Chapter 1

The Arrow-Debreu model

One of two central paradigms in modern general equilibrium theory is the Walrasian general equilibrium model of an economy with a finite number of commodities and a finite number of households and firms, as formulated by K. J. Arrow and G. Debreu.

In this chapter, we shall investigate the existence and optimality of Walrasian (or competitive) equilibrium in the Arrow-Debreu model.

In the classical Arrow-Debreu model only a finite number of commodities are exchanged, produced or consumed. It is useful to think of physical commodities such as steel or wheat or apples that are available at different times or in different locations or in different states of the world as different commodities. We suppose that there are m such commodities. Inputs for production are negatively signed and outputs of production are positively signed. Any two commodity bundles (vectors) can be added to produce a new commodity bundle (vector) and any scalar multiple of a commodity bundle (vector) is a commodity bundle (vector) . Hence, it is natural to view the commodity space E as the finite dimensional vector space $\mathbb{R}^m$.

The terms at which good j can be exchanged in the market for good i is defined by the ratio of the prices $\frac{p_i}{p_j}$, where p_i and p_j are nonnegative real numbers and $p_j > 0$. That is, $\frac{p_i}{p_j}$ is the amount of good j that can be exchanged for a unit amount of good i at prices $\mathbf{p} = (p_1, p_2, \ldots, p_m)$. Given a price vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ and a commodity vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, the "value" of $\mathbf{x}$ at prices $\mathbf{p}$ is given by $\mathbf{p} \cdot \mathbf{x} = \sum_{i=1}^{m} p_i x_i$ (here " $\cdot$ " is the inner product on the space). Hence, each price vector defines a linear and continuous functional on the commodity space E and we define the price space as the dual space of E, denoted by E'. Indeed, if p is a given price vector, then

$$f : E \to \mathbb{R}, \quad E \ni x \mapsto f(x) := p \cdot x \in \mathbb{R}$$

is a linear and continuous functional on the commodity space E. The space of all linear and continuous functional on E is the dual space, denoted by E'. For the case $E = \mathbb{R}^m$, we see that $E' = \mathbb{R}^m$.

In addition to the linear structure of the commodity space, we impose a topology on E such that the linear operations of vector addition and scalar multiplication are continuous. In the finite dimensional case this enables us to show (under some additional hypotheses) that the supply and demand functions depend continuously on prices - and thus, capturing the economic intuition that a "small" change in prices results in a "small" change in demand and supply. In some other models one require that the commodity space is a topological vector space E and the price space to be the topological dual E', i.e., the space of continuous linear functionals on E. This formal duality between commodities and price was introduced by G. Debreu.

The behavioral assumption that consumers prefer more to less has important implications for equilibrium analysis. One consequence is that equilibrium prices must be positive. The natural partial ordering on $\mathbb{R}^m$ makes precise the notion that commodity bundle (vector) $\mathbf{x}$ "has more" than commodity bundle (vector) $\mathbf{y}$, i.e., $\mathbf{x} > \mathbf{y}$. The Euclidean space $\mathbb{R}^m$ together with the natural partial ordering is an ordered vector space. In this chapter, we use the natural order structure of $\mathbb{R}^m$ to formulate the notions of monotone preferences - agents who prefer more to less - and positive linear functionals - positive prices. Later, we restrict our attention to Riesz spaces (or vector lattices) as models of the commodity and the price spaces. That is, we require $\langle E, E' \rangle$ the dual pair of topological vector spaces that define the commodity and price spaces to be dual topological Riesz spaces.

1.1 Preferences and utility functions

The basic tenet of economic theory is that economic agents are rational in the sense that they know their own interests and act in a way to maximize their own welfare. This assumption is made precise by hypothesizing an opportunity set for the individual over which it is assumed that the agent can make consistent pairwise choices. One consistency requirement is that if the chooses a over b and b over c, then she will choose a over c. Formally, we suppose the opportunities comprise some (non-empty) set X and individual tastes or preferences are represented by a binary relation on X. In this section, we shall discuss the basic properties of preferences in a general setting with particular emphasis on preferences defined on subsets of finite dimensional commodity spaces.

