \to Y$ defined by $c(x) = f(x, x)$, for all $x \in Y$. Of course $c(x) = f(x, x) \in A(f(x, x)) + B(x) = A(c(x)) + B(x)$, $x \in Y$.

The rest of the proof is now identically with that of Theorem 2.8.4. □

Remark 2.8.7. Suppose that the conditions ii) and iii) from Theorem 2.8.6. hold. If there exists $r > 0$ such that for $Y = \{x \in X|\ \|x\| \leq r\}$ we have $B(Y) \subset Y$ and $\|y\| \leq D(y, A(y))$, $y \in Y$ then the conclusion of Theorem 2.8.6. holds.

Indeed, let $y \in A(y) + B(x)$, $x \in Y$. Then there exists $u \in A(y)$ such that $y - u \in B(x)$, $x \in Y$. Thus $\|y\| \leq D(y, A(y)) \leq \|y - u\| \leq \|B(x)\| \leq$

r. Hence $y \in Y$. $\square$

As applications, some existence results for integral inclusions are now presented.

Using Theorem 2.8.5. we have the following existence result for a Fredholm-Volterra integral inclusion:

Theorem 2.8.8. *Let us consider the following inclusion:*

$$y(t) \in \lambda_1 \int_a^t K_1(s, y(s))ds + \lambda_2 \int_a^b K_2(t, s, y(s))ds, \quad t \in [a, b]$$

(where $\lambda_1, \lambda_2 \in \mathbb{R}$).

We assume that:

i) $K_1 : [a, b] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ is a l.s.c., measurable and integrably bounded multi-valued operator.

ii) $K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ is an u.s.c., measurable and integrably bounded (by an integrable function m_{K_2}) multi-valued operator.

iii) there exists $L > 0$ such that

$$H(K_1(t, u_1), K_1(t, u_2)) \leq L\|u_1 - u_2\|, \text{ for each } (t, u_1), (t, u_2) \in [a, b] \times \mathbb{R}^n.$$

iv) λ_2 satisfy the following relation:

$$|\lambda_2| \leq \frac{R}{2M_{K_2}(b-a)}, \text{ where } R \geq \frac{\delta}{1 - \dfrac{|\lambda_1|L}{2\tau}}$$

(with $M_{K_2} = \max_{t \in [a,b]} m_{K_2}(t)$, $\tau > |\lambda_1|L$ and δ is an upper bound for the set of continuous selections for the multi-valued operator $t \mapsto \lambda_2 \int_a^b K_2(t, s, y(s))ds$, with $y \in C[a, b]$).

Then, there exists $y_0 \in C[a, b]$ such that the integral inclusion has at least a solution $y^* \in \tilde{B}(y_0, R) \subset C[a, b]$.

Proof. Let $A, B : C[a, b] \to \mathcal{P}(C[a, b])$ be two multi-valued operators given by

$$A(y) = \left\{ u \in C[a, b] | \ u(t) \in \lambda_1 \int_a^t K_1(s, y(s))ds \text{ a.e. on } [a, b] \right\}$$

$$B(y) = \left\{ v \in C[a,b] |\ v(t) \in \lambda_2 \int_a^b K_2(s, y(s))ds \text{ a.e. on } [a,b] \right\}$$

Obviously $y^* \in Fix(A+B)$ if and only if y^* is a solution for the considered integral inclusion. We need to show that the multi-valued operators A and B satisfies the assumptions of Theorem 2.8.5.

Clearly, from the Ascoli-Arzela theorem we have that $A : C[a,b] \to P_{cp,cv}(C[a,b])$.

We shall prove that A is a multi-valued contraction. To see this, let $y, z \in C[a,b]$ be and $u_1 \in A(y)$. Then $u_1 \in C[a,b]$ and $u_1(t) \in \lambda_1 \int_a^t K_1(s, y(s))ds$ a.e. on $[a,b]$. It follows that there is a mapping $k_y^1 \in S^1_{K_1(\cdot, y(\cdot))}$ such that $u_1(t) = \lambda_1 \int_a^t k_y^1(s)ds$ a.e. on $[a,b]$. Since $H(K_1(t, y(t)), K_1(t, z(t)) \leq L\|y(t) - z(t)\|$, one obtain that there exists $w \in K_1(t, z(t))$ such that $\|k_y^1(t) - w\| \leq L\|y(t) - z(t)\|$. Thus the multi-valued operator G defined by $G(t) = K_z^1(t) \cap K(t)$ (where $K_z^1(t) = K_1(t, z(t))$ and $K(t) = \{w |\ \|k_y^1(t) - w\| \leq L\|y(t) - z(t)\|\}$ has nonempty values and is measurable. Let k_z^1 be a measurable selection for G (which exists by Kuratowski and Ryll Nardzewski'selection theorem). Then $k_z^1(t) \in K_1(t, z(t))$ and $\|k_y^1(t) - k_z^1(t)\| \leq L\|y(t) - z(t)\|$ a.e. on $[a,b]$.

Define $u_2(t) = \lambda_1 \int_a^t k_z^1(s)ds$. It follows that $u_2 \in A(z)$ and

$$\|u_1(t) - u_2(t)\| \leq |\lambda_1| \int_a^t \|k_y^1(s) - k_z^1(s)\|ds \leq |\lambda_1| L \int_a^t \|y(s) - z(s)\|ds =$$

$$= \lambda_1 | L \int_a^t \|y(s) - z(s)\| e^{-\tau(s-a)} e^{\tau(s-a)} ds \leq |\lambda_1| L \|y - z\|_B \int_a^t e^{\tau(s-a)} ds \leq$$

$$\leq |\lambda_1| L \frac{1}{\tau} e^{\tau(t-a)} \|y - z\|_B.$$

(Here $\|\cdot\|_B$ denote the Bielecki-type norm on $C[a,b]$.) Finally, we have that $\|u_1 - u_2\|_B \leq \dfrac{|\lambda_1| L}{\tau} \|y - z\|_B$. From this and the analogous inequality obtained by interchanging the roles of y and z we get that

$H_B(A(y), A(z)) \leq \frac{|\lambda_1|}{\tau} L\|y-z\|_B$, for each $y, z \in C[z, b]$. Taking $\tau > |\lambda_1|L$ it follows that A is multi-valued contraction.

By Covitz-Nadler fixed point theorem one obtain $y_0 \in Fix\, A$.

