Chapter 1

Optimization problems

Let X, Y be topological vector spaces. Consider $A \subset X$, $B, C \subset Y$, $f : X \to Y$ a singlevalued operator and $F : Y \to P(Y)$ a multivalued operator.

Let us show now that maximization with respect to a cone, which subsumes ordinary and Pareto optimization, is equivalent to a strict fixed point problem of the following type:

$$\text{find } y \in Y \text{ such that } \{y\} = F(y).$$

Pareto efficiency, or Pareto optimality, is a concept in economics with applications in engineering and social sciences. The term is named after Vilfredo Pareto, an Italian economist who used the concept in his studies of economic efficiency and income distribution.

An economic system that is not Pareto efficient implies that a certain change in allocation of goods (for example) may result in some individuals being made "better off" with no individual being made "worse off", and therefore can be made more Pareto efficient through a Pareto improvement. Here "better off" is often interpreted as "put in a preferred position." It is commonly accepted that outcomes that are not Pareto efficient are to be avoided, and therefore Pareto efficiency is an important criterion for evaluating economic systems and public policies.

If economic allocation in any system is not Pareto efficient, there is potential for a Pareto improvement-an increase in Pareto efficiency: through reallocation, improvements to at least one participant's well-being can be made without reducing any other participant's well-being.

In the real world ensuring that nobody is disadvantaged by a change aimed at improving economic efficiency may require compensation of one or more parties. For instance, if a change in economic policy dictates that a legally protected monopoly ceases to exist and that market subsequently becomes competitive and more efficient, the monopolist will be made worse off. However, the loss to the monopolist will be more than offset by the gain in efficiency. This means the monopolist can be compensated for its loss while still leaving an efficiency gain to be realized by others in the economy. Thus, the requirement of nobody being made worse off for a gain to others is met. In real-world practice compensations have substantial frictional costs. They can also lead to incentive distortions over time since most real-world policy changes occur with players who are not atomistic, rather who have considerable market power (or political power) over time and may use it in a game theoretic manner. Compensation attempts may therefore lead to substantial practical problems of misrepresentation and moral hazard and considerable inefficiency as players behave opportunistically and with guile.

Under certain idealized conditions, it can be shown that a system of free markets will lead to a Pareto efficient outcome. This is called the first welfare theorem. It was first demonstrated mathematically by economists Kenneth Arrow and Gerard Debreu. However, the result does not rigorously establish welfare results for real economies because of the restrictive assumptions necessary for the proof (markets exist for all possible goods, all markets are in full equilibrium, markets are perfectly competitive, transaction costs are negligible, there must be no externalities, and market participants must have perfect information). Moreover, it has since been demonstrated mathematically that, in the absence of perfect information or complete markets, outcomes will generically be Pareto inefficient (the Greenwald-Stiglitz theorem).

Recall that a a set $C \subset Y$ is a cone if $\lambda y \in C$, for all $y \in C$ and each $\lambda \geq 0$. A convex cone is a cone for which $\lambda_1 y_1 + \lambda_2 y_2 \in C$, for all $y_1, y_2 \in C$ and each $\lambda_1, \lambda_2 \geq 0$. A cone is called pointed if $C \cap (-C) = \{\theta\}$. For a pointed cone we write $y \geq z$ if and only if $y - z \in C$ and $y > z$ if and only if $y - z \in C - \{\theta\}$.

An element $y^* \in B$ is a maximal element of B with respect to C and we will denote this by:

$$y^* = \max(B; C)$$

if and only

if there is no $y \in B$ for which $y^* < y$.

Now, for a specified pointed cone C we consider the problem:

$$\text{maximize } f(x) \text{ subject to } x \in A, \qquad (*)$$

of determining all $x^* \in A$ for which $f(x^*) \in \max[f(A); C]$. Such an element x^* is said to be a maximal point for the considered problem.

This abstract problem has been studied in several papers by Borwein and others. When $X = \mathbb{R}^n, Y = \mathbb{R}^m, f_1, \cdots, f_m : \mathbb{R}^n \to \mathbb{R}$, with $f(x) = (f_1(x), \cdots, f_m(x))$ and $C = \mathbb{R}^m_+$, then the previous abstract problem becomes a Pareto maximization problem, which has been considered by numerous authors.