We begin our discussion by recalling some basic properties of binary relations. Recall that a **binary relation** on a (non-empty) set X is a non-empty subset $\succeq$ of $X \times X$. The membership $(x, y) \in \succeq$ is usually written as $x \succeq y$. A binary relation $\succeq$ on a set X is said to be:

1. **Reflexive**; whenever $x \succeq x$ holds for all $x \in X$.

2. **Complete**; whenever for each pair x, y of elements of X either $x \succeq y$ or $y \succeq x$ holds.

3. **Transitive**; whenever $x \succeq y$ and $y \succeq z$ imply $x \succeq z$.

Definition 1.1.1 *A preference relation on a set is a reflexive, complete and transitive relation on the set.*

Let $\succeq$ be a preference relation on a set X. The notation $x \succeq y$ is read "the bundle (vector) x is at least as good as the bundle (vector) y" or that "x is no worse than y". The notation $x \succ y$ (read "x is preferred to y" or that "x is better than y") means that $x \succeq y$ and $y \not\succeq x$. When $x \succeq y$ and $y \succeq x$ both hold at the same time, then we write $x \sim y$ and say that "x is indifferent to y". If x is an element of X, then the set $\{y \in X : \; y \succ x\}$ is called *the better than set of* x and the set $\{y \in X : \; x \succ y\}$ is called *the worse than set of* x. Analogous names are given to the sets $\{y \in X : \; y \succ x\}$ and $\{y \in X : \; x \succ y\}$.

When X has a topological structure (i.e., X is a topological space), the continuity of preferences is defined as follows.

Definition 1.1.2 *A preference relation $\succeq$ on a topological space X is said to be*

a) **upper semicontinuous***, if for each $x \in X$ the set $\{y \in X : \; y \succeq x\}$ is closed;*

b) **lower semicontinuous**, *if for each* $x \in X$ *the set* $\{y \in X : x \succeq y\}$ *is closed; and*

c) **continuous**, *whenever* $\succeq$ *is both upper and lower semicontinuous, i.e., whenever for each* $x \in X$ *the sets*

$$\{y \in X : y \succeq x\} \quad \textit{and} \quad \{y \in X : x \succeq y\}$$

are both closed.

Since the complements of the sets $\{y \in X : y \succeq x\}$ and $\{z \in X : x \succeq z\}$ are $\{z \in X : z \succeq x\}$ and $\{y \in X : y \succeq x\}$ respectively, it should be immediate that a preference relation $\succeq$ on a topological space X is continuous if and only if for each $x \in X$ the sets

$$\{y \in X : y \succ x\} \quad \text{and} \quad \{z \in X : x \succ z\}$$

are both open.

The continuous preferences are characterized as follows.

Theorem 1.1.3 *For a preference relation* $\succeq$ *on a topological space* X *the following statements are equivalent.*

a) The preference $\succeq$ *is continuous.*

b) The preference $\succeq$ *(considered as a subset of* $X \times X$*) is closed in* $X \times X$.

c) If $x \succ y$ *holds in* X*, then there exist disjoint neighborhoods* U_x *and* U_y *of* x *and* y *respectively, such that* $a \in U_x$ *and* $b \in U_y$ *imply* $a \succ b$.

Proof. (a) $\Rightarrow$ (c). Let $x \succ y$. We have two cases.

I. There exists some $z \in X$ such that $x \succ z \succ y$. In this case, the two neighborhoods $U_x = \{a \in X : a \succ z\}$ and $U_y = \{b \in X : z \succ b\}$ satisfy the desired properties.

II. There is no $z \in X$ satisfying $x \succ z \succ y$. In this case, take $U_x = \{a \in X : \ a \succ y\}$ and $U_y = \{b \in X : \ x \succ b\}$.

(c) $\Rightarrow$ (b). Let $\{(x_\alpha, y_\alpha)\}$ be a set of $\succeq$ satisfying $(x_\alpha, y_\alpha) \to (x, y)$ in $X \times X$. If $y \succ x$ holds, then there exist two neighborhoods U_x and U_y of x and y respectively, such that $a \in U_x$ and $b \in U_y$ imply $b \succ a$. In particular, for all sufficiently large α, we must have $y_\alpha x_\alpha$, which is a contradiction. Hence, $x \succeq y$ holds, and so (x, y) belongs to $\succeq$. That is, $\succeq$ is a closed subset of $X \times X$.

