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DISCUSSION OF HYBRID JS-CONTRACTIONS IN b-METRIC SPACES WITH APPLICATIONS TO THE EXISTENCE OF SOLUTIONS FOR INTEGRAL EQUATIONS

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Abstract. The purpose of this work is to introduce the new concept of a hybrid JS-contraction in b-metric spaces which includes various contractions in such spaces. The existence and uniqueness of a fixed point for self mappings satisfying the proposed contractive condition on b-metric spaces are studied. As an application, the existence result of a unique solution for integral equations is also given in order to illustrate the effectiveness of the obtained results.
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1. Introduction

The Banach contraction principle is one of famous fixed point theorems and appeared in the explicit form in Banach's thesis in 1922. This principle was used to establish the existence and uniqueness of a solution for an integral equation. So far, according to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle.

In 1971, Ćirić [5, 6] defined and investigated a class of generalized contractions, which includes the Banach's contractions as follows:

Definition 1.1 ([5]). Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be a Ćirić contraction if the following condition holds:

$$d(fx, fy) \leq td(x, y) + ud(x, fx) + vd(y, fy) + w[d(x, fy) + d(y, fx)]$$

for all $x, y \in X$, where t, u, v, w are nonnegative real numbers with $t+u+v+2w < 1$.

Definition 1.2 ([6]). Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be a quasi-contraction if there exists $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq \lambda M(x, y)$$

for all $x, y \in X$, where

$$M(x,y) := \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\Big\}.$$

In 2014, Jleli and Samet [9] introduced a new type of the contractive condition and established a new fixed point theorem for such mappings on the setting of generalized metric spaces as follows:

Definition 1.3 ([9]). Let (X, d) be a generalized metric space. A mapping $f : X \to X$ is said to be a JS-contraction if there exists $\lambda \in (0, 1)$ such that

$$\psi(d(fx, fy)) \leq [\psi(d(x, y))]^{\lambda}$$

for all $x, y \in X$ with $fx \neq fy$, where $\psi : (0, \infty) \to (1, \infty)$ is a nondecreasing function satisfying the following conditions:

- for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
- there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = l$.

Theorem 1.4 ([9]). *Let (X, d) be a complete generalized metric space and $f : X \to X$ be a JS-contraction mapping. Then f has a unique fixed point.*

It is well-known that each metric space is also a generalized metric space (see [9]). So they obtained the following result.

Theorem 1.5 (Corollary 2.1 in [8]). *Let (X, d) be a complete metric space and $f : X \to X$ be a JS-contraction mapping. Then f has a unique fixed point.*

On the other hand, Hussain et al. [8] introduced a new generalization of the Banach contraction and so they obtained sufficient conditions for the existence of a fixed point for such mappings on complete metric spaces as follows:

Definition 1.6 ([8]). Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be a JS-Ćirić contraction mapping if the following condition holds:

$$\psi(d(fx, fy)) \leq [\psi(d(x, y))]^t [\psi(d(x, fx))]^u [\psi(d(y, fy))]^v [\psi(d(x, fy) + d(y, fx))]^w$$

for all $x, y \in X$, where t, u, v, w are nonnegative real numbers with $t + u + v + 2w < 1$ and $\psi : [0, \infty) \to [1, \infty)$ is a nondecreasing function satisfying the following conditions:

(Ψ_1) $\psi(t) = 1$ if and only if $t = 0$;

(Ψ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

(Ψ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = l$;

(Ψ_4) $\psi(t + s) \leq \psi(t)\psi(s)$ for all $t, s > 0$.

Theorem 1.7 ([8]). *Let (X, d) be a complete metric space and $f : X \to X$ be a continuous JS-Ćirić contraction mapping. Then f has a unique fixed point.*

On the other hand, the concept of a b-metric space was introduced by Bakhtin [2] and then extensively used by Czerwik [7]. One of the special case of b-metric space is a metric space. This is due to the fact that every metric space is a b-metric space. However, a b-metric space does not necessarily to be a metric space. There exist many examples in the literature showing that the class of b-metrics is effectively larger than that of metric spaces. To avoid the repetition, we refer the same terminology, notations and basic facts about b-metric spaces as having been utilized in [2, 7]. For more details about the definitions, well-known examples of b-metric spaces, the concepts of b-convergence, b-Cauchy sequence and b-completeness in b-metric spaces, one can also refer to [4, 11]. In recent years, a number of fixed point results in b-metric spaces have been studied extensively in [3, 13, 12, 14] and references therein.