Considers $Y = \tilde{B}(y_0, R)$. We can choose $R > 0$ such that $A(Y) \subset \tilde{B}\left(y_0, \frac{R}{2}\right)$ (namely, one take $R \geq \frac{diam(A(y_0))}{1 - \frac{|\lambda_1|L}{2\tau}}$ where $\tau > |\lambda_1|L$).

The multi-valued operator B is u.s.c. and compact. Let choose $\lambda_2 \in \mathbb{R}$ such that $B(Y) \subset \tilde{B}\left(0, \frac{R}{2}\right)$. Let $y \in Y$ and $v \in B(y)$ be arbitrarily chosen. Then $v(t) \in \lambda_2 \int_a^b K_2(t, s, y(s))ds$ a.e. on $[a, b]$. It follows that $v(t) = \lambda_2 \int_a^b f_y(t, s)ds$, where $f_y(t, s) \in K_2(t, s, y(s))$ a.e. on $[a, b] \times [a, b]$.

Clearly $\|v(t)\| \leq |\lambda_2| \int_a^b \|f_y(t, s)\|ds \leq |\lambda_2|(b - a)M_{K_2} \leq \frac{R}{2}$. So $v \in \tilde{B}\left(0, \frac{R}{2}\right)$.

Then, the multi-valued operator $T = A + B$ has the property $T : Y \to P_{cp,cv}(Y)$, i.e. $A(y) + B(y) \subset Y$ for each $y \in Y$.

The conclusion follows by Theorem 2.8.5. $\square$

An auxiliary result is:

Lema 2.8.9. (Rybinski [233]) *Let S be a complete measurable space, X a complete separable metric space and Y a separable Banach space. Suppose that $F : S \times X \to P_{cl,cv}(Y)$ is measurable and $F(t, \cdot)$ is l.s.c., for each $t \in S$.*

Then, there exists $f : S \times X \to Y$ selection for F such that f is measurable and $f(t, \cdot)$ is continuous, for each $t \in S$.

Some existence results for Fredholm and Volterra type integral inclusions via fixed point technique are the following theorems.

Theorem 2.8.10. *Consider the following Fredholm-type integral in-*

clusion:

$$x(t) \in \int_a^b K(t, s, x(s))ds + g(t), \quad t \in [a, b],$$

where $K : [a, b] \times [a, b] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ *and* $g : [a, b] \to \mathbb{R}^n$.

If the following conditions are satisfied:

(i) there exists an integrable function $M : [a, b] \to \mathbb{R}_+$ *such that for each* $t \in [a, b]$ *and each* $u \in \mathbb{R}^n$ *we have* $K(t, s, u) \subset M(s)B(0; 1)$, *a. e.* $s \in [a, b]$ *(i. e. for each* $t \in [a, b]$ *and each* $u \in \mathbb{R}^n$ *if* $v(s) \in K(t, s, u)$, $s \in [a, b]$ *then we have* $\|v(s)\| \leq M(s)$, *a. e.* $s \in [a, b]$*)*

(ii) for each $x \in C([a, b], \mathbb{R}^n)$ *we have that the multivalued operator* $K_x(t, s) := K(t, s, x(s)) : [a, b] \times [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ *is measurable*

(iii) for each $(s, u) \in [a, b] \times \mathbb{R}^n$ *the multivalued operator* $K(\cdot, s, u) : [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ *is l.s.c.*

(iv) $g \in C([a, b], \mathbb{R}^n)$

(v) $H(K(t, s, u), K(t, s, v)) \leq l(t, s)\|u - v\|$, *for each* $t, s \in [a, b]$ *and* $u, v \in \mathbb{R}^n$, *where* $l \in C[a, b] \times [a, b]$ *and* $\max\limits_{t \in [a,b]} \int_a^b l(t, s)ds < 1$,

then the integral inclusion has at least a solution in $C[a, b]$ *and its solution set is stable with respect to small perturbations of the free term.*

Proof. Consider the multi-valued operator $T : C[a, b] \to \mathcal{P}(C[a, b])$, given by the formula:

$$T(x) = \left\{ v \in C[a, b] |\ v(t) \in \int_a^b K(t, s, x(s))ds + g(t),\ t \in [a, b] \right\}.$$

We prove successively:

a) $T(x) \neq \emptyset$, for each $x \in C[a, b]$.

Indeed, from Lemma 2.8.9 we have that for each $x \in C[a, b]$ there is $k : [a, b] \times [a, b] \to \mathbb{R}^n$ such that $k(t, s)$ is a selection of $K_x(t, s) := K(t, s, x(s))$, $t, s \in [a, b]$ with k measurable and $k(\cdot, s)$ continuous, for each $s \in [a,b]$. From (i) we get that $k(t,\cdot)$ is integrable and so

$$v(t) = \int_a^b k(t,s)ds + g(t) \in \int_a^b K(t,s,x(s))ds + g(t), \quad t \in [a,b].$$

Hence, (see also (iv)) $v \in T(x)$.

b) $T(x) \in P_{cl}(C[a,b])$, for each $x \in C[a,b]$.

The fact that $T(x)$ is closed for each $x \in C[a,b]$ follows from (i), (ii) and Teorema 8.6.3. from Aubin-Frankowska [16].

(indeed, let $(x_n)_{n\geq 0} \in T(x)$ such that $x_n \overset{C[a,b]}{\longrightarrow} \tilde{x}$. Then $\tilde{x} \in C[a,b]$ and $x_n(t) \in \int_a^b K(t,s,x(s))ds + g(t)$, for each $t \in [a,b]$. Because $\int_a^b K(t,s,x(s))ds$ is compact, for every t then $x_n(t) \to \tilde{x}(t) \in \int_a^b K(t,s,x(s))ds + g(t)$, $t \in [a,b]$. So $\tilde{x} \in T(x)$).

c) $H(T(x_1),T(x_2)) \leq L\|x_1 - x_2\|$, for each $x_1, x_2 \in C[a,b]$ (where $L < 1$).