Let us show now that the considered problem is equivalent to a strict fixed point problem.

Theorem. *Let $f : X \to Y$ and $F : Y \to \mathcal{P}(Y)$, be defined by $F(y) = \{f(x) | x \in A, f(x) \in C + y\}$.*

Then x^* is a maximal element for problem (*) if and only if $\{f(x^*)\} = F(f(x^*))$.

Proof. First suppose that x^* is a maximal element for $(*)$. Then, there is no $x \in A$ such that $f(x^*) < f(x)$, i. e. there is no $x \in A$ such that $f(x) - f(x^*) \in C - \{\theta\}$. Also, we can observe that $\{f(x^*)\} \in F(f(x^*))$. We have to show now that $\{f(x^*)\} = F(f(x^*))$. If there exists another element $f(x)$ of $F(f(x^*))$, with $f(x) \neq f(x^*)$, then since x is feasible to $(*)$ it satisfies $\theta \neq f(x) - f(x^*) \in C$, contrary to our assumption. Thus the equality $\{f(x^*)\} = F(f(x^*))$ is established. Next, suppose that $\{f(x^*)\} = F(f(x^*))$ holds. Then, there is no $x \in A$ such that $f(x) \in F(f(x^*))$, with $f(x) \neq f(x^*)$. So, there is no $x \in A$ such that $f(x) \in C + f(x^*)$, with $f(x) \neq f(x^*)$. As consequence, there is no $x \in A$ such that $f(x) - f(x^*) \in C - \{\theta\}$. Since $f(x) - f(x^*) \in C - \{\theta\}$ cannot hold for any feasible x to $(*)$, we get the desired conclusion: x^* is a maximal point. $\square$

Definition. Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued operator. A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be a sequence of successive approximatuons for T starting from $(x, y) \in Graph(T)$ if $x_0 = x, x_1 = y$ and $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}^*$.

An aswer to the above optimization problem is the following.

Theorem. *Let (X, d) be a complete metric space, $Y \in P_{cl}(X)$ and $T : Y \to P_b(Y)$ be a H-continuous multivalued operator.*

We suppose that:

i) there exists $\alpha \in [0, 1[$ and a sequence $(y_n)_{n\in\mathbb{N}} \subset Y$ of successive approximations for T starting from some $(y_0, y_1) \in Graph(T)$ such that

$$diam T(y_{n+1}) \leq \alpha \; diam T(y_n), \; \textit{for all } n \in \mathbb{N};$$

ii) $y \in T(y), \quad (\forall)\, y \in Y$.

In these conditions, $(SF)_T = \{y^*\}$.

Proof. Let $y_0 \in Y, y_1 \in T(y_0)$ and $y_{n+1} \in T(y_n)\,, \quad (\forall)\, n \in \mathbb{N}^*$. From i) we have:

$$diam(T(y_n)) \leqslant \alpha diam(T(y_{n-1}))) \leq \cdots \leq \alpha^n(diam(T(y_0))) \to 0, \text{ as } n \to \infty.$$

This implies that $diam(T(y_n)) \to 0, \quad \text{as} \quad n \to \infty.$

Since $diam(T(y_n)) \to 0$, we immediately get that $d(y_n, y_{n+p}) \to 0$, as $n \to \infty$. Indeed,

$$d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + \cdots + d(y_{n+p-1}, y_{n+p}) \leq$$

$$\leq diam(T(y_n)) + \cdots + diam(T_{y_{n+p-1}}) \leq \alpha^n diam(T(y_0) + \cdots + \alpha^{n+p-1} diam(T(y_0)$$
$$\leq \frac{\alpha^n}{1-\alpha} diam(T(y_0).$$

It follows that the sequence $(y_n)_{n\in\mathbb{N}}$ is Cauchy in the complete metric space (Y, d). Hence, there exists $y^* \in X$ such that $\lim_{n\to\infty} y_n = y^*$. Moreover, Y being closed set, $y^* \in Y$.

By the hypothesis ii) we have that $y^* \in T(y^*)$. Since $diam(T(y_n)) \to diam(T(y^*))$ as $n \to \infty$ on one hand, and $diam(T(y_n)) \to 0$ as $n \to \infty$ on the other hand, we get that $diam(T(y^*)) = 0$. Thus we conclude that $\{y^*\} = T(y^*)$. $\square$