(b) $\Rightarrow$ (a). Let $\{y_\alpha\}$ be a net of $\{y \in X : \ y \succeq x\}$ satisfying $y_\alpha \to z$ in X. Then the net $\{(y_\alpha, x)\}$ of $\succeq$ satisfies $(y_\alpha, x) \to (z, x)$ in $X \times X$. Since $\succeq$ is closed in $X \times X$, we see that $(z, x) \in \succeq$. Thus, $z \succeq x$ holds, proving that the set $\{y \in X : \ y \succeq x\}$ is a closed set.

In a similar fashion, we can show that the set $\{y \in X : \ x \succeq y\}$ is a closed set for each $x \in X$, and the proof of the theorem is complete. $\square$

Throughout this book we shall employ the symbol $\mathbb{R}$ to indicate the set of real numbers. Any function $u : X \to \mathbb{R}$ defines a preference relation on X by saying that

$$x \succeq y \quad \text{if and only if} \quad u(x) \geq u(y).$$

In this case $x \succ y$ is, of course, equivalent to $u(x) > u(y)$.

A function $u : X \to \mathbb{R}$ is said to be a **utility function** representing a preference relation $\succeq$ on a set X whenever $x \succeq y$ holds if and only if $u(x) \geq u(y)$. The utility functions are not uniquely determined. For instance, if a function u represents a preference relation, then so do the functions $u + c$, u^3, u^5 and e^u.

When a preference relation can be represented by a utility function ? The next theorem tells us that a very general class of preference relations can be represented by utility functions.

Theorem 1.1.4 *Every continuous preference on a topological space with a countable base of open sets can be represented by a continuous utility function.*

Convexity is used to express the behavioral assumption that the more an agent has of commodity i, the less willing she is to exchange a unit of commodity i for an additional unit of commodity i, i.e., convexity represents the notion of diminishing marginal rate of substitution. Several convexity properties of preference relation are defined next.

Definition 1.1.5 *A preference relation $\succeq$ defined on a convex subset X of a vector space is said to be:*

a) **Convex***; whenever $y \succeq x$ and $z \succeq x$ in X and $0 < \alpha < 1$ imply $\alpha y + (1 - \alpha)z \succeq x$.*

b) **Strictly convex***; whenever $y \succeq x$, $z \succeq x$ and y different than z imply $\alpha y + (1 - \alpha)z \succ x$ for all $0 < \alpha < 1$.*

It should be clear that a preference relation $\succeq$ defined on a convex set X is convex if and only if the set $\{y \in X : \ y \succeq x\}$ is convex for each $x \in X$.

A utility function that gives rise to a convex preference is referred to as a *quasiconcave function*. Similarly, a utility function that gives rise to a strictly convex preference is known as a *strictly quasi-concave function*. Their definition is as follows.

Definition 1.1.6 *A function $u : C \to \mathbb{R}$ defined on a non-empty convex subset C of a vector space is said to be:*

1. **Quasi-concave***; whenever for each* $x, y \in C$ *with* $x \neq y$ *and each* $0 < \alpha < 1$ *we have*

$$u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}.$$

2. **Strictly quasi-concave***; whenever for each pair* $x, y \in C$ *with* $x \neq y$ *and each* $0 < \alpha < 1$ *we have*

$$u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\}.$$

3. **Concave***; whenever for each* $x, y \in C$ *with* $x \neq y$ *and each* $0 < \alpha < 1$ *we have*

$$u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y).$$

4. **Strictly concave***; whenever for each* $x, y \in C$ *with* $x \neq y$ *and each* $0 < \alpha < 1$ *we have*

$$u(\alpha x + (1-\alpha)y) > \alpha u(x) + (1-\alpha)u(y).$$

The concavity properties can also be expressed in terms of convex combinations. For instance, it can be shown easily by mathematical induction that a function $u : C \to \mathbb{R}$ defined on a convex subset of vector space is quasi-concave if and only if

$$u\left(\sum_{i=1}^{n} \alpha_i x_i\right) \geq \min\{u(x_i) : \ i = 1, \ldots, n\}$$

holds for each convex combination $\sum_{i=1}^{n} \alpha_i x_i$ of elements of C. Similar statements hold true for the other concavity properties.