Let $x_1, x_2 \in C[a,b]$ and $v_1 \in T(x_1)$. Then $v_1(t) \in \int_a^b K(t,s,x(s))ds + g(t)$, $t \in [a,b]$. It follows that there exists $k_1(t,s) \in K_{x_1}(t,s) = K(t,s,x_1(s))$, such that $v_1(t) = \int_a^b k_1(t,s)ds + g(t)$, $t \in [a,b]$. From (iii) it follows that $H(K(t,s,x_1(s)), K(t,s,x_2(s))) \leq l(t,s)\|x_1(s) - x_2(s)\|$ and hence there is $w \in K(t,s,x_2(s))$ such that $\|k_1(t,s) - w\| \leq l(t,s)\|x_1(s) - x_2(s)\|$, $(t,s) \in [a,b]\times[a,b]$. Consider $U : [a,b]\times[a,b] \to \mathcal{P}(\mathbb{R}^n)$, given by

$$U(t,s) = \{w| \ \|k_1(t,s) - w\| \leq l(t,s)\|x_1(s) - x_2(s)\|\}.$$

Define the multi-valued operator $V(t,s) = U(t,s) \cap K_{x_2}(t,s)$. From the Proposition 3.4 a) from Deimling ([80], pp. 25) we have that V is measurable. From Proposition 15.6. from Gorniewicz [92], we get that $V(\cdot,s)$ is l.s.c. Hence, from Lemma 2.8.9, there exists k_2 a measurable selection for V such that $k_2(\cdot,s)$ is continuous. So, $k_2(t,s) \in K_{x_2}(t,s) = K(t,s,x_2(s))$ and $\|k_1(t,s) - k_2(t,s)\| \leq l(t,s)\|x_1(s) - x_2(s)\|$, for each $t, s \in [a,b]$.

Let us define $v_2(t) = \int_a^b k_2(t,s)ds + g(t) \in C[a,b]$.

One have

$$\|v_1(t) - v_2(t)\| \leq \int_a^b \|k_1(t,s) - k_2(t,s)\| ds \leq \int_a^b l(t,s)\|x_1(s) - x_2(s)\| ds \leq$$

$$\leq \|x_1 - x_2\| \int_a^b l(t,s)ds \leq \left(\sup_{t\in[a,b]} \int_a^b l(t,s)ds \right) \|x_1 - x_2\|, \text{ for each } t \in [a,b].$$

Consider $L = \max_{t\in[a,b]} \int_a^b l(t,s)ds < 1$. Hence $\|v_1 - v_2\| \leq L\|x_1 - x_2\|$. By the analogous relation obtained by interchanging the roles of x_1 and x_2 it follows that $H(T(x_1), T(x_2)) \leq L\|x_1 - x_2\|$. So T satisfy all the hypothesis of Covitz-Nadler fixed point principle, having a fixed point, let say $x^* \in T(x^*)$. Then $x^*(t) \in \int_a^b K(t,s,x^*(s))ds$, $t \in [a,b]$.

So, the integral inclusion has at least a solution.

If S_g is the solution set for the considered integral inclusion and S_h is the solution set for

$$x(t) \in \int_a^b K(t,s,x(s))ds + h(t), \quad t \in [a,b]$$

we estimate the distance $H(S_g, S_h)$.

Since $S_g = FixT_g$ (where T_g is given by (3.2)) and $S_h = FixT_h$ using Lemma 1.1 from Lim [132] one obtain:

$$H(S_g, S_h) = H(FixT_g, FixT_h) \leq \frac{1}{1-L} \sup_{x\in C[a,b]} H(T_g(x), T_h(x)).$$

Let $x \in C[a,b]$ and $v \in T_g(x)$. Then $v \in C[a,b]$ and $v(t) \in \int_a^b K(t,s,x(s))ds + g(t)$, $t \in [a,b]$. Of course, $v(t) = \int_a^b k(t,s)ds + g(t)$, with $k(t,s) \in K(t,s,x(s))$, $(t,s) \in [a,b] \times [a,b]$. Consider $w(t) = \int_a^b k(t,s)ds + h(t) \in \int_a^b K(t,s,x(s))ds + h(t)$, $t \in [a,b]$.

Then $\|v(t) - w(t)\| = \|g(t) - h(t)\|$ and hence $\|v - w\| = \|g - h\|$. From the analogous relation obtained for each $w \in T_h(x)$ it follows that $\sup\limits_{x \in C[a,b]} H(T_g(x), T_h(x)\| = \|g - h\|$.

As consequence $H(S_g, S_h) \leq \dfrac{1}{1-L}\|g - h\|$, showing the stability of the solution set with respect to small perturbation of the free term. □

Remark 2.8.11. If K is a single-valued operator then Theorem 2.8.9. is Theorem 1 from Constantin [65].

Theorem 2.8.12. *Consider the following Volterra-type integral inclusion:*

$$x(t) \in \int_a^t K(t, s, x(s))ds + g(t), \quad t \in [a, b],$$

where $K : [a, b] \times [a, b] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ and $g : [a, b] \to \mathbb{R}^n$.

If the following conditions are satisfied:

(i) there exists an integrable function $M : [a, b] \to \mathbb{R}_+$ such that for each $t \in [a, b]$ and each $u \in \mathbb{R}^n$ we have $K(t, s, u) \subset M(s)B(0; 1)$, a. e. $s \in [a, b]$

(ii) for each $x \in C([a, b], \mathbb{R}^n)$ we have that the multivalued operator $K_x(t, s) := K(t, s, x(s)) : [a, b] \times [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ is measurable

(iii) for each $(s, u) \in [a, b] \times \mathbb{R}^n$ the multivalued operator $K(\cdot, s, u) : [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ is l.s.c.

(iv) $g \in C([a, b], \mathbb{R}^n)$

(v) $H(K(t, s, u), K(t, s, v)) \leq k(s)\|u - v\|$, for each $t, s \in [a, b]$ and $u, v \in \mathbb{R}^n$ (where $k \in L^1[a, b]$),

then the integral inclusion has at least a solution in $C[a, b]$ and the solution set is stable with respect to small perturbations of the free term.

Proof. Consider the multi-valued operator $T : C[a, b] \to \mathcal{P}(C[a, b])$

given by:

$$T(x)=\left\{v\in C[a,b]|\ v(t)\in\int_a^t K(t,s,x(s))ds+g(t),\ t\in[a,b]\right\}.$$

We prove successively:

a) $T(x)\neq\emptyset$, for each $x\in C[a,b]$.