In this paper, we introduce the new concept of a hybrid JS-contraction in b-metric spaces and prove some fixed point results for self mappings satisfying this contractive condition on b-metric spaces. Our main results generalize, extend and improve the corresponding results on the topics given in the literature. The illustrative example is furnished which demonstrates the validity of the hypotheses and degree of utility of our results. We also show that many fixed point results for several contraction mappings in b-metric spaces can be obtained from our main results. Finally, we investigate the existence and uniqueness result of a solution for the linear/nonlinear integral equations by using our main results.

2. Main results

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, non-negative real numbers and real numbers, respectively. In the sequel, we use the following notion to prove the fixed point theorems in our main results.

Definition 2.1. Let f be a self mapping on a nonempty set X. A point x in X is said to be a periodic point of f if and only if

$$f^n x = x$$

for some $n \in \mathbb{N}$.

Let (X, d) be a b-metric space with the coefficient $s \geq 1$ and $f : X \to X$ be a self-mapping. Throughout this paper, unless otherwise stated, for all $x, y \in X$, let

$$M_s(x,y) := \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2s}\Big\}. \tag{2.1}$$

Here, we first introduce the new contractive condition in b-metric spaces which is called a hybrid JS-contraction as follows:

Definition 2.2. Let (X, d) be a b-metric space with the coefficient $s \geq 1$. A mapping $f : X \to X$ is called a hybrid JS-contraction if there exists a function $\psi : [0,\infty) \to [1,\infty)$ such that

$$\psi(s^3 d(fx, fy)) \leq [\psi(M_s(x,y))]^\lambda \tag{2.2}$$

for all $x, y \in X$ with $fx \neq fy$, where $0 < \lambda < 1$ and ψ is a nondecreasing continuous function with $\psi(t) = 1$ if and only if $t = 0$.

Theorem 2.3. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and $f : X \to X$ be a continuous hybrid JS-contraction mapping. Then f has a unique fixed point in X.*

Proof. First of all, we will show that f has a periodic point. Suppose this to contrary that f does not have a periodic point. Let x_0 be a fixed element in X. Then we get

$$f^p x_0 \neq x_0 \tag{2.3}$$

for all $p \in \mathbb{N}$. This implies that

$$f^m x_0 \neq f^n x_0 \tag{2.4}$$

for all $m, n \in \mathbb{N} \cup \{0\}$ with $m \neq n$. Construct a sequence $\{x_n\}$ in X such that $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. From (2.4), we get

$$d(x_m, x_n) > 0 \tag{2.5}$$

for all $m, n \in \mathbb{N} \cup \{0\}$ with $m \neq n$. From (2.2) and (2.5), we have

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \psi(s^3 d(x_n, x_{n+1})) \\ &\leq [\psi(M_s(x_{n-1}, x_n))]^\lambda \end{aligned} \tag{2.6}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} &M_s(x_{n-1}, x_n) \\ &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2s}\right\} \\ &= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}. \end{aligned}$$

If $M_s(x_{n^*-1}, x_{n^*}) = d(x_{n^*}, x_{n^*+1})$ for some $n^* \in \mathbb{N}$, then

$$\psi(d(x_{n^*}, x_{n^*+1})) \leq [\psi(M_s(x_{n^*-1}, x_{n^*}))]^\lambda = [\psi(d(x_{n^*}, x_{n^*+1}))]^\lambda < \psi(d(x_{n^*}, x_{n^*+1})),$$

which is a contradiction. Consequently, we get $M_s(x_{n-1}, x_n) = d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. From (2.6), we obtain

$$\psi(d(x_n, x_{n+1})) \leq [\psi(M_s(x_{n-1}, x_n))]^\lambda = [\psi(d(x_{n-1}, x_n))]^\lambda < \psi(d(x_{n-1}, x_n)).$$

