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GEOMETRICAL PROPERTIES OF l_p SPACES

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Abstract. In this work, some geometrical properties of Hilbert spaces are investigated in l_p spaces, for $p \geq 2$. As an application, we obtain an extension of the Banach Contraction Principle for best proximity points. The case of nonexpansive mappings is also discussed.
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1. Introduction

If the fixed point equation $Tx = x$ of a given mapping T does not have a solution, then it is of interest to find an approximate solution for the fixed point equation. In other words, we are searching for an element in the domain of the mapping, whose image is as close to it as possible. This situation motivates to develop the notion called best proximity point theory (see, [1, 5, 4, 6, 9, 10, 11, 12, 13]).

Recently, based on geometrical properties of Hilbert spaces, Raj [11] introduced the so-called P-property. Using this property, some best proximity point results were proved for various classes of non-self mappings in Banach and metric spaces [1, 11].

In this work, we investigate the P-property in l_p spaces, for $p \geq 2$. As an application, we obtain an extension of the Banach Contraction Principle for best proximity points. The case of nonexpansive mappings is also discussed.

2. The P-property and l_p spaces

Let A, B be nonempty subsets of a metric space (M, d). Then the proximity pair associated with the pair (A, B), denoted by (A_0, B_0), is defined by

$$A_0 = \{x \in A : d(x, y) = d(A, B); \text{ for some } y \in B\},$$

and

$$B_0 = \{y \in B : d(x, y) = d(A, B); \text{ for some } x \in A\},$$

where $d(A, B) = \inf\{d(x, y); \ (x, y) \in A \times B\}$.

Definition 2.1. [11] A pair (A, B) of nonempty subsets of a metric space (M, d), with $A_0 \neq \emptyset$, is said to have the P-property if and only if

$$\left.\begin{array}{l} d(a,b) = d(A,B) \\ d(x,y) = d(A,B) \end{array}\right\} \Longrightarrow d(a,x) = d(b,y),$$

whenever $a, x \in A_0$ and $b, y \in B_0$.

It is clear that A_0 is not empty if and only if B_0 is not empty. As an example of a metric space where the P-property holds, one may consider any pair of closed convex bounded subsets of a real Hilbert vector space [11].

The attention was focused on proximinal sets and convexity while studying the P-property. Therefore, trying to investigate this property, one has to consider the nearest point projections.

Let (M, d) be a metric space. Let C be a nonempty subset of M. Define the nearest point projection $P_C : M \to 2^C$ by

$$P_C(x) = \Big\{c \in C;\ d(x,c) = \inf\{d(x,c) :\ c \in C\}\Big\}.$$

If $P_C(x)$ is reduced to one point, for every x in M, then C is said to be a Chebyshev set. In this case, the mapping P_C is not seen as a multivalued mapping but a singlevalued mapping, i.e., $P_C : M \to C$ defined by

$$d(x, P_C(x)) = \inf\{d(x,c) :\ c \in C\},$$

for any $x \in M$.

It is well known that the nearest point projection onto a closed convex set in strictly convex reflexive Banach space is well defined singlevalued mapping.

Now let us show that the P-property holds in Banach spaces l_p, $p \geq 2$.

Theorem 2.2. *Let (A, B) be a pair of nonempty, bounded and closed convex subsets of the Banach space l_p, $p \geq 2$. Then the pair (A, B) has the P-property.*

Proof. The Banach space l_p, for $p \geq 2$, is a reflexive strictly convex Banach space. Therefore, if B is nonempty, closed, and convex subset of the space l_p, then B is a Chebyshev subset. Let P_B be the nearest point projection onto B, consider the set

$$A_n = \Big\{x \in A;\ d(x,B) = ||x - P_B(x)|| \leq d(A,B) + \frac{1}{n}\Big\},$$

for any $n \geq 1$. From the definition of $d(A, B)$ and the continuity and the convexity of the function $x \to d(x, B)$, we know that A_n is a nonempty, bounded, closed and convex subset of A, for any $n \geq 1$. Obviously $\{A_n\}$ is decreasing. Using Smulian's characterization of reflexivity [14], we conclude that $A_\infty = \bigcap_{n\geq 1} A_n \neq \emptyset$. Let $u \in A_\infty$. Hence

$$d(u,B) = \|u - P_B(u)\| \leq d(A,B) + \frac{1}{n},$$

for any $n \geq 1$, which implies that $d(u, B) = \|u - P_B(u)\| \leq d(A, B)$. Since by definition of $d(A, B)$, we have $d(A, B) \leq \|u - P_B(u)\|$, we get $\|u - P_B(u)\| = d(A, B)$, i.e., $u \in A_0$ and $P_B(u) \in B_0$. Therefore, A_0 and B_0 are nonempty. In order to Show that the pair (A, B) has the P-property, let $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ such that