Indeed, let $x\in C[a,b]$ be arbitrarily. Since the multi-valued operator $K_x(t,s)=K(t,s,x(s))$ is (jointly) measurable for $(t,s)\in[a,b]\times[a,b]$ and $K_x(\cdot,s)$ is l.s.c., for each $s\in[a,b]$, we get from Lemma 2.8.9 that there exists a measurable selection of K_x, say $k(t,s)\in K_x(t,s)$, for each $(t,s)\in[a,b]\times[a,b]$ such that $k(\cdot,s)$ is continuous for each $s\in[a,b]$. From (i) each measurable selection of $K_x(t,s)$ is integrable with respect to s. Let

$$v(t)=\int_a^t k(t,s)ds+g(t)\in\int_a^t K(t,s,x(s))ds+g(t),\quad t\in[a,b].$$

So, $v\in T(x)$.

b) $T(x)\in P_{cl}(C[a,b])$. The proof is similar with the proof of b) from Theorem 2.8.10.

c) $H(T(x_1),T(x_2))\leq L\|x_1-x_2\|$, for each $x_1,x_2\in C[a,b]$ (where $L<1$).

Let $x_1,x_2\in C[a,b]$ and $v_1\in T(x_1)$. Then $v_1(t)\in\displaystyle\int_a^t K(t,s,x_1(s))ds+$ $g(t)$, $t\in[a,b]$. It follows that $v_1(t)=\displaystyle\int_a^b k_1(t,s)ds+g(t)$, $t\in[a,b]$, where $k_1(t,s)\in K_{x_1}(t,s)$, $(t,s)\in[a,b]\times[a,b]$. From (iii) it follows $H(K(t,s,x_1(s)),K(t,s,x_2(s))\leq k(s)\|x_1(s)-x_2(s)\|$ and hence there exists $w\in K(t,s,x_2(s))$ such that $\|k_1(t,s)-w\|\leq k(s)\|x_1(s)-x_2(s)\|$, $t,s\in[a,b]$.

Consider $U:[a,b]\times[a,b]\to\mathcal{P}(\mathbb{R}^n)$, given by the formula $U(t,s)=\{w|\ \|k_1(t,s)-w\|\leq k(s)\|x_1(s)-x_2(s)\|\}$. Since the multi-valued operator $V(t,s)=U(t,s)\cap K_{x_2}(t,s)$ is measurable and $V(\cdot,s)$ is l.s.c., for each $s\in[a,b]$, there exists $k_2(t,s)$ a measurable selection for V such that

$k_2(\cdot, s)$ is continuous for each $s \in [a, b]$. So, $k_2(t, s) \in K(t, s, x_2(s))$ and $\|k_1(t, s) - k_2(t, s)\| \leq k(s)\|x_1(s) - x_2(s)\|$ for each $t, s \in [a, b]$.

Let us define $v_2(t) = \int_a^t k_2(t, s)ds + g(t)$, $t \in [a, b]$. One have:

$$\|v_1(t) - v_2(t)\| \leq \int_a^t \|k_1(t, s) - k_2(t, s)\|ds \leq \int_a^t k(s)\|x_1(s) - x_2(s)\|ds =$$

$$= \int_a^t k(s)e^{-\tau p(s)}e^{\tau p(s)}\|x_1(s) - x_2(s)\|ds \leq \|x_1 - x_2\|_B \int_a^t k(s)e^{\tau p(s)}ds =$$

$$= \|x_1 - x_2\|_B \frac{1}{\tau} \int_a^t (e^{\tau p(s)})' ds = \frac{\|x_1 - x_2\|_B}{\tau} e^{\tau p(s)} \Bigg|_a^t \leq \frac{\|x_1 - x_2\|_B}{\tau} e^{\tau p(t)},$$

for each $t, s \in [a, b]$ (here $p(t) = \int_a^t k(s)ds$, $t \in [a, b]$ and $\tau > 1$).

Then $\|v_1 - v_2\|_B \leq \frac{1}{\tau}\|x_1 - x_2\|_B$. By the analogous inequality obtained by interchanging the roles of x_1 and x_2 it follows that

$$H(T(x_1), T(x_2)) \leq \frac{1}{\tau}\|x_1 - x_2\|_B$$

(here $\|\cdot\|_B$ is the Bielecki-type norm on $C[a, b]$ given by the formula $\|v\|_B = \max_{t \in [a,b]} \|v(t)\|e^{-\tau p(t)}$).

So T satisfy the hypothesis of Covitz-Nadler fixed point principle and hence there is $x^* \in T(x^*)$. Then x^* is a solution for the integral inclusion. As before, one can prove the stability of the solution set with respect to small perturbation of the free term. $\square$

Let us consider now the following integral inclusion with delay:

$$\text{(2.8.1.)} \quad \begin{cases} x(t) \in \int_{t-\tau}^t F(s, x(s))ds, & t \in [0, T] \\ x(t) = \varphi(t), & t \in [-\tau, 0] \end{cases}$$

where

$$
\text{(2.8.2.)}\quad \left.\begin{array}{l} F : [-\tau, T] \times \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}_+), \\ \varphi : [-\tau, 0] \to \mathbb{R}_+ \text{ is continuous, such that } \varphi(0) = \displaystyle\int_{-\tau}^{0} F(s, \varphi(s))ds \\ \text{and } T, \tau > 0 \text{ are given.} \end{array}\right\}
$$

A solution for this integral inclusion is a continuous function $x : [-\tau, T] \to \mathbb{R}_+$ such that x satisfies the relations (2.8.1.) for each t.

An existence result for (2.8.1.) is the following:

Theorem 2.8.13. *Consider the problem (2.8.1.), with F, φ, T, τ satisfying (2.8.2.).*

If $F : [-\tau, T] \times \mathbb{R}_+ \to P_{b,cl}(\mathbb{R}_+)$ is a multi-valued operator such that:

i) F is measurable and integrably bounded

ii) there exists $k \in L^1[-\tau, T]$ such that

$$H(F(s, u), F(s, v)) \leq k(s)|u - v|,$$

for each $s \in [-\tau, T]$ and every $u, v \in \mathbb{R}_+$.
Then the problem (2.8.1.) has at least a solution.

