Chapter 1

The Arrow-Debreu model

1.1 Maximal elements

Let $\succeq$ be a preference relation on a set X and let A be a non-empty subset of X. Then we say that an element $a \in A$ is a **maximal element** for $\succeq$ on A whenever there is no element $b \in A$ satisfying $b \succ a$. Since $\succeq$ (as a preference relation) is complete, note that an element $a \in A$ is a maximal element if and only if $a \succeq x$ holds for each $x \in A$. It may happen that $\succeq$ need not have any maximal elements on a given set A. The next results describe some basic properties of maximal elements.

Theorem 1.1.1 *For a preference relation $\succeq$ on a set X and a non-empty subset A of X the following statements hold.*

1. All maximal elements of A for $\succeq$ lie in the same indifference set; and

2. If $X = \mathbb{R}^m_+$ and $\succeq$ has a strictly desirable bundle (vector) , then no interior point of A can be a maximal element.

Proof. (1) Let a be a maximal element for $\succeq$ on A. If $b \in A$ is another maximal element, then $a \succeq b$ and $b \succeq a$ both hold, and so $a \sim b$. This means that the maximal elements of A for $\succeq$ lie in the same indifference set.

(2) Let v be an extremely desirable bundle (vector) for $\succeq$ and let a be an interior point of A. Then for some sufficiently small $\alpha > 0$ we must have $a + \alpha v \in A$. Now the relation $a + \alpha v \succ a$ shows that a cannot be a maximal element for $\succeq$ on A. $\square$

Recall that a preference relation $\succeq$ on a topological space X is said to be **upper semicontinuous** whenever for each $x \in X$ the set $\{y \in X : \ y \succeq x\}$ is a closed set. Remarkably, upper semicontinuous preference relations on compact topological spaces always have maximal elements. The details are included in the next theorem.

Theorem 1.1.2 *The set of all maximal elements of an upper semicontinuous preference relation on a compact topological space is non-empty and compact.*

Proof. Let $\succeq$ be an upper semicontinuous preference on a compact topological space X. For each $x \in X$ let $C_x = \{y \in X : \ y \succeq x\}$. Since $\succeq$ is upper semicontinuous, the (non-empty) set C_x is closed - and hence compact. Now note that the set of all maximal elements of $\succeq$ is the compact set $\bigcap_{x \in X} C_x$. We shall show that $\bigcap_{x \in X} C_x \neq \emptyset$.

To this end, let $x_1, x_2, \ldots, x_n \in X$. Since $\succeq$ is a complete binary relation, the set $\{x_1, x_2, \ldots, x_n\}$ is completely ordered. We can assume that $x_1 \succeq x_2 \succeq \ldots \succeq x_n$. This implies $C_{x_1} \subseteq C_{x_2} \subseteq \ldots \subseteq C_{x_n}$, and so $\bigcap_{i=1}^{n} C_{x_i} = C_{x_1} \neq \emptyset$. Thus the collection of closed sets $\{C_x : \ x \in X\}$ has the finite intersection property. By the compactness of X, the set

$\bigcap_{x\in X} C_x$ is non-empty. $\square$

When does a preference relation have a unique maximal element on a set? The next result provides an answer.

Theorem 1.1.3 *For an upper semicontinuous convex preference $\succeq$ on a convex compact subset X of a topological vector space, the following statements hold.*

a) The set of all maximal elements of $\succeq$ in X is a non-empty, convex and compact.

b) If, in addition, $\succeq$ is strictly convex, then $\succeq$ has exactly one maximal element in X.

Proof. (a) By Theorem 1.1.2, we know that the set of all maximal elements of $\succeq$ is non-empty and compact. To see that this set is also convex, let a and b be two maximal elements of $\succeq$ in X and let $0 < \alpha < 1$. Then $\alpha a+(1-\alpha)b \in X$ and by the convexity of $\succeq$, we see that $\alpha a+(1-\alpha)b \succeq a$. On the other hand, by the maximality of a, we have $a \succeq \alpha a + (1-\alpha)b$ and therefore, $\alpha a + (1-\alpha)b$ is also a maximal element of $\succeq$.

(b) Assume that $\succeq$ is also strictly convex. If a and b are two distinct maximal elements, then $\frac{1}{2}a + \frac{1}{2}b \in X$ and $\frac{1}{2}a + \frac{1}{2}b \succ a$ must hold (why?), contrary to the maximality property of a. This shows that $\succeq$ has exactly one maximal element in X.

$\square$