Since ψ is a nondecreasing function, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded from below. Then, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Taking the limit as $n \to \infty$ in (2.6), we obtain

$$\psi(r) \leq \psi(s^3 r) \leq [\psi(r)]^\lambda.$$

Since $0 < \lambda < 1$, we have $\psi(r) = 1$ and hence $r = 0$. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

Next, we will prove that $\{x_n\}$ is a b-Cauchy sequence in X. Assume this to contrary that there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) \geq k$ and

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon. \tag{2.8}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ satisfying (2.8). It implies that

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{2.9}$$

From (2.8), (2.9) and the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\ &\leq s[d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})] \\ &< s[\epsilon + d(x_{n(k)-1}, x_{n(k)})]. \end{aligned} \tag{2.10}$$

Taking the limit superior as $k \to \infty$ in (2.10) and using (2.7), we obtain

$$\epsilon \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq s\epsilon. \tag{2.11}$$

From the triangle inequality, we get

$$d(x_{m(k)}, x_{n(k)}) \leq s[d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)})] \tag{2.12}$$

and

$$d(x_{m(k)+1}, x_{n(k)}) \leq s[d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)})]. \tag{2.13}$$

Taking the limit superior as $k \to \infty$ in (2.12) and (2.13), it follows from (2.7) and (2.11) that

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) \quad \text{and} \quad \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) \leq s^2\epsilon,$$

which implies that

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) \leq s^2\epsilon. \tag{2.14}$$

Using the above process again, we have

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{n(k)+1}, x_{m(k)}) \leq s^2\epsilon. \tag{2.15}$$

Finally, we can see that

$$d(x_{m(k)}, x_{n(k)+1}) \leq s[d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1})]. \tag{2.16}$$

Taking the limit superior as $k \to \infty$ in (2.16), it follows from (2.7) and (2.15) that

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}). \tag{2.17}$$

Similarly, we obtain

$$\limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq s^3\epsilon. \tag{2.18}$$

It follows from (2.17) and (2.18) that

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq s^3\epsilon. \tag{2.19}$$

From (2.2), we have

$$\psi(s^3 d(x_{m(k)+1}, x_{n(k)+1})) = \psi(s^3 d(fx_{m(k)}, fx_{n(k)})) \leq [\psi(M_s(x_{m(k)}, x_{n(k)}))]^\lambda, \tag{2.20}$$

where

$$\begin{aligned} M_s(x_{m(k)}, x_{n(k)}) &= \max\Big\{d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, fx_{m(k)}), d(x_{n(k)}, fx_{n(k)}), \\ &\qquad \frac{d(x_{m(k)}, fx_{n(k)}) + d(x_{n(k)}, fx_{m(k)})}{2s}\Big\} \\ &= \max\Big\{d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}), \\ &\qquad \frac{d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})}{2s}\Big\}. \end{aligned}$$

Taking the limit superior as $k \to \infty$ in the above equation and using (2.7), (2.11), (2.14) and (2.15), we have

$$\epsilon = \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}\right\} \leq \limsup_{k\to\infty} M_s(x_{m(k)}, x_{n(k)}) \leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon.$$

Taking the limit superior as $k \to \infty$ in (2.20), by using (2.19) and the continuity of ψ, we get

$$\begin{aligned} \psi(s\epsilon) &= \psi\Big(s^3\Big(\frac{\epsilon}{s^2}\Big)\Big) \\ &\leq \psi\Big(s^3 \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1})\Big) \\ &\leq \Big[\psi\Big(\limsup_{k\to\infty} M_s(x_{m(k)}, x_{n(k)})\Big)\Big]^\lambda \\ &\leq [\psi(s\epsilon)]^\lambda. \end{aligned} \tag{2.21}$$

Since $0 < \lambda < 1$, we have $\psi(s\epsilon) = 1$ and so $\epsilon = 0$, which is a contradiction. Therefore, $\{x_n\}$ is a b-Cauchy sequence in X. By the completeness of a b-metric space X, there exists $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

and hence

$$\lim_{n\to\infty} d(fx_n, x) = \lim_{n\to\infty} d(x_{n+1}, x) = 0. \tag{2.22}$$

Since f is a continuous mapping, we obtain

$$\lim_{n\to\infty} d(fx_n, fx) = 0.$$

From the triangle inequality, we have

$$d(x, fx) \leq s[d(x, fx_n) + d(fx_n, fx)] \tag{2.23}$$

for all $n \in \mathbb{N}$. Letting limit as $n \to \infty$ in above inequality, we get

$$d(x, fx) = 0.$$

This contradicts with (2.3). Therefore, f has a periodic point. Then we have

$$f^p x = x \tag{2.24}$$

for some $p \in \mathbb{N}$ and for some $x \in X$.