A function $u : C \to \mathbb{R}$ defined on a convex subset C of a vector space is said to be **convex** whenever $-u$ is concave, i.e., whenever for each $x, y \in C$ and each $0 < \alpha < 1$ we have

$$u(\alpha x + (1-\alpha)y) \leq \alpha u(x) + (1-\alpha)u(y).$$

Similarly, a function u is said to be **strictly convex** whenever $-u$ is strictly concave.

Every concave function is quasi-concave. Indeed, if $u : C \to \mathbb{R}$ is a concave function and $x, y \in C$ and $0 < \alpha < 1$, then put $m = \min\{u(x), u(y)\}$ and note that

$$u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y) \geq \alpha m + (1-\alpha)m = m.$$

The converse is false. For instance, the function $u : [0, \infty) \to \mathbb{R}$ defined by the formula $u(x) = x^2$ is quasi-concave (in fact, strictly quasi-concave) but it is not a concave function (why?). In a similar manner, we can establish that a strictly concave function is strictly quasi-concave.

The concave twice differentiable functions are precisely the ones having non-positive second derivatives. The details follow.

Theorem 1.1.7 *Let (a, b) be an open interval of $\mathbb{R}$ and let $f : (a, b) \to \mathbb{R}$ be a twice differentiable function. Then f is concave (resp. convex) if and only if $f''(x) \leq 0$ (resp. $f''(x) \geq 0$) holds for all $x \in (a, b)$.*

Proof. Assume first that $f : (a, b) \to \mathbb{R}$ is a concave function and let $x \in (a, b)$. Pick h small enough such that $x + h$ and $x - h$ both belong to (a, b). Using Taylor's second order formula we have

$$f(x+h) - f(x) = f'(x)h + \frac{1}{2}f''(x)h^2 + o(h^2)$$

and

$$f(x-h) - f(x) = -f'(x)h + \frac{1}{2}f''(x)h^2 + o(h^2).$$

Thus,

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + \frac{o(h^2)}{h^2},$$

and so

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \tag{$*$}$$

Since,

$$f(x+h) - 2f(x) + f(x-h)$$

$$= 2\left[\frac{1}{2}(x+h) + \frac{1}{2}f(x-h) - f\left(\frac{1}{2}(x+h) + \frac{1}{2}(x-h)\right)\right] \le 0$$

holds, it follows from $(*)$ that $f''(x) \le 0$.

Now assume that $f''(x) \le 0$ holds for all x. Fix s and t in (a,b) such that $s < t$, and let $0 < \alpha < 1$. Put $r = \alpha s + (1-\alpha)t$. By the Mean Value Theorem there exist ζ, ξ and τ satisfying $s < \zeta < r < \tau < t$ and $\zeta < \xi < \tau$ such that

$$\begin{aligned}
\alpha f(s) + (1-\alpha)f(t) - f(r) &= \alpha[f(s) - f(r)] + (1-\alpha)[f(t) - f(r)] \\
&= \alpha f'(\zeta)(s-r) + (1-\alpha)f'(\tau)(t-\tau) \\
&= \alpha f'(\zeta)(1-\alpha)(s-t) + (1-\alpha)f'(\tau)\alpha(t-s) \\
&= \alpha(1-\alpha)(t-s)[f'(\tau) - f'(\zeta)] \\
&= \alpha(1-\alpha)(t-s)(\tau-\zeta)f''(\xi) \le 0.
\end{aligned}$$

That is, $f(r) \ge \alpha f(s) + (1-\alpha)f(t)$ holds, which shows that f is a concave function.

The above proof also shows that if $f''(x) < 0$ holds for all $x \in (a,b)$, then f is strictly concave. $\square$

The following theorem characterizes the quasi-concave and strictly quasi-concave functions.

Theorem 1.1.8 *For a convex subset C of a vector space and a function $u : C \to \mathbb{R}$ the following statement hold.*

a) The function u is quasi-concave if and only if the preference relation defined by u is convex.

b) The function u is strictly quasi-concave if and only if the preference relation defined by u is strictly convex.

Proof. We shall prove (a) and leave the identical arguments for proving (b) to the reader. Assume first that u is a quasi-concave function and let $x \succeq y$ and $z \succeq y$ hold in C (i.e., $u(x) \geq u(y)$ and $u(z) \geq u(y)$) and let $0 < \alpha < 1$. Since u is quasi-concave, we have

$$u(\alpha x + (1-\alpha)z) \geq \min\{u(x), u(z)\} \geq u(y),$$

which means that $\alpha x + (1-\alpha)z \succeq y$.

