Chapter 1

The Arrow-Debreu model

1.1 Demand functions

Preferences and utility are not observable in the market place. What we do observe are agents making transactions at market prices, i.e., demanding and supplying commodities at these prices. This suggests an alternative primitive formulation of economic behavior in terms of demand functions. In this section, we derive demand functions from utility maximization subject to a budget constraint. Consequently, the demand functions satisfy certain restrictions which play a critical role in equilibrium analysis.

Before starting our discussion in this section, let us introduce some standard notation. Boldface letters will denote vectors. For instance, the boldface letter $\mathbf{x}$ will represent the vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $\mathbf{p}$ the vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$. The symbol $\mathbf{x} \gg 0$ means that $x_i > 0$ holds for each i, i.e., all components of $\mathbf{x}$ are positive real numbers. Similarly, the notation $\mathbf{x} \gg \mathbf{y}$ means that $x_i > y_i$ holds for each i. Any vector $\mathbf{x}$

that satisfies $\mathbf{x} \gg 0$ is called a **strictly positive vector**.

Now fix a vector $\mathbf{p} \in \mathbb{R}^m_+$ - which we shall call a **price**. The **budget set** for $\mathbf{p}$ corresponding to a vector $\omega \in \mathbb{R}^m_+$ is the set

$$\mathcal{B}_\omega(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^m_+ : \ \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \omega\}.$$

A **budget set** for $\mathbf{p}$ is any set of the form $\mathcal{B}_\omega(\mathbf{p})$. The **budget line** of a budget set $\mathcal{B}_\omega(\mathbf{p})$ is the set $\{\mathbf{x} \in \mathcal{B}_\omega(\mathbf{p}) : \ \mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \omega\}$. Recall that the inner product $\mathbf{p} \cdot \mathbf{x}$ of two vectors is defined by

$$\mathbf{p} \cdot \mathbf{x} = p_1 x_1 + p_2 x_2 + \ldots + p_m x_m = \sum_{i=1}^{m} p_i x_i.$$

It is well known that the function $(\mathbf{p}, \mathbf{x}) \mapsto \mathbf{p} \cdot \mathbf{x}$ - from $\mathbb{R}^m_+ \times \mathbb{R}^m_+$ into $\mathbb{R}$ - is (jointly) continuous. An immediate consequence of the continuity of the dot product function $(\mathbf{p}, \mathbf{x}) \mapsto \mathbf{p} \cdot \mathbf{x}$ is that all budget sets are closed.

When does a price have bounded budget sets? It turns out that either all budget sets for a price are bounded or else all are unbounded. The condition for boundedness or unboundedness of the budget sets is included in the next theorem.

Theorem 1.1.1 *For a price $\mathbf{p} \in \mathbb{R}^m_+$ the following statements hold.*

1. All budget sets for $\mathbf{p}$ are bounded if and only if $\mathbf{p} \gg 0$.

2. All budget sets for $\mathbf{p}$ are unbounded if and only if $\mathbf{p}$ has at least one component equal to zero.

Proof. We establish (1) and leave the identical proof of (2) for the reader. To this end, assume first that every budget set for a price $\mathbf{p}$ is bounded. Then, we claim that $p_i > 0$ holds for each i. Indeed, if some $p_i = 0$, then the vectors $n\mathbf{e}_i$ $(i = 1, \ldots, m)$ - where $\mathbf{e}_i$ denotes the standard unit vector in the i^{th} direction - belong to every budget set (since $\mathbf{p}\cdot\mathbf{e}_i = 0$), proving that every budget set is unbounded.

Now assume that $\mathbf{p} \gg 0$ and let $\omega \in \mathbb{R}^m_+$. Put $r = \min\{p_1, p_2, \ldots, p_m\} > 0$. If $\mathbf{x} \in \mathcal{B}_\omega(\mathbf{p})$, then for each i we have

$$0 \le p_i x_i \le \sum_{k=1}^{m} p_k x_k = \mathbf{p}\cdot\mathbf{x} \le \mathbf{p}\cdot\omega,$$

and therefore

$$0 \le x_i \le \frac{\mathbf{p}\cdot\mathbf{x}}{p_i} \le \frac{\mathbf{p}\cdot\omega}{r} < \infty$$

holds for each $i = 1, 2, \ldots, m$. This shows that the budget set $\mathcal{B}_\omega(\mathbf{p})$ is bounded. $\square$

Since all budget sets are closed (and compactness in a finite dimensional vector space is equivalent to closedness and boundedness), the first part of Theorem 1.1.1 can be restated as follows: *All budget sets for a price* $\mathbf{p}$ *are compact if and only if* $\mathbf{p} \gg 0$. In particular, from this conclusion and Theorem 1.1.3 we have the following result.