Next, we will claim that f has a fixed point. From (2.24), if $p = 1$, then it is easy to see that x is a fixed point of f. In another way, we may assume that $p > 1$

and then we will show that $f^p x$ is a fixed point of f. Suppose this to contrary that $f^{p+1}x \neq f^p x$. From (2.2), we have

$$\begin{aligned} \psi(d(f^p x, f^{p+1}x)) &\leq \psi(s^3 d(f^p x, f^{p+1}x)) \\ &\leq [\psi(M_s(f^{p-1}x, f^p x))]^\lambda, \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} M_s(f^{p-1}x, f^p x) &= \max\Big\{d(f^{p-1}x, f^p x), d(f^{p-1}x, f^p x), d(f^p x, f^{p+1}x), \\ &\qquad \frac{d(f^{p-1}x, f^{p+1}x) + d(f^p x, f^p x)}{2s}\Big\} \\ &= \max\left\{d(f^{p-1}x, f^p x), d(f^p x, f^{p+1}x)\right\}. \end{aligned}$$

If $M_s(f^{p-1}x, f^p x) = d(f^p x, f^{p+1}x)$, then

$$\begin{aligned} \psi(d(f^p x, f^{p+1}x)) &\leq [\psi(M_s(f^{p-1}x, f^p x))]^\lambda \\ &= [\psi(d(f^p x, f^{p+1}x))]^\lambda \\ &< \psi(d(f^p x, f^{p+1}x)), \end{aligned}$$

which is a contradiction. Therefore, we obtain

$$M_s(f^{p-1}x, f^p x) = d(f^{p-1}x, f^p x).$$

It is impossible that

$$d(f^{p-1}x, f^p x) = 0.$$

By using (2.25), we see that

$$\begin{aligned} \psi(d(f^p x, f^{p+1}x)) &\leq [\psi(M_s(f^{p-1}x, f^p x))]^\lambda \\ &= [\psi(d(f^{p-1}x, f^p x))]^\lambda \\ &< \psi(d(f^{p-1}x, f^p x)). \end{aligned}$$

From above inequality, we have

$$d(f^p x, f^{p+1}x) < d(f^{p-1}x, f^p x).$$

By repeating this process, we get

$$d(f^q x, f^{q+1}x) < d(f^{q-1}x, f^q x). \tag{2.26}$$

for all $q \in \mathbb{N}$ with $q \leq p$. From (2.24) and (2.26), we get

$$d(x, fx) = d(f^p x, f^{p+1}x) < d(f^{p-1}x, f^p x) < \ldots < d(fx, f^2 x) < d(x, fx),$$

which is a contradiction. Therefore, $f^p x$ is a fixed point of f.

Finally, we will show that f has a unique fixed point. Let y be a distinct fixed point of f. From (2.2), we have

$$\begin{aligned}
\psi(d(x,y)) &= \psi(d(fx,fy)) \\
&\leq \psi(s^3 d(fx,fy)) \\
&\leq [\psi(M_s(x,y))]^\lambda \\
&= \left[\psi\left(\max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2s}\right\}\right)\right]^\lambda \\
&= [\psi\left(d(x,y)\right)]^\lambda \\
&< [\psi\left(d(x,y)\right)],
\end{aligned}$$

which is a contradiction. Therefore, $x = y$ and hence f has a unique fixed point. The proof is completed.