Proof. Let us define the multi-valued operator $A : C[0, T] \to \mathcal{P}(C[0, T])$ by the formula

$$Ax = \left\{ v \in C[0, T] |\ v(t) \in \int_{t-\tau}^{t} F(s, \tilde{x}(s))ds,\ t \in [0, T] \right\}$$

where

$$\tilde{x}(s) = \begin{cases} \varphi(s), & s \in [-\tau, 0] \\ x(s), & s \in [0, T]. \end{cases}$$

Let $Y = \{v \in C[0, T] |\ v(t) \geq 0$ for each $t \in [0, T]$ and $v(0) = \varphi(0)\}$. We shall prove:

a) $Ax \in Y$, for each $x \in Y$. Indeed, let $x \in Y$ be arbitrarily and $v \in Ax$. Then $v \in C[0, T]$ and $v(t) = \displaystyle\int_{t-\tau}^{t} f_{\tilde{x}}(s)ds$, where $f_{\tilde{x}}(s) \in F(s, \tilde{x}(s))$,

$s \in [-\tau, T]$. Since $F(s, \tilde{x}(s)) \in P(\mathbb{R}_+)$, for each $s \in [-\tau, T]$ we have that $v(t) \geq 0$ for each $t \in [0, T]$.

On the other side:

$$v(0) = \int_{-\tau}^{0} f_{\tilde{x}}(s)ds \in \int_{-\tau}^{0} F(s, \tilde{x}(s))ds = \int_{-\tau}^{0} F(s, \varphi(s))ds = \varphi(0).$$

b) $A : Y \to P_{cl}(Y)$ is a multi-valued contraction. Indeed, by standard argument, one can prove that $A(x) \in P_{cl}(Y)$, for every $x \in Y$.

Let us demonstrate that there exist $L \in]0, 1[$ such that $\|Ax - Ay\|_B \leq L\|x - y\|_B$ for each $x, y \in Y$ (where $\| \cdot \|_B$ is the classical Bielecki-type norm on $C[0, T]$).

Let $v_1 \in Ax$ be arbitrarily. Then $v_1(t) \in \displaystyle\int_{t-\tau}^{t} F(s, \tilde{x}(s))ds$, $t \in [0, T]$ and hence $v_1(t) = \displaystyle\int_{t-\tau}^{t} f_{\tilde{x}}(s)ds$, with $f_{\tilde{x}}(s) \in F(s, \tilde{x}(s))$, $s \in [-\tau, T]$. Using the condition ii) we obtain that there exist $w \in F(s, \tilde{y}(s))$ such that $\|f_{\tilde{x}}(s) - w\| \leq k(s)\|\tilde{x}(s) - \tilde{y}(s)\|$. As before, we can construct an integrable selection $f_{\tilde{y}}$ for $F(s, \tilde{y}(s))$, such that

$$\|f_{\tilde{x}}(s) - f_{\tilde{y}}(s)\| \leq k(s)\|\tilde{x}(s) - \tilde{y}(s)\|, \text{ for } s \in [-\tau, T].$$

Let us define $v_2(t) = \displaystyle\int_{t-\tau}^{t} f_{\tilde{y}}(s)ds$, $t \in [0, T]$. Then

$$\|v_1(t) - v_2(t)\| \leq \int_{t-\tau}^{t} \|f_{\tilde{x}}(s) - f_{\tilde{y}}(s)\|ds \leq$$

$$\leq \int_{t-\tau}^{t} k(s)\|\tilde{x}(s) - \tilde{y}(s)\|ds = \int_{t-\tau}^{t} \|\tilde{x}(s) - \tilde{y}(s)\|e^{-\tau p(s)}e^{\tau p(s)}k(s)ds \leq$$

$$\leq \|\tilde{x} - \tilde{y}\|_B \int_{t-\tau}^{t} k(s)e^{\tau p(s)}ds = \|\tilde{x} - \tilde{y}\|_B \frac{1}{\tau} \int_{t-\tau}^{t} (e^{\tau p(s)})'ds =$$

$$= \frac{1}{\tau}\|\tilde{x} - \tilde{y}\|_B[e^{\tau p(t)} - e^{\tau p(t-\tau)}] \leq \frac{1}{\tau}\|\tilde{x} - \tilde{y}\|_B e^{\tau p(t)}$$

where $p(s) = \displaystyle\int_{-\tau}^{s} k(u)du$, $s \in [-\tau, T]$.

Hence $\|v_1 - v_2\|_B \leq \frac{1}{\tau}\|\tilde{x} - \tilde{y}\|_B = \frac{1}{\tau}\|x - y\|_B$.

From the analogous inequality obtained by interchanging the roles of x and y, we get the conclusion.

From a), b) and using Covitz-Nadler's fixed point theorem we have that there exists $x^* \in Y$ such that $x^* \in Ax^*$. Then

$$x^*(t) \in \int_{t-\tau}^{t} F(s, \tilde{x}^*(s))ds, \quad t \in [0, T]. \quad \square$$

Bibliographical comments. This section uses results from Petruşel [175] and [176]. For the theory of integral inclusions via fixed point principles we refer to: Appell- de Pascale-Nguyêñ-Zabreiko [13], Burton [49], Constantin [65], Corduneanu [67], Couchouron-Precup [70], Czerwik [74], Himmelberg-van Vleck [101], Kannai [121], Petruşel [192], [193], Precup [208], O'Regan [161], O'Regan-Precup [163], O'Regan-Precup [166].

2.9 Fixed points and differential inclusions.

Let X be a nonempty set, $F : X \to P(X)$ be a multi-valued operator and $f : X \to X$ be a selection for F. By Fix$F$ (respectively Fix$f$) we denote the fixed points set of the multi-valued (respectively single-valued) operator. Obviously Fix$f \subset$ Fix$F$ and hence the following implication holds:

$$(2.9.1.) \quad \mathrm{Fix}f \neq \emptyset \ \Rightarrow \ \mathrm{Fix}F \neq \emptyset.$$

We are now interested about the reverse implication:

$$(2.9.2.) \quad \mathrm{Fix}F \neq \emptyset \ \Rightarrow \ \mathrm{Fix}f \neq \emptyset,$$

where f is an arbitrary selection with a certain property.

The first purpose of this section is to give some abstract results for the problem (2.9.2.) in the setting of the multi-valued fixed points structures (see Rus [222]).

Then, we will apply some of these abstract results to the problem of the topological dimension of the fixed point set of some contractive type multi-valued operators. Further applications to some multi-valued Cauchy and Darboux problems are also discussed.

Our results generalize and extend some theorems of this type obtained by J. Saint Raymond in [237] and Z. Dzedzej and B. D. Gelman in [85].