Theorem 1.1.2 *For a price* $\mathbf{p} \gg 0$ *and a continuous preference* $\succeq$ *on* $\mathbb{R}^m_+$ *the following statements hold.*

1. If $\succeq$ *is also convex, then on every budget set for* $\mathbf{p}$ *the preference* $\succeq$ *has at least one maximal element.*

2. If $\succeq$ *is strictly convex, then on every budget set for* $\mathbf{p}$ *the preference* $\succeq$ *has a unique maximal element.*

3. If $\succeq$ *has an extremely desirable bundle (vector) and is strictly convex, then on every budget set for* $\mathbf{p}$ *the preference* $\succ$ *has exactly one maximal element lying on the budget line.*

For the rest of the discussion in this section all preference relations will be assumed defined on some $\mathbb{R}^m_+$. You should keep in mind that the interior of $\mathbb{R}^m_+$ is precisely the set of all strictly positive vectors and the boundary of $\mathbb{R}^m_+$ consists of all vectors of $\mathbb{R}^m_+$ having at least one component equal to zero.

Theorem 1.1.3 *For a price $\mathbf{p} \in \partial\mathbb{R}^m_+$ and a preference relation $\succeq$ on $\mathbb{R}^m_+$ the following statements hold.*

1. If $\succeq$ is strictly monotone, then $\succeq$ does not have any maximal element in any budget set for $\mathbf{p}$.

2. If $\succeq$ is strictly monotone on $\operatorname{Int}(\mathbb{R}^m_+)$ such that everything in the interior is preferred to anything on the boundary and if an element $\omega \in \mathbb{R}^m_+$ satisfies $\mathbf{p} \cdot \omega > 0$, then $\succeq$ does not have any maximal element in $\mathcal{B}_\omega(\mathbf{p})$.

Proof. Let $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m_+$ be a price having at least one component zero. We can assume that $p_1 = 0$.

(1) Suppose that $\succeq$ is strictly monotone and let $\mathbf{x}$ be a vector in some budget set $\mathcal{B}_\omega(\mathbf{p})$. Then, $\mathbf{y} = (x_1 + 1, x_2, \ldots, x_m) \in \mathcal{B}_\omega(\mathbf{p})$ and $\mathbf{y} > \mathbf{x}$. The strict monotonicity of $\succeq$ implies $\mathbf{y} \succ \mathbf{x}$. This shows that $\succeq$ does not have a maximal element in $\mathcal{B}_\omega(\mathbf{p})$.

(2) Now assume that $\succeq$ satisfies the stated properties and that $\mathbf{p} \cdot \omega > 0$. From $\mathbf{p} \cdot \omega > 0$, it follows that the budget set $\mathcal{B}_\omega(\mathbf{p})$ contains strictly positive elements and so if $\succeq$ has a maximal element in $\mathcal{B}_\omega(\mathbf{p})$, then this element must be strictly positive. However, if $\mathbf{x}$ is any strictly positive element in $\mathcal{B}_\omega(\mathbf{p})$, then $\mathbf{y} = (x_1 + 1, x_2, \ldots, x_m)$ is also a strictly positive element in $\mathcal{B}_\omega(\mathbf{p})$ satisfying $\mathbf{y} > \mathbf{x}$. Since $\succeq$ is strictly monotone on $\operatorname{Int}(\mathbb{R}^m_+)$, we see that $\mathbf{y} \succ \mathbf{x}$ must hold, which shows that $\succeq$ does not have a maximal element in $\mathcal{B}_\omega(\mathbf{p})$. □

Now consider a continuous strictly convex preference relation $\succeq$ on some $\mathbb{R}^m_+$ having an extremely desirable bundle (vector) . Also, let $0 < \omega \in \mathbb{R}^m_+$ be a fixed vector - referred to as *the initial endowment*. Then, by Theorem 1.1.2(3), for each price $\mathbf{p} \in \mathrm{Int}(\mathbb{R}^m_+)$ the preference relation $\succeq$ has exactly one maximal element in the budget set $\mathcal{B}_\omega(\mathbf{p})$. This maximal element is called **the demand vector** of the preference $\succeq$ at prices $\mathbf{p}$ and will be denoted by $\mathbf{x}_\omega(\mathbf{p})$. If, in a given situation, ω is fixed and clarity is not at stake, then the subscript ω will be dropped and the demand vector $\mathbf{y}_\omega(\mathbf{p})$ will be denoted simply by $\mathbf{x}(\mathbf{p})$. Thus, in this case, a function