Example 2.4. Let $X = [0,1]$. Define the mapping $d : X \times X \to [0,\infty)$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

Clearly, (X,d) is a complete b-metric space with the coefficient $s = 2$. Define a mapping $f : X \to X$ and a function $\psi : [0,\infty) \to [1,\infty)$ by

$$fx = \frac{x^4}{3}$$

for all $x \in X$ and

$$\psi(t) = e^{te^t}$$

for all $t \in [0,\infty)$. It is easy to see that ψ is a nondecreasing continuous function and $\psi(t) = 1$ if and only if $t = 0$. Next, we will show that f satisfies the condition (2.2). Let $x, y \in X$ with $fx \neq fy$. Without loss of generality, we may assume that $x < y$. Then we have

$$\begin{aligned}
\psi(s^3 d(fx,fy)) = e^{\frac{8\left(x^4+y^4\right)^2}{9} e^{\frac{8\left(x^4+y^4\right)^2}{9}}} &\leq \left(e^{(x+y)^2 e^{(x+y)^2}}\right)^{\frac{8}{9}} \\
&\leq \left(e^{M_s(x,y) e^{M_s(x,y)}}\right)^{\frac{8}{9}} = [\psi(M_s(x,y))]^{\frac{8}{9}}. \qquad (2.27)
\end{aligned}$$

It yields that f is a hybrid JS-contraction mapping with $\lambda = \frac{8}{9} \in (0,1)$. Now, all hypotheses in Theorem 2.3 hold. So we can conclude that f has a unique fixed point. In this case, 0 is a unique fixed point of f.

Remark 2.5. The Banach contraction mapping principle with the usual metric d can not be applied in Example 2.4. Indeed, for $x = 1$ and $y = 0.9$, we get

$$d(fx,fy) = d(f(1), f(0.9)) = \frac{0.3439}{3} > 0.1 = d(1,0.9) = d(x,y) \geq kd(x,y)$$

for all $k \in [0,1)$.

Theorem 2.6. *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$ and $f : X \to X$ be a given mapping. Assume that there exist a nondecreasing continuous* *function* $\psi : [0, \infty) \to [1, \infty)$ *with* $\psi(t) = 1$ *if and only if* $t = 0$ *and nonnegative real number* t, u, v, w *with* $t + u + v + 2w < 1$ *and*

$$\psi(s^3 d(fx, fy)) \leq [\psi(d(x, y))]^t [\psi(d(x, fx))]^u [\psi(d(y, fy))]^v \left[\psi\left(\frac{d(x, fy) + d(y, fx)}{2s}\right)\right]^{2w} \tag{2.28}$$

for all $x, y \in X$. *Then* f *has a unique fixed point in* X.

Proof. We will divide this proof into two cases.

Case I: Suppose that $t + u + v + 2w = 0$. By (2.28), we have

$$\psi(s^3 d(fx, fy)) = 1$$

and so

$$d(fx, fy) = 0$$

for all $x, y \in X$. It implies that f is a constant function. Hence, f has a unique fixed point.

Case II: Suppose that $t + u + v + 2w \in (0, 1)$. It is easy to see that f is a hybrid JS-contraction mapping with $\lambda := t + u + v + 2w \in (0, 1)$. By Theorem 2.3, f has a unique fixed point. The proof is completed.

Next, we will show that Theorem 2.3 and Theorem 2.6 can be obtained the several fixed point results for various kinds of contractive conditions in b-metric spaces.

Corollary 2.7. *Let* (X, d) *be a complete b-metric space with coefficient* $s \geq 1$ *and* $f : X \to X$ *be an s-Ćirić-contraction mapping, that is,*

$$s^3 d(fx, fy) \leq t d(x, y) + u d(x, fx) + v d(y, fy) + w \left[\frac{d(x, fy) + d(y, fx)}{s}\right] \tag{2.29}$$

for all $x, y \in X$, *where* t, u, v, w *are nonnegative real numbers with* $t + u + v + 2w < 1$. *Then* f *has a unique fixed point in* X.

Proof. Defining a function $\psi : [0, \infty) \to [1, \infty)$ by $\psi(t) = e^t$ for all $t \geq 0$, we obtain f satisfies the contractive condition (2.28) with $\lambda := t + u + v + 2w \in [0, 1)$. By using Theorem 2.6, we get this result.

Corollary 2.8. *Let* (X, d) *be a complete b-metric space with coefficient* $s \geq 1$ *and* $f : X \to X$ *be an s-quasi-contraction mapping, that is,*

$$s^3 d(fx, fy) \leq \lambda M_s(x, y) \tag{2.30}$$

for all $x, y \in X$, *where* $\lambda \in [0, 1)$. *Then* f *has a unique fixed point in* X.