For the beginning we need some known notions and results.

Definition 2.9.1. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping. Then φ is said to be:

i) a comparison function if φ is strictly increasing and $\lim\limits_{n\to\infty} \varphi^n(t) = 0$, for each $t \in]0, \infty[$;

ii) a strong comparison function if φ is strictly increasing and $\sum\limits_{n=1}^{\infty} \varphi^n(t) < \infty$, for each $t \in]0, \infty[$;

iii) a strict comparison function if φ is a strong comparison function and $\lim\limits_{t\to\infty} (t - \varphi(t)) = +\infty$.

Remark 2.9.2. i) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = at$ (where $a \in]0, 1[$) is a strict comparison function.

ii) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \ln(1 + t)$ is a strong comparison function.

Definition 2.9.3. Let (X, d) be a metric space. A multi-valued operator $F : X \to P_{cl}(X)$ is called:

i) φ-contraction if there exists a comparison function φ such that $H(F(x), F(y)) \le \varphi(d(x, y))$, for each $x, y \in X$.

ii) (α, φ)-contraction if there exists a comparison function φ such that $\alpha(F(A)) \le \varphi(\alpha(A))$, for each $A \in P_b(X) \cap I(F)$ (where α denote an abstract measure of noncompactness for example α is α_K or α_H, Kuratowski and respectively Hausdorff measure of noncompactness).

The notion of fixed point structure for multi-valued mappings has been introduced by Rus in [222]. For the convenience of the reader we recall some basic notions and results. For this purpose, let X, Y be two nonempty sets and $\mathbb{M}^0(X,Y)$ be the set of all multi-valued operators T from X to Y. Denote by $\mathbb{M}^0(X) := \mathbb{M}^0(X,X)$.

Definition 2.9.3. A triple (X, S, M^0) is a fixed point structure if:

(i) $S \subset P(X), S \neq \emptyset$.

(ii) $M^0 : P(X) \multimap \bigcup_{Y \in P(X)} \mathbb{M}^0(Y)$, $Y \multimap M^0(Y)$, is a mapping such that if $Z \subset Y, Z \neq \emptyset$, then $M^0(Z) \supset \{T|_Z | T \in M^0(Y) \text{ and } Z \in I(T)\}$.

(iii) every $Y \in S$ has the strict fixed point property with respect to $M^0(Y)$.

Definition 2.9.4. Let (X, S, M^0) be a fixed point structure, $\theta : Z \to \mathbb{R}_+$ (where $S \subset Z \subset P(X)$) and $\mu : P(X) \to P(X)$. The pair (θ, μ) is a compatible pair with (X, S, M^0) if:

a) μ is a closure operator, $S \subset \mu(Z) \subset Z$ and $\theta(\mu(Y)) = \theta(Y)$, for each $Y \in Z$.

b) $Fix\mu \cap Z_\theta \subset S$.

Definition 2.9.5. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function and $\theta : Z \to \mathbb{R}_+$. A multi-valued mapping $T : Y \to P(X)$ is said to be a (θ, φ)- contraction if:

i) $A \in P(Y) \cap Z$ implies $T(A) \in Z$.

ii) $\theta(T(A)) \leq \varphi(\theta(A))$, for each $A \in P(Y) \cap Z \cap I(T)$.

Definition 2.9.5. A multi-valued operator $T : Y \to P(X)$ is said to be θ-condensing if:

i) $A \in P(Y) \cap Z$ implies $T(A) \in Z$.

ii) $A \in P(Y) \cap Z, \theta(A) \neq \emptyset$ implies $\theta(T(A)) < \theta(A)$.

Definition 2.9.6. Let X be a nonempty set, $Z \subset P(X), Z \neq \emptyset$ and $\theta : Z \to \mathbb{R}_+$. Then θ has the intersection property if $Y_n \in Z, Y_{n+1} \subset Y_n$,

$n \in \mathbb{N}$ and $\theta(Y_n) \to 0$, as $n \to \infty$ implies $\bigcap\limits_{n\in\mathbb{N}} Y_n \neq \emptyset$.

Theorem 2.9.7. ([222]) *Let $(X, S(X), M^0)$ be a fixed point structure and (θ, μ) a compatible pair with $(X, S(X), M^0)$. Let $Y \in \mu(Z)$ and $F \in M^0(Y)$. We suppose that:*

(i) $\theta|_{\mu(Z)}$ has the intersection property

ii) F is a (θ, φ)-contraction.

Then $FixF \neq \emptyset$.

Theorem 2.9.8. ([222]) *Let $(X, S(X), M^0)$ be a fixed point structure and (θ, μ) a compatible pair with $(X, S(X), M^0)$. Let $Y \in \mu(Z)$ and $F \in M^0(Y)$. We suppose that:*

i) $A \in Z$, $x \in Y$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$

ii) F is θ-condensing.

Then $FixF \neq \emptyset$.

An auxiliary result is:

Lemma 2.9.9 *Let X be a nonempty set, μ a closure operator, $Y \in Fix\mu$ and $T : Y \to P(Y)$ a multi-valued operator. Let $A \subset Y$ be a nonempty subset of Y. Then there exists $A_0 \subset Y$ such that:*

a) $A \subset A_0$

b) $A_0 \in Fix\mu$

c) $A_0 \in I(T)$

d) $\mu(T(A_0) \cup A) = A_0$.

If $F : X \to P(Y)$ is a multi-valued operator and $f : X \to Y$ is a selection for F, then we denote by $\tilde{f}$ the multi-valued operator defined by $\tilde{f}(x) = \{f(x)\}$, $x \in X$. The first general result is:

Theorem 2.9.10. *Let $(X, S(X), M^0)$ be a fixed point structure and (θ, μ) a compatible pair with $(X, S(X), M^0)$. Let $Y \in \mu(Z)$ and $F \in M^0(Y)$. We suppose that:*

i) $\theta|_{\mu(Z)}$ *has the intersection property*

ii) F *is a* (θ, φ)*-contraction*

iii) f *is a selection of* F *such that* $\tilde{f} \in M^0(Y)$.

Then $Fixf \neq \emptyset$.