$$\mathbf{x}_\omega : \mathrm{Int}(\mathbb{R}^m_+) \to \mathbb{R}^m_+$$

is defined by saying that $\mathbf{x}_\omega(\mathbf{p})$ is the demand vector of $\succeq$ at prices $\mathbf{p}$. The function $\mathbf{x}_\omega(\cdot)$ is known as the **demand function** corresponding to the preference $\succ$. Two important properties of the demand function should be noted immediately.

1) Since [by Theorem 1.1.2(3)], $\mathbf{x}_\omega(\mathbf{p})$ lies on the budget line, for each $\mathbf{p} \in \mathrm{Int}(\mathbb{R}^m_+)$ we always have $\mathbf{p} \cdot \mathbf{x}_\omega(\mathbf{p}) = \mathbf{p} \cdot \omega$.

2) The demand function is a homogeneous function of degree zero, i.e., for each $\lambda > 0$ and each $\mathbf{p} \gg 0$ we have $\mathbf{x}_\omega(\mathbf{p}) = \mathbf{x}_\omega(\lambda\mathbf{p})$.

This follows immediately from the budget identity $\mathcal{B}_\omega(\lambda\mathbf{p}) = \mathcal{B}_\omega(\mathbf{p})$.

Observe that a continuous preference $\succeq$ on $\mathbb{R}^m_+$ need not be strictly convex in order for the demand function $\mathbf{x}_\omega(\cdot)$ to be defined. The hypothesis of strict convexity may be relaxed. For example, the preference relation on $\mathbb{R}^2_+$ defined by the utility function $u(x, y) = xy$ is strictly monotone on $\mathrm{Int}(\mathbb{R}^2_+)$ but not strictly convex on $\mathbb{R}^2_+$. For each price $\mathbf{p} \gg 0$ the preference relation defined by this utility function has exactly one maximal element in the budget set $\mathcal{B}_\omega(\mathbf{p})$. Therefore, it is easy to check that the demand function $\mathbf{x}_\omega(\cdot)$ for this preference is well defined and satisfies the above two properties.

Our immediate objective is to study the properties of the demand functions. Since the demand functions are defined for certain preferences, let us give a name to these preferences that ill be useful in the economic analysis in this chapter.

Definition 1.1.4 *A continuous preference relation* $\succeq$ *on some* $\mathbb{R}^m_+$ *is said to be a* **neoclassical preference** *whenever either*

1) $\succeq$ *is strictly monotone and strictly convex; or else*

2) $\succeq$ *is strictly monotone and strictly convex on* $\mathrm{Int}(\mathbb{R}^m_+)$*, and everything in the interior is preferred to anything on the boundary.*

The next example illustrates how neoclassical preferences arise from common utility functions.

Example 1.1.5 *We exhibit two neoclassical preferences defined by utility functions* u_1 *and* u_2*. Preference* $\succeq_1$ *will satisfy condition (1) but not condition (2) of Definition 1.1.4, and preference* $\succeq_2$ *will satisfy condition (2) but not condition (1). Preferences such as (1) typically have demands on the boundary of* $\mathbb{R}^m_+$*, but preferences of type (2) always have demands on the interior of* $\mathbb{R}^m_+$*.*

(1) Consider the utility function defined on $\mathbb{R}^2_+$ *by the function*

$$u_1(x, y) = \sqrt{x} + \sqrt{y}.$$

Then the utility function is continuous, strictly monotone, and strictly convex on $\mathbb{R}^2_+$*. However, this utility function does not have the property that everything in the interior of* $\mathbb{R}^2_+$ *is preferred to anything on the boundary. Since the element* $(1, 0) \in \partial\mathbb{R}^2_+$ *is clearly preferred to* $\left(\frac{1}{9}, \frac{1}{9}\right)$ *which is in the interior.*

(2) Now consider the preference defined by the formula

$$u_2(x, y) = xy.$$

This utility function is strictly convex and strictly monotone on $\mathrm{Int}(\mathbb{R}^2_+)$ *however, it is not strictly convex on the boundary* $\partial\mathbb{R}^2_+$ *since every vector on the boundary is indifferent to the origin. For this very reason,* $\mathbf{x} \in \partial\mathbb{R}^2_+$ *and* $\mathbf{y} \in \mathrm{Int}(\mathbb{R}^2_+)$ *imply* $\mathbf{y} \succ \mathbf{x}$*, i.e., everything in the interior is preferred to anything on the boundary.*

It should be noted that strictly positive vectors are always extremely desirable vectors for neoclassical preferences. Our immediate objective is to study the properties of the demand functions that correspond to neoclassical preferences. The next theorem is the first step in establishing the continuity of demand functions.