Now assume that the preference relation defined by u is convex and let $x, y \in C$. Without loss of generality, we can suppose that $u(x) \geq u(y)$ (i.e., $x \succeq y$). From $x \succeq y$ and $y \succeq y$ and the convexity of $\succeq$, we see that $\alpha x + (1-\alpha)y \succeq y$. Therefore,

$$u(\alpha x + (1-\alpha)y) \geq u(y) = \min\{u(x), u(y)\},$$

and the proof of theorem is finished. $\square$

We now turn our attention to monotonicity of preferences. Usually, in such a case the preference is defined on a subset of a (partially) ordered vector space.

An *ordered vector space* E is a real vector space E together with an order relation $\geq$ that satisfies the following two properties connecting the algebraic and order structures.

i) If $x \geq y$ holds in E, then $x + z \geq y + z$ also holds for all $z \in E$; and

ii) If $x \geq y$ holds in E, then $\alpha x \geq \alpha y$ also holds for all $\alpha \geq 0$.

The symbol $x > y$ is used to designate that $x \geq y$ and $x \neq y$ both hold. The set $E^+ = \{x \in E : \ x \geq 0\}$ is known as the *positive cone* of E and its elements are referred to as the positive vectors.

The important example for this chapter will be the ordered vector space $E = \mathbb{R}^m$. The ordering is defined by $\mathbf{x} = (x_1, x_2, \ldots, x_m) \geq \mathbf{y} = (y_1, y_2, \ldots, y_m)$ if and only if $x_i \geq y_i$ holds for all $i = 1, 2, \ldots, m$. The positive cone of $\mathbb{R}^m$ is denoted by $\mathbb{R}^m_+$. Clearly,

$$\mathbb{R}^m_+ = \{\mathbf{x} = (x_1, x_2, \ldots, x_m) : \ x_i \geq 0 \text{ holds for all } i = 1, 2, \ldots, m\}.$$

Note that $\mathbf{x} > \mathbf{y}$ holds in $\mathbb{R}^m$ if and only if $x_i \geq y_i$ holds for all i and $x_i > y_i$ holds for at least one i.

Definition 1.1.9 *A preference relation $\succeq$ on a non-empty subset X of an ordered vector space is said to be:*

a) **Monotone***; whenever $x, y \in X$ and $x > y$ imply $x \succeq y$; and*

b) **Strictly monotone***; whenever $x, y \in X$ and $x > y$ imply $x \succ y$.*

A strictly monotone preference is clearly monotone. However, a monotone preference need not be strictly monotone. For example, consider the preference on $\mathbb{R}^2_+$ defined by the utility function $u(x, y) = xy$. Clearly, $(x_1, y_1) > (x_2, y_2)$ implies

$$u(x_1, y_1) = x_1 y_1 \geq x_2 y_2 = u(x_2, y_2).$$

On the other hand note that $(2,0) > (1,0)$ and $(2,0) \not\succ (1,0)$ hold.

The level curves of a strictly monotone quasi-concave function are "convex to the origin". Recall that a *level curve* of a function $u : C \to \mathbb{R}$ is any set of the form $\{x \in C : \ u(x) = c\}$, where c is any fixed real number - in economics the level curves are known, of course, as **indifference curves**. Intuitively, a curve is said to be "convex to the origin" whenever its graph has the shape shown in Figure ??. Mathematically, a "convex to the origin" curve is described by saying that if A and B are any two points on the curve, then a ray passing through the origin O and any point X of the line segment AB will meet the curve at most at one point D between O and X. The notion of diminishing marginal rate of substitution is clearly seen by observing the slopes at points A and B.

Theorem 1.1.10 *Let $u : C \to \mathbb{R}$ be a function defined on a convex subset C of the positive cone of some ordered vector space. If u is strictly monotone and quasi-concave, then its level curves are convex to the origin.*

Proof. Assume that $x, y \in C$ satisfy $u(x) = u(y) = c$ and let $z = \alpha x + (1-\alpha)y$ for some $0 < \alpha < 1$. Since u is quasi-concave, we see that

$$u(z) \geq \min\{u(x), u(y)\} = c.$$

Since u is strictly monotone, we see that the ray $\{\lambda z : \ \lambda \geq 0\}$ cannot meet the level set $\{a \in C : \ u(a) = c\}$ at any point outside the line segment joining 0 and z. This shows that the level curves of u are convex to the origin. $\square$

We continue our discussion with the introduction of the extremely desirable bundles (vectors).