$$\|x_1 - y_1\| = \|x_2 - y_2\| = d(A, B).$$

Recall what is known as the Clarkson's inequality [3] in l_p:

$$\|x + y\|^p + \|x - y\|^p \leq 2^{p-1} \Big(\|x\|^p + \|y\|^p \Big), \tag{2.1}$$

for any x, y in l_p, for $p \geq 2$. Applying this inequality for $x = x_1 - y_1$ and $y = x_2 - y_2$, yields:

$$\|(x_1 - y_1) + (x_2 - y_2)\|^p + \|(x_1 - y_1) - (x_2 - y_2)\|^p \leq \\ 2^{p-1}\Big(\|(x_1 - y_1)\|^p + \|(x_2 - y_2)\|^p \Big).$$

Or,

$$\left\| \frac{x_1 + x_2}{2} - \frac{y_1 + y_2}{2} \right\|^p \leq d^p(A, B) - \left\| \frac{x_1 - y_1}{2} - \frac{x_2 - y_2}{2} \right\|^p$$

Using the convexity of A and B, we get

$$\left\| \frac{x_1 - y_1}{2} - \frac{x_2 - y_2}{2} \right\| = 0.$$

Hence $x_1 - y_1 = x_2 - y_2$ which implies $\|x_1 - x_2\| = \|x_1 - x_2\|$. Therefore, the pair (A, B) has the P-property. □

3. Best proximity points in l_p Spaces

The Banach Contraction Principle is considered as one of the most beautiful and fundamental fixed point theorems ever proved. This result deals with Lipschitzian mappings. Recall a mapping $T : M \to M$ is said to be Lipschitzian if there is a constant $k \geq 0$ such that for all $x, y \in M$, we have

$$d(T(x), T(y)) \leq k\, d(x, y).$$

The smallest number k for which the above holds is called the Lipschitz constant of T. A Lipschitzian mapping with Lipschitz constant in $[0, 1)$ is known as a contraction.

Theorem 3.1. *(Banach Contraction Principle) Let (M, d) be a complete metric space and let $T : M \to M$ be a contraction. Then T has a unique fixed point.*

If the fixed point equation $Tx = x$ of a given mapping T does not have a solution, then it is of interest to find an approximate solution for the fixed point equation.

Definition 3.2. Let A and B be nonempty subsets of a metric space (M, d), and $T : A \to B$ be a mapping. A point $x \in A$ is said to be a best proximity point of T if

$$d(x, Tx) = d(A, B) = \inf\{d(a, b);\ a \in A,\ b \in B\}.$$

Note that if $A \cap B \neq \emptyset$, then x is a best proximity point of T if $T(x) = x$, i.e., x is a fixed point of T.

As an application to the P-property, we obtain an extension of the Banach Contraction Principle for best proximity points. Our result is similar to Theorem 4.1 of [12].

Theorem 3.3. *Let (A, B) be a pair of nonempty, bounded and closed convex subsets of the Banach space l_p, $p \geq 2$. Let $T : A \to B$ be a contraction mapping such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point x in A.*

Proof. First, we prove the existence of a best proximity point of T in A.

By Theorem 2.2, $A_0 \neq \emptyset$, we pick $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, $Tx_0 \in B_0$. So, there exists an element $x_1 \in A_0$ such that $\|x_1 - Tx_0\| = d(A, B)$. Again, since $Tx_1 \in B_0$, there exists an element $x_2 \in A_0$ such that $\|x_2 - Tx_1\| = d(A, B)$. By induction, we construct a sequence $\{x_n\}$ such that

(i) $x_{2n} \in A_0$ and $x_{2n+1} \in B_0$, for any $n \in \mathbb{N}$;
(ii) $\|x_{n+1} - T(x_n)\| = d(A, B)$.

By Theorem 2.2, the pair (A, B) has the P-property, we have

$$\left.\begin{array}{l} \|x_{n+1} - T(x_n)\| = d(A, B) \\ \|x_n - T(x_{n-1})\| = d(A, B) \end{array}\right\} \Longrightarrow \|x_{n+1} - x_n\| = \|T(x_n) - T(x_{n-1})\|,$$