Proof. From Lemma 2.9.9. we have that there exists $A_0 \in I(F) \cap S(X) \cap Z$. From (iii) it follows that $\tilde{f}|_{A_0} \in M^0(A_0)$. This implies (see Theorem 2.9.7.) $Fix\tilde{f} \neq \emptyset$ and hence $Fixf \neq \emptyset$. □

Another result of this type is:

Theorem 2.9.11. *Let* $(X, S(X), M^0)$ *be a fixed point structure and* (θ, μ) *a compatible pair with* $(X, S(X), M^0)$. *Let* $Y \in \mu(Z)$ *and* $F \in M^0(Y)$. *We suppose that:*

i) $A \in Z$, $x \in Y$ *implies* $A \cup \{x\} \in Z$ *and* $\theta(A \cup \{x\}) = \theta(A)$

ii) F *is* θ*-condensing*

iii) f *is a selection of* F *such that* $\tilde{f} \in M^0(Y)$.

Then $Fixf \neq \emptyset$.

Proof. From Lemma 2.9.9. we have that there exists $A_0 \in I(F) \cap S(x) \cap Z$. From iii) it follows that $\tilde{f}|_{A_0} \in M^0(A_0)$ and hence $Fix\tilde{f} \neq \emptyset$ (see Theorem 2.9.8.). This imply $Fixf \neq \emptyset$. □

Some important consequences of these abstract results are:

Corollary 2.9.12. *Let* X *be a real Banach space and* $F : X \to P_{cp}(X)$ *be a multi-valued* φ*-contraction with a strict comparison function* φ. *If* $f : X \to X$ *is a continuous selection of* F *then* $Fixf \neq \emptyset$.

Proof. Let $S(X) = P_{cl}(X)$ and $M^0(Y)$ be the set of the operators $F : Y \to P_{cp}(Y)$ such that F is a multi-valued φ-contraction, with φ a strong comparison function. Then $(X, S(X), M^0)$ is a fixed point structure. F being a multi-valued φ-contraction is θ-condensing with respect to $\theta = \alpha_H$ and $\mu(Y) = \overline{conv}Y$. The conclusion follows now from Theorem 2.9.11. □

Corollary 2.9.13. *Let X be a real Banach space, Y a nonempty, bounded, closed, convex subset of X and $F : Y \to P_{clc,cv}(Y)$ a u.s.c. multi-valued (α, φ)-contraction. If $f : Y \to Y$ is a continuous selection of F then $Fixf \neq \emptyset$.*

Proof. Let $S(X) = P_{cl,cv}(X)$, $z = P_b(X)$ and $M^0(Y) = \{F : Y \to P_{cl,cv}(Y)|\ F \text{ is u.s.c.}\}$. Let $\theta = \alpha$ (an abstract measure of noncompactness) and $\mu(Y) = \overline{conv}Y$. The result is an easy application of Theorem 2.9.10. □

Other results of this type are:

Theorem 2.9.14. *Let X be a real Banach space, $Y \in P_{cl}(X)$ and $F : Y \to P_{cp}(X)$ be weakly inward multi-valued a-contraction. If $f : Y \to X$ is a continuous selection of F then* $\text{Fix}f \neq \emptyset$.

Proof. By Theorem 11.4 from Deimlimg [80] on have that $\text{Fix}F \neq \emptyset$. Let $x_0 \in \text{Fix}F$. The closed ball $\tilde{B}(x_0, R) = \{y \in Y|\ \|x_0 - y\| \leq R\}$, with $R \geq \frac{1}{1-a}\delta(F(x_0))$, is invariant with respect to F and so $F : \tilde{B}(x_0; R) \to P_{b,cl}(\tilde{B}(x_0, R))$. F being a multi-valued a-contraction is α_H-condensing and therefore f is α_H-condensing. The conclusion that $\text{Fix}f \neq \emptyset$ follows by Sadovskii's fixed point theorem, see for example [81]. □

Theorem 2.9.15. *Let X be a real Banach space, $Y \in P_{cp,cv}(X)$ and $F : Y \to P_{cl,cv}(X)$ be an u.s.c. weakly inward multi-valued operator. If $f : Y \to X$ is a continuous selection of F then* $\text{Fix}f \neq \emptyset$.

Proof. By Halpern's fixed point theorem (see [95]) it follows that $\text{Fix}F \neq \emptyset$. Let $f : Y \to X$ be a continuous selection of F. Using the fact that F is weakly inward one obtain that f is weakly inward (since $f(x) \in F(x) \subset \tilde{I}_Y(x)$, for each $x \in Y$). The conclusion follows from a fixed point theorem given by Deimling (see [81] pp.210). □

The following problem appear in Saint Raymond [237] and Dzedzej-Gelman [85] :

If X *is a Banach space and* $F : X \to P(X)$ *is a multi-valued operator then when* $\dim F(x) \geq n$*, for each* $x \in X$ *imply that* $\dim \mathrm{Fix} F \geq n$*? (By* $\dim Y$ *we denote the topological (covering) dimension of the space* Y*).*

Some answers to this question are given in what follows.

An auxiliary result is the following:

Lemma 2.9.16. ([85]). *Let* X *be a Banach space,* T *be a compact metric space with* $\dim T < n$ *and* $F : T \to P_{b,cl,cv}(X)$ *be a l.s.c. operator such that* $0 \in F(x)$ *and* $\dim F(x) \geq n$*, for each* $x \in T$*. Then there exists a continuous selection* f *of* F *such that* $f(x) \neq 0$ *for each* $x \in T$*.*

The following results generalize and complete some results given in [237] and [85].

Theorem 2.9.17. *Let* X *be a Banach space and* $F : X \to P_{cp,cv}(X)$ *be a multi-valued* φ*-contraction, with* φ *a strict comparison function. If* $\dim F(x) \geq n$ *for each* $x \in X$ *then* $\dim \mathrm{Fix} F \geq n$*.*

Proof. Since $F : X \to P_{cp,cv}(X)$ is a multi-valued φ-contraction and $FixF$ is bounded (see Proposition 3.1 from [85]) it follows that $\mathrm{Fix}F$ is compact (otherwise $\alpha_H(\mathrm{Fix}F) \leq \alpha_H(F(\mathrm{Fix}F)) < \alpha_H(\mathrm{Fix}F)$, a contradiction). Consider the multi-valued operator $G : \mathrm{Fix}F \to P_{cp,cv}(X)$ given by $G(x) = x - F(x)$, for each $x \in X$.