Proof. If $\lambda = 0$, we get f is a constant mapping and hence it has a unique fixed point. On the other hand, we may assume that $\lambda \in (0, 1)$. Defining a function $\psi : [0, \infty) \to [1, \infty)$ by $\psi(t) = e^t$ for all $t \geq 0$, we obtain that f is a hybrid JS-contraction mapping with $\lambda \in (0, 1)$. By using Theorem 2.3, we get this result.

Corollary 2.9. *Let* (X, d) *be a complete b-metric space with coefficient* $s \geq 1$ *and* $f : X \to X$ *be an s-JS-contraction mapping, that is,*

$$\psi(s^3 d(fx, fy)) \leq [\psi(d(x, y))]^\lambda \tag{2.31}$$

for all $x, y \in X$ *with* $fx \neq fy$, *where* $\lambda \in (0, 1)$ *and* $\psi : [0, \infty) \to [1, \infty)$ *is a nondecreasing continuous function with* $\psi(t) = 1$ *if and only if* $t = 0$. *Then* f *has a unique fixed point in* X.

Proof. It is easy to see that the contractive condition (2.31) implies the contractive condition (2.2). Therefore, f is a hybrid JS-contraction mapping with $\lambda \in (0,1)$. By using Theorem 2.3, we get this result.

Corollary 2.10. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be an s-JS-Ćirić contraction mapping, that is,*

$$\psi(s^3 d(fx,fy)) \leq [\psi(d(x,y))]^t [\psi(d(x,fx))]^u [\psi(d(y,fy))]^v \left[\psi\left(\frac{d(x,fy)+d(y,fx)}{s}\right)\right]^w \tag{2.32}$$

for all $x,y \in X$, where t,u,v,w are nonnegative real numbers with $t+u+v+2w<1$ and $\psi : [0,\infty) \to [1,\infty)$ is a nondecreasing function satisfying the following conditions:

(Ψ_1) *$\psi(t)=1$ if and only if $t=0$;*

(Ψ_2) *for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;*

(Ψ_1) *there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = l$;*

(Ψ_4) *$\psi(t+s) \leq \psi(t)\psi(s)$ for all $t,s>0$.*

Then f has a unique fixed point in X.

Proof. Since ψ is a nondecreasing function satisfying (Ψ_4), we have

$$\left[\psi\left(\frac{d(x,fy)+d(y,fx)}{s}\right)\right]^w \leq \left[\psi\left(\frac{d(x,fy)+d(y,fx)}{2s}\right)\right]^{2w}$$

for all $x,y \in X$.
Therefore, the contractive condition (2.28) holds with $\lambda := t+u+v+2w \in [0,1)$. By using Theorem 2.6, we get this result.

Remark 2.11. From Corollaries 2.7, 2.8, 2.9 and 2.10, we obtain the following results:

- fixed point results of Ćirić in [5] provided that f is a 1-Ćirić-contraction mapping;
- fixed point results of Ćirić in [6] provided that f is a 1-quasi-contraction mapping;
- fixed point result of Jleli and Samet in [8, Corollary 2.1] (Theorem 1.5) provided that f is a 1-JS-contraction mapping;
- fixed point result of Hussain et al. in [8] (Theorem 1.7) provided that f is a 1-JS-Ćirić contraction mapping.

The reader can see the relation between hybrid JS-contractions and various kind of contractive conditions in the Figure 1.