Theorem 1.1.6 *Let* $\succeq$ *be a neoclassical preference on some* $\mathbb{R}^m_+$ *and let* ω *and* $\mathbf{p}$ *in* $\mathbb{R}^m_+$ *satisfy* $\mathbf{p} \cdot \omega > 0$*. If a sequence* $\{\mathbf{p}_n\}$ *of* $\mathrm{Int}(\mathbb{R}^m_+)$ *satisfies* $\mathbf{p}_n \to \mathbf{p}$ *and* $\mathbf{x}_\omega(\mathbf{p}_n) \to \mathbf{x}$*, then we have:*

a) $\mathbf{p} \gg 0$*, i.e.,* $\mathbf{p} \in \mathrm{Int}(\mathbb{R}^m_+)$*;*

b) $\mathbf{x} \in \mathcal{B}_\omega(\mathbf{p})$*; and*

c) $\mathbf{x} = \mathbf{x}_\omega(\mathbf{p})$*.*

Proof. From $\mathbf{p}_n \cdot \mathbf{x}_\omega(\mathbf{p}_n) = \mathbf{p}_n \cdot \omega$ and the continuity of the dot product, it follows that $\mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \omega$, and so $\mathbf{x} \in \mathcal{B}_\omega(\mathbf{p})$. Next, we claim that $\mathbf{x}$ is a maximal element for $\succeq$ in $\mathcal{B}_\omega(\mathbf{p})$. To see this, let $\mathbf{y} \in \mathcal{B}_\omega(\mathbf{p})$. Then $\mathbf{p} \cdot \mathbf{y} \le \mathbf{p} \cdot \omega$ holds, and so (since $\mathbf{p} \cdot \omega > 0$) for each $0 < \lambda < 1$, we have $\mathbf{p} \cdot (\lambda \mathbf{y}) < \mathbf{p} \cdot \omega$. From $\mathbf{p}_n \to \mathbf{p}$ and the continuity of the dot product, we see that there exists some n_0 satisfying $\mathbf{p}_n \cdot (\lambda \mathbf{y}) < \mathbf{p}_n \cdot \omega = \mathbf{p}_n \cdot \mathbf{x}_\omega(\mathbf{p}_n)$

for all $n \geq n_0$. Thus, $\mathbf{x}_\omega(\mathbf{p}_n) \succeq \lambda y$ holds for all $n \geq n_0$, and this (in view of continuity of $\succeq$) implies $\mathbf{x} \succeq \lambda \mathbf{y}$ for all $0 < \lambda < 1$. Letting $\lambda \uparrow 1$ (and using the continuity of $\succeq$ once more), we see that $\mathbf{x} \succeq \mathbf{y}$. This shows that $\mathbf{x}$ is a maximal element in $\mathcal{B}_\omega(\mathbf{p})$.

Now a glance at Theorem 1.1.3 reveals that $\mathbf{p} \gg 0$ must hold, in which case Theorem 1.1.2(2) guarantees that $\mathbf{x} = \mathbf{x}_\omega(\mathbf{p})$, and the proof of the theorem is finished. $\square$

To obtain the continuity of the demand functions we need the Closed Graph Theorem for continuous functions.

Lemma 1.1.7 (The Closed Graph Theorem) *Let $f : X \to Y$ be a function between two topological spaces with Y Hausdorff compact. Then f is continuous if and only if its graph $G_f = \{(x, f(x)) : \ x \in X\}$ is a closed subset of $X \times Y$.*

Proof. If f is continuous, then its graph G_f is clearly a closed subset of $X \times Y$.