Suppose, by contradiction, that $\dim \mathrm{Fix}F < n$. Then by Lemma 2.9.16. there is a continuous selection g of G such that $g(x) \neq 0$ for each $x \in \mathrm{Fix}F$. It follows that there exists a continuous selection f_0 of $F|_{\mathrm{Fix}F}$ with no fixed points. Using Michael's selection theorem we extend f_0 to a map $f : X \to X$ which is a selection of F without fixed points, contradiction with Corollary 2.9.11.. $\square$

By the same technique one obtain the following results on the dimension of the fixed points set of some contractive type multi-valued operators.

Theorem 2.9.18. *Let* X *be a Banach space,* $Y \in P_{cl}(X)$ *and*

$F : Y \to P_{cp,cv}(X)$ be a weakly inward multi-valued a-contraction. If $\dim F(x) \geq n$*, for each $x \in X$ then* $\dim \mathrm{Fix} F \geq n$.

Theorem 2.9.19. *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and $F : Y \to P_{cp,cv}(X)$ be a continuous γ-condensing and weakly inward multi-valued operator (where γ is α_K or α_H). If* $\dim F(x) \geq n$ *for each $x \in X$ then* $\dim \mathrm{Fix} F \geq n$.

Finally, we give some examples illustrating the usefulness of these theorems.

Example 2.9.20. Consider the multi-valued Cauchy problem:

$$x'(t) \in F(t, x(t)), \quad x(0) = x^0,$$

where $F : [0, a] \times \mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ is a multi-valued operator satisfying the following conditions:
(a) F is upper semi-continuous and integrably bounded
(b) $F(\cdot, x) : [0, h] \to P_{cp,cv}(\mathbb{R}^n)$ is measurable, for all $x \in \mathbb{R}^n$
(c) $H(F(t, u), F(t, v)) \leq k(t)\varphi(|u - v|)$, for each $t \in [0, a]$, for all $u, v \in \mathbb{R}^n$ and some φ a strict comparison function, where for each $t \in [0, a]$, $k(t) \in L^1[0, a]$ and $\sup\limits_{t\in[0,a]} \int_0^t k(s)ds \leq 1$.

Denote by S_{x_0} the solutions set for the Cauchy problem . Consider also, the following integral operator: $T : C([0, a], \mathbb{R}^n) \to P_{cp,cv}(C([0, a], \mathbb{R}^n))$ defined by

$$T(x) = \left\{ v \in C([0, a], \mathbb{R}^n) |\ v(t) \in x^0 + \int_0^t F(s, x(s))ds,\ t \in [0, a] \right\}$$

Obviously $FixT = S_{x_0}$. By standard arguments on have that for every $x_1, x_2 \in C([0, a], \mathbb{R}^n)$ and every $x_1, x_2 \in C([0, a], \mathbb{R}^n)$ and every $v_1 \in T(x_1)$ there exists $v_2 \in T(x_2)$ such that

$$|v_1(t) - v_2(t)| \leq \varphi(\|x_1 - x_2\|) \int_0^t k(s)ds, \text{ for every } t \in [0, a].$$

Then $\|v_1 - v_2\| \leq \varphi(\|x_1 - x_2\|)$. By the analogous inequality obtained by interchanging the roles of x_1 and x_2 we obtain $H(T(x_1), T(x_2)) \leq \varphi(\|x_1 - x_2\|)$ for each $x_1, x_2 \in C([0,a], \mathbb{R}^n)$.

From the above relation, using Lemma 2.6. and the same argument as in Theorem 2.7. (both results in Dzedzej-Gelman [85]), from Theorem 2.9.17. we obtain:

Theorem 2.9.21. *Let $F : [0,a] \times \mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ be a multi-valued operator satisfying the assertions (a)-(c). Assume that $\mu(\{t| \ \dim F(t,x) < 1$, for any $x \in \mathbb{R}^n\}) = 0$. Then the solution set for the Cauchy problem has an infinite dimension for all $x^0 \in \mathbb{R}^n$.*

Example 2.9.22. Consider the following multi-valued Darboux problem:

$$\frac{\partial^2 u}{\partial x \partial y} \in F(x, y, u(x,y)), \quad u(x,0) = 0, \quad u(0,y) = 0,$$

where $\quad (x,y) \in \overline{D} = [0,a] \times [0,b]$.

As before, the following result on the topological dimension of the solutions set is an application of Theorem 2.9.17.

Theorem 2.9.23. *Let $F : \overline{D} \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ be a multi-valued operator satisfying the following conditions:*

i) F is u.s.c. and integrably bounded;

ii) $F(\cdot, \cdot, u) : \overline{D} \to P_{cp,cv}(\mathbb{R})$ is measurable, for all $u \in \mathbb{R}$;

iii) $H(F(t,s,u), F(t,s,v)) \leq k(t,s)\varphi(|u-v|)$, for each $(t,s) \in \overline{D}$, for all $u, v \in \mathbb{R}$ and for some φ a strict comparison function (where for each $(t,s) \in \overline{D} \quad k(t,s) \in L^1(\overline{D})$ and $\sup\limits_{(x,y)\in\overline{D}} \int_0^x \int_0^y k(x,t)dsdt \leq 1$).

iv) $\mu(\{(t,s) \in \overline{D}| \ \dim F(t,s,u) < 1$, for all $u \in \mathbb{R}\}) = 0$.

Then the solutions set S for the Darboux problem has an infinite dimension.

Proof. If we consider the multi-valued operator $T : C(\overline{D}) \to$

$P_{cp,cv}(C(\overline{D}))$, given by

$$T(u) = \left\{ z \in C(\overline{D}) |\ z(x,y) \in \int_0^x \int_0^y F(s,t,u(s,t))dsdt,\ (x,y) \in \overline{D} \right\}$$

then it is obviously that $FixT = S$. T is a multi-valued φ-contraction and the conclusion follows from Lemma 2.6. in [85]. $\square$

Bibliographical comments. Following the method from Saint Raymond [237] and Dzedzej-Gelman [85], results of this section can be found in Petruşel [177] and [180]. We mention also the papers: Anello [3], Antosiewicz-Cellina [12], Aubin-Cellina [15], Cellina-Colombo [55], Cernea [56], [58], Constantin [65], Deimling [80], Kisielewicz [127], Lakshmikantham-Wen-Zhang [130], Naselli Ricceri [156], Petruşel [178], [180], Ricceri [218], for similar results and interesting applications.

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