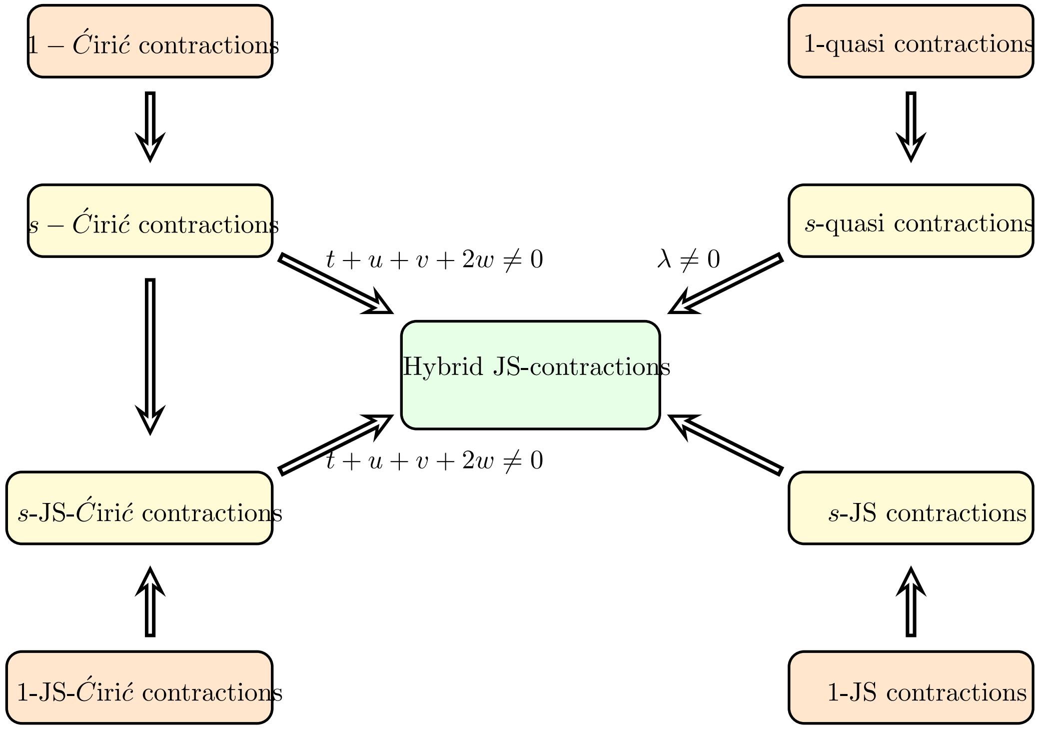

FIGURE 1. Relation between hybrid JS-contractions and various kind of contractive conditions

3. APPLICATIONS TO LINEAR/NONLINEAR INTEGRAL EQUATIONS

In this section, we present the existence and uniqueness result of a solution for linear/nonlinear integral equations by using the result in the previous section.

Theorem 3.1. *Consider the following linear/nonlinear integral equation:*

$$x(t) = \int_a^b K(t,r)g(r,x(r))dr, \tag{3.1}$$

where $a, b \in \mathbb{R}$ with $a < b$, $x \in C[a,b]$ (the set of all continuous real value functions defined on $[a,b]$) and $K : [a,b] \times [a,b] \to [0,\infty)$ and $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ are given mappings. Assume that the following conditions hold:

(A_1) $K : [a,b] \times [a,b] \to [0,\infty)$ and $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous;

(A_2) there exists $p > 1$ such that

$$\max_{a \le t \le b} \left(\int_a^b |K(t,r)|^{\frac{p}{p-1}} dr \right) \quad \le \quad \frac{1}{16(b-a)}$$

and for each $x, y \in C[a,b]$ and $r, t \in [a,b]$, we have

$$\psi\left(\frac{2^{3-p}M_s(x,y)}{p^2}\right) \le [\psi(M_s(x,y))]^{\lambda} \tag{3.2}$$

and

$$p^2|g(r,x(r)) - g(r,y(r))|^p \le 2\xi(t,r)M_s(x,y), \tag{3.3}$$

where $\lambda \in (0,1)$, $\psi : [0,\infty) \to [1,\infty)$ is a nondecreasing continuous function such that $\psi(t) = 1$ if and only if $t = 0$, $\xi : [a,b] \times [a,b] \to [0,\infty)$ is a continuous function such that

$$\max_{t\in[a,b]}\Big(\int_a^b \xi(t,r)dr\Big) \le 2(b-a)^{p-1},$$

and $M_s(x,y)$ is defined as (2.1) with the b-metric d on $C[a,b]$ which is given by

$$d(u,v) = \max_{t\in[a,b]}|u(t) - v(t)|^p$$

for all $u, v \in C[a,b]$. Then the integral equation (3.1) *has a unique solution.*

Proof. Let $X = C[a,b]$ and $f : X \to X$ be a mapping which is defined for each $x \in X$ by

$$(fx)(t) = \int_a^b K(t,r)g(r,x(r))dr \quad \text{for all } t \in [a,b].$$

It is easy to see that a function x is a unique solution of (3.1) if and only if it is a unique fixed point of the mapping f. Moreover, (X,d) is a complete b-metric space with coefficient $s = 2^{p-1}$.