Definition 1.1.11 *Let $\succeq$ be a preference relation defined on a subset X of a vector space E. Then a vector $v \in E$ is said to be an* **extremely desirable bundle** *(or* **vector***) for $\succeq$ whenever*

1. $x + \alpha v \in X$ holds for all $x \in X$ and $\alpha > 0$; and

2. $x + \alpha v \succ x$ holds for all $x \in X$ and $\alpha > 0$.

Note that if $v > 0$ is an extremely desirable bundle (vector) then so is λv for each $\lambda > 0$. It was mentioned before that quite often preferences are represented by utility functions. The next theorem is an important representation theorem for preferences defined on the positive cone of a finite dimensional vector space.

Theorem 1.1.12 *For a continuous preference $\succeq$ defined on the positive cone $\mathbb{R}^m_+$ of some $\mathbb{R}^m$ the following statements hold:*

1. If $\succeq$ is convex, monotone with an extremely desirable bundle (vector), then $\succeq$ can be represented by a continuous, monotone and quasi-concave utility function.

2. If $\succeq$ is strictly convex and strictly monotone with an extremely desirable bundle (vector), then $\succeq$ can be represented by a continuous, strictly monotone and strictly quasi-concave utility function.

Proof. We shall prove (1) and leave the identical proof of part (2) for the reader. So, let $\succeq$ be a continuous, convex and monotone preference relation having an extremely desirable bundle (vector) v. Replacing v by $e = v + (1, 1, \ldots, 1)$, we see (by the monotonicity of $\succeq$) that e is also extremely desirable. Thus, we can assume that there exists an extremely desirable bundle (vector) $e = (e_1, e_2, \ldots, e_m)$ satisfying $e_i > 0$ for each i.

Now for each $x \in \mathbb{R}^m_+$, we put

$$u(x) = \inf\{\alpha > 0 : \ \alpha e \succeq x\}.$$

Since all components of e are positive, there exists some $\alpha > 0$ such that $\alpha e > x$, and so by the monotonicity $\alpha e \succeq x$ must hold for some $\alpha > 0$. Thus, $u(x)$ is well defined.

We claim that $x \sim u(x)e$. Since the set $\{y \in \mathbb{R}^m_+ : \ y \succeq x\}$ is closed, it easily follows that $u(x)e \succeq x$ holds. On the other hand, if $u(x) > 0$, then for all $\varepsilon > 0$ sufficiently small we must have $x \succeq (u(x) - \varepsilon)e$, and so by letting $\varepsilon \to 0$, we see that $x \succeq u(x)e$ also holds. Consequently, if $u(x) > 0$, then $u(x)e \sim x$. If $u(x) = 0$, then from $x \geq 0$ and the monotonicity of $\succeq$, we infer that $x \succeq 0 = u(x)e$. That is, $x \sim u(x)e$ is also true in this case.

Now observe that if $\alpha \geq 0$ and $\beta \geq 0$, then $\alpha e \succeq \beta e$ if and only if $\alpha \geq \beta$. Indeed, if $\alpha e \succeq \beta e$, then $\beta > \alpha$ implies $\beta e = \alpha e + (\beta - \alpha)e \succeq \alpha e$, which is impossible. In particular, the above show that for each x in $\mathbb{R}^m_+$ there exists exactly one scalar - the number $u(x)$ - such that $x \sim u(x)e$.

Now it should be clear that the function $u : \mathbb{R}^m_+ \to \mathbb{R}$ defined above is a utility function representing $\succeq$. The continuity of u follows from the identities

$$\{x \in \mathbb{R}^m_+ : \ u(x) \leq r\} = \{x \in \mathbb{R}^m_+ : \ x \preceq re\}$$

and

$$\{x \in \mathbb{R}^m_+ : \ u(x) \geq r\} = \{x \in \mathbb{R}^m_+ : \ x \succeq re\}$$

and the continuity of $\succeq$. $\square$