For the converse, assume that G_f is a closed subset of $X \times Y$. Let $\{x_\alpha\}$ be a net of X satisfying $x_\alpha \to x$. We have to show that $f(x_\alpha) \to f(x)$. To this end, assume by way of contradiction that $f(x_\alpha) \not\to f(x)$. Then there exist an open neighborhood V of $f(x)$ and a subset $\{y_\lambda\}$ of $\{x_\alpha\}$ satisfying $f(y_\lambda) \notin V$ for each λ. Since Y is a compact topological space, there exists a subnet $\{z_\sigma\}$ of $\{y_\lambda\}$ (and hence, a subnet of $\{x_\alpha\}$) with $f(z_\sigma) \to u$ in Y. Clearly, $u \notin V$ and so $u \neq f(x)$. On the other hand, we have $(z_\sigma, f(z_\sigma)) \to (x, u)$ in $X \times Y$, and by the closedness of G_f, we infer that $u = f(x) \in V$, which is impossible. This contradiction shows that the function f is continuous at x, and hence continuous everywhere on X. $\square$

If Y is not compact, then the closedness of the graph G_f need not imply the continuity of f. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if} \quad x \neq 0; \\ 0, & \text{if} \quad x = 0. \end{cases}$$

has a closed graph but it is not continuous. It is, also interesting to know that there are examples of functions with closed graphs that are discontinuous at every point. To construct such an example, consider $X = \mathbb{R}$ with the Euclidean topology and $Y = \mathbb{R}$ with the discrete topology (i.e., every subset is open). Then the function $f : X \to Y$ defined by $f(x) = x$ has a closed graph but fails to be continuous at any point of X.

We are now ready to establish the continuity of the demand functions. Intuitively, the continuity of a demand function expresses the fact that "small changes in the price vector result in small changes in the demand vector".

Theorem 1.1.8 *Every demand function corresponding to a neoclassical preference is continuous.*

Proof. Let $\succeq$ be a neoclassical preference on some $\mathbb{R}^m_+$ and let $\omega \in \mathbb{R}^m_+$ be fixed. For simplicity, we shall denote the demand function $\mathbf{x}_\omega(\cdot)$ by

$$\mathbf{x}(\cdot) = (x_1(\cdot), x_2(\cdot), \ldots, x_m(\cdot)).$$

Now, let $\mathbf{p} \gg 0$ be fixed. Note first that $\mathbf{p}$ is in the interior of a "box" $[\mathbf{r}, \mathbf{s}]$ with $\mathbf{r} \gg 0$.[1] Let $r = \min\{r_1, r_2, \ldots, r_m\} > 0$. If

[1]The "box" $[\mathbf{r}, \mathbf{s}]$ is the set $[\mathbf{r}, \mathbf{s}] = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{r} \leq \mathbf{x} \leq \mathbf{s}\}$. In mathematical terminology, a "box" is known as an order interval.

$\mathbf{q} = (q_1, q_2, \ldots, q_m) \in [\mathbf{r}, \mathbf{s}]$, then we have

$$q_i x_i(\mathbf{q} \leq \sum_{k=1}^{m} q_k x_k(\mathbf{q}) = \mathbf{q} \cdot \mathbf{x}(\mathbf{q}) = \mathbf{q} \cdot \omega \leq \mathbf{s} \cdot \omega,$$

and consequently

$$x_i(\mathbf{q}) \leq \frac{\mathbf{s} \cdot \omega}{q_i} \leq \frac{\mathbf{s} \cdot \omega}{r} = M < \infty \tag{$*$}$$

holds for each $i = 1, 2, \ldots, m$. This implies that the function $\mathbf{x}(\cdot)$ is bounded on $[\mathbf{r}, \mathbf{s}]$, and so the set $Y = \overline{\mathbf{x}([\mathbf{r}, \mathbf{s}])}$ - where bar denotes closure - is a compact subset of $\mathbb{R}^m_+$. To show that $\mathbf{x}(\cdot)$ is continuous at $\mathbf{p}$, it suffices to establish that $\mathbf{x} : [\mathbf{r}, \mathbf{s}] \to Y$ is continuous. By Lemma 1.1.7, it suffices to show that the function $\mathbf{x} : [\mathbf{r}, \mathbf{s}] \to Y$ has a closed graph.