Next, we will show that f is a hybrid JS-contraction mapping. Let $0 < q < 1$ with $\frac{1}{p} + \dfrac{1}{q} = 1$ and let $x, y \in X$. From condition (A_2), for each $t \in [a,b]$, we have

$$\begin{aligned}
&\psi\left(2^{3p-3}|(fx)(t) - (fy)(t)|^p\right)\\
&= \psi\left(2^{3p-3}\left[\int_a^b |K(t,r)g(r,x(r)) - K(t,r)g(r,y(r))|\,dr\right]^p\right)\\
&= \psi\left(2^{3p-3}\left[\int_a^b |K(t,r)|\,|g(r,x(r)) - g(r,y(r))|\,dr\right]^p\right)\\
&\le \psi\left(2^{3p-3}\left[\left(\int_a^b |K(t,r)|^q\,dr\right)^{\frac{1}{q}}\left(\int_a^b |g(r,x(r)) - g(r,y(r))|^p\,dr\right)^{\frac{1}{p}}\right]^p\right)\\
&\le \psi\left(2^{3p-3}\left(\max_{a\le t\le b}\int_a^b |K(t,r)|^{\frac{p}{p-1}}\,dr\right)^{\frac{p}{q}}\left(\int_a^b |g(r,x(r)) - g(r,y(r))|^p\,dr\right)\right)\\
&\le \psi\left(2^{3p-3}\left(\frac{1}{16(b-a)}\right)^{p-1}\left(\int_a^b |g(r,x(r)) - g(r,y(r))|^p\,dr\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \psi\left(\frac{1}{(2(b-a))^{p-1}}\int_a^b \frac{2\xi(t,r)M_s(x,y)}{p^2}dr\right)\\
&= \psi\left(\frac{2^{2-p}M_s(x,y)}{p^2(b-a)^{p-1}}\int_a^b \xi(t,r)dr\right)\\
&\leq \psi\left(\frac{2^{3-p}M_s(x,y)}{p^2}\right)\\
&\leq [\psi\left(M_s(x,y)\right)]^{\lambda}.
\end{aligned}
$$

This implies that

$$\max_{t\in[a,b]}\psi\left(s^3|(fx)(t)-(fy)(t)|^p\right) \leq [\psi\left(M_s(x,y)\right)]^{\lambda}.$$

It follows that

$$
\begin{aligned}
\psi(s^3 d(fx,fy)) &= \psi\left(s^3\max_{t\in[a,b]}|(fx)(t)-(fy)(t)|^p\right)\\
&\leq \max_{t\in[a,b]}\psi\left(s^3|(fx)(t)-(fy)(t)|^p\right)\\
&\leq [\psi\left(M_s(x,y)\right)]^{\lambda}.
\end{aligned}
$$

Therefore, the condition (2.2) holds. By using Theorem 2.3, there exists $\widehat{x}\in X$ which is a unique fixed point of f. Hence $\widehat{x}$ is a unique solution for the nonlinear integral equation (3.1). This completes the proof.

4. Conclusions

The concept of hybrid JS-contractions in b-metric spaces was first introduced in this work. Based on this concept, we studied the existence and uniqueness results of a fixed point for mappings satisfying the hybrid JS-contractive condition in b-metric spaces. We also shown that our main results are generalization of many fixed point results for mappings satisfying various contractive conditions in b-metric spaces. Moreover, fixed point results of Ćirić in [5, 6], the fixed point result of Jleli and Samet in [8, Corollary 2.1] (Theorem 1.5) and the fixed point result of Hussain et al. in [8] (Theorem 1.7) are special cases of our results in this work. Finally, we obtained the existence and uniqueness result of the solution for nonlinear integral equations under some suitable condition from the fixed point result.

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