for any $n \geq 1$. Since T is a contraction, there exists $k < 1$ such that

$$\|x_{n+1} - x_n\| = \|T(x_n) - T(x_{n-1})\| \leq k \ \|x_n - x_{n-1}\|,$$

for any $n \geq 1$, which implies that

$$\|x_{n+1} - x_n\| \leq k^n \ \|x_1 - x_0\|,$$

for any $n \geq 1$. Hence $\sum_{n \in \mathbb{N}} \|x_{n+1} - x_n\|$ is convergent which implies that $\{x_n\}$ is Cauchy. Thus, there exists x such that $\{x_n\}$ converges to x. Since $\{x_n\} \subset A$ and A is closed, we conclude that $x \in A$. Since T is a continuous mapping and B is closed, we have $\{T(x_n)\}$ converges to $T(x) \in B$. Since $\|x_{n+1} - Tx_n\| = d(A, B)$, we get $\|x - T(x)\| = d(A, B)$, i.e., $x \in A_0$ and $T(x) \in B_0$. Clearly x is a best proximity point of T in A. Next, we prove that T has a unique best proximity point in A. Suppose that there exist $x, y \in A$ such that

$$\|x - T(x)\| = \|y - T(y)\| = d(A, B).$$

Since the pair (A, B) satisfies the P-property and T is a contraction mapping, we get

$$\|x - y\| = \|Tx - Ty\| < \|x - y\|,$$

which implies $\|x - y\| = 0$, i.e., $x = y$. Hence T has a unique best proximity point in A. $\square$

One may wonder what happens to the conclusion of Theorem 3.3 if T is not assumed to be a contraction. In particular, what happens when we assume T is nonexpansive. Nonexpansive mappings are those mappings which have Lipschitz constant equal to one. First, recall that uniformly convex Banach spaces have the fixed point property for nonexpansive mappings by Browder [2], Göhde [7]

Theorem 3.4. *(Browder-Göhde's Theorem)*
If K is a bounded, closed and convex subset of a uniformly convex Banach space E and $T : K \to K$ is nonexpansive, then T has a fixed point. Moreover, the fixed point set of T is a closed and convex subset of K.

Armed with Theorem 3.4, we are ready to extend the conclusion of Theorem 3.3 to nonexpansive mappings.

Theorem 3.5. *Let (A, B) be a pair of nonempty, bounded and closed convex subsets of the Banach space l_p, $p \geq 2$. Let $T : A \to B$ be a nonexpansive mapping such that $T(A_0) \subseteq B_0$. Then T has a best proximity point x in A.*

Proof. Since that A_0 is a Chebyshev subset. Then $x \in A$ is a best proximity point of T if and only if $P_{A_0}(T(x)) = x$, i.e., x is a fixed point of $P_{A_0} \circ T$. Indeed, let $x \in A$ be a best proximity point of T, i.e., $d(x, T(x)) = d(A, B)$. In particular, we have $x \in A_0$. Since $T(A_0) \subset B_0$, then $T(x) \in B_0$. Since $d(T(x), x) = d(T(x), A_0)$, we conclude that $x = P_{A_0}(T(x))$. Conversely, assume that x is a fixed point of $P_{A_0} \circ T$, i.e., $x = P_{A_0}(T(x))$. Then, we have $x \in A_0$ and $T(x) \in B_0$. Hence

$$d(x, T(x)) = d(P_{A_0}(T(x)), T(x)) = d(T(x), A_0) = d(A, B),$$

since $T(x) \in B_0$. Therefore, x is a best proximity point of T in A.

Next, consider the mapping $P_{A_0} \circ T : A_0 \to A_0$. By Theorem 2.2, the pair (A, B) has the P-property. It follows that the restriction of the nearest point projection P_{A_0} to B_0 is an isometry. Our assumption on the mapping T implies that $P_{A_0} \circ T$ is nonexpansive.

Finally, since A_0 is a closed nonempty convex subset of A, then A_0 is bounded. Theorem 3.4 implies that $P_{A_0} \circ T$ has a fixed point. Therefore, T has a best proximity point in A.

□

4. Conclusions and further study

Strict convexity and reflexivity of Banach spaces l_p, for $p \geq 2$, are crucial in the proof of the P-property. As an application to the latter geometrical property, we have obtained an extension of the Banach Contraction Principle for best proximity points. Furthermore, we have extended the fixed point property for nonexpansive mappings by Browder and Göhde for the setting of best proximity points.

Open Problem. Trying to investigate the P-property in nonlinear metric spaces, one has to consider the concept of convexity. Several attempts have been made to introduce a convex structure on a metric space. One such convex structure is available in $CAT_p(0)$ spaces [8], which are considered to be a nonlinear version of Banach spaces l_p, for $p \geq 2$. On the other hand, it was shown in [8], that complete $CAT_p(0)$ spaces, with $p \geq 2$, have the property (R) which is a nonlinear metric analogue of the Smulian characterization of reflexivity [14]. From this point of view, one may wonder if the

P-property holds in complete $CAT_p(0)$ spaces. It is our aim to try to investigate this in a future work.

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