To this end, let a sequence $\{\mathbf{q}_n\} \subseteq [\mathbf{r}, \mathbf{s}]$ satisfy $\mathbf{q}_n \to \mathbf{q}$ and $\mathbf{x}(\mathbf{q}_n) \to \mathbf{x}$. By Theorem 1.1.6, it follows that $\mathbf{x} = \mathbf{x}(\mathbf{q})$. This shows that the function $\mathbf{x}[\mathbf{r}, \mathbf{s}] \to Y$ has a closed graph, and the proof of the theorem is finished. $\square$

Now let us give an economic interpretation of the discussion so far. The vector space $\mathbb{R}^m_+$ can be thought of as representing the commodity space of our economy - where, of course, the number m represents the number of available commodities. A preference relation can be thought of as representing the "taste" of a consumer and the vector ω as her initial endowment. The vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ represents the prevailing prices, p_i is the price (usually per unit) of commodity i. Then the demand vector $\mathbf{x}(\mathbf{p})$ represents the commodity bundle that maximizes the consumer's utility function subject to her budget constraint. If $\mathbf{x}_\omega(\mathbf{p}) = \mathbf{x}(\mathbf{p}) = (x_1(\mathbf{p}), x_2(\mathbf{p}), \ldots, x_m(\mathbf{p}))$ is the demand vector, then the real number

$$\sum_{i=1}^{m} x_i(\mathbf{p})$$

represents the total number of units of goods demanded by the individual - for a vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ the number $\sum_{i=1}^{m} |x_i|$ is called the m_1-*norm* of the vector and is denoted by $||\mathbf{x}||_1$, i.e., $||\mathbf{x}||_1 = \sum_{i=1}^{m} |x_i|$. Thus, the number $||\mathbf{x}(\mathbf{p})||_1$ is the aggregate number of units of goods demanded by the consumer.

As prices go to the boundary, some goods become (relatively) cheap and consequently demand for some commodities must become "very large". The details of this statement are given in the next theorem.

Theorem 1.1.9 *Consider a neoclassical preference $\succ$ on $\mathbb{R}^m_+$, a vector $\omega \in \mathbb{R}^m_+$ and denote by $\mathbf{x}(\cdot) = (x_1(\cdot), x_2(\cdot), \ldots, x_m(\cdot))$ the demand function corresponding to $\succeq$. Also, assume that a sequence $\{\mathbf{p}_n\}$ of strictly positive vectors satisfies*

$$\mathbf{p}_n = (p_1^n, p_2^n, \ldots, p_m^n) \to \mathbf{p} = (p_1, p_2, \ldots, p_m).$$

Then, we have:

1) If $p_i > 0$ holds for some i, then the sequence $\{x_i(\mathbf{p}_n)\}$ - the i^{th} components of the demand sequence $\{\mathbf{x}(\mathbf{p}_n)\}$ - is a bounded sequence.

2) If $\mathbf{p} \in \partial\mathbb{R}^m_+$ and $\mathbf{p} \cdot \omega > 0$, then

$$\lim_{n\to\infty} ||\mathbf{x}(\mathbf{p}_n)||_1 = \lim_{n\to\infty} \sum_{i=1}^{m} x_i(\mathbf{p}_n) = \infty.$$

Proof. Assume that $\{\mathbf{p}_n\}$ is a sequence of strictly positive prices satisfying the hypothesis of the theorem. Pick some $\mathbf{q} \gg 0$ such that $\mathbf{p}_n \leq \mathbf{q}$ holds for all n.

(1) Assume that $p_i > 0$ holds for some i. From $\mathbf{p} \gg 0$ and $\lim_{n\to\infty} p_i^n = p_i$, we infer that there exists some $\delta > 0$ such that $p_i^n > \delta$ holds for each n.

Now note that the inequality

$$p_i^n x_i(\mathbf{p}_n) \le \sum_{k=1}^{m} p_k^n x_k(\mathbf{p}_n) = \mathbf{p}_n \cdot \mathbf{x}(\mathbf{p}_n) = \mathbf{p}_n \cdot \omega \le \mathbf{q} \cdot \omega,$$

implies that

$$x_i(\mathbf{p}_n) \le \frac{\mathbf{q} \cdot \omega}{p_i^n} \le \frac{\mathbf{q} \cdot \omega}{\delta} < \infty$$

holds for each n. Therefore, $\{x_i(\mathbf{p}_n)\}$ is a bounded sequence.

(2) If $\{\mathbf{x}(\mathbf{p}_n)\}$ has a bounded subsequence, then by passing to a subsequence (and relabelling), we can assume that $\mathbf{x}(\mathbf{p}_n) \to \mathbf{x}$ holds in $\mathbb{R}_+^m$. In such a case, theorem 1.1.6 implies that $\mathbf{p} \gg 0$ must hold, which contradicts $\mathbf{p} \in \partial\mathbb{R}_+^m$, and our conclusion follows. $\square$

Part (2) of the preceding theorem asserts that when prices drop to zero, then the demand collectively tends to infinity. However, it should be noted that when the individual price of a commodity drops to zero, the demand for that particular commodity does not necessarily tend to infinity.