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RATE OF CONVERGENCE OF MODIFIED MANN ITERATION FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. Modified Mann iteration has been studied extensively for the approximation of fixed points of asymptotically nonexpansive mappings by many authors and known to be weakly convergent in the infinite-dimensional space. Our aim in this present paper is to provide a nonasymptotic $O(1/n)$ convergence rate result for a modified Mann iteration for asymptotically nonexpansive mappings in real Hilbert spaces.
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1. Introduction

Throughout this paper, we consider the real Hilbert space setting: H denotes a real Hilbert space with scalar product $\langle .,. \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty, closed and convex subset of H. A self-mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty), k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \quad \forall x, y \in C, n \geq 1,$$

and T is a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C.$$

We denote the set of fixed points of T by

$$F(T) := \{x \in X \mid Tx = x\}.$$

This following example as given in [12] shows that the class of asymptotically nonexpansive mappings properly contains the class of nonexpansive mappings.

Example 1.1. Let B denote the unit ball in the Hilbert space ℓ^2 and let T be defined as follows:

$$T : (x_1, x_2, x_3, \ldots) \to (0, x_1^2, a_2 x_2, a_3 x_3, \ldots),$$

where $\{a_n\}$ is a sequence of real numbers such that $0 < a_n < 1$ and

$$\prod_{n=2}^{\infty} a_n = \frac{1}{2}.$$

Then T is Lipschitz and $\|Tx - Ty\| \leq 2\|x - y\|,\ \forall x, y \in B$. Moreover,

$$\|T^n x - T^n y\| \leq 2 \prod_{n=2}^{\infty} a_n \|x - y\|,\ \forall n = 2, 3, \ldots$$

Therefore,

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} 2 \prod_{n=2}^{\infty} a_n = 1.$$

Clearly, T is not a nonexpansive mapping.

It is proved in [12] that if C is a nonempty, closed, convex, and bounded subset of a uniformly convex Banach space, and if $T : C \to C$ is asymptotically nonexpansive, then T has a fixed point. See [3, 4, 10, 11, 16, 17, 19, 20] and other related papers for other conditions under which the fixed points of asymptotically nonexpansive mappings exist.

In [25], Schu introduced the modified Mann iteration, which generates a sequence $\{x_n\}$ in the following way:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n \quad \forall n = 1, 2, \ldots \tag{1.1}$$

where $\alpha_n \in (0, 1)$ satisying certain conditions and $T : C \to C$ is an asymptotically nonexpansive mapping. Schu [25] proved the following theorem.

Theorem 1.2. (see [25, Thm. 2.1]) *Let X be a uniformly convex Banach space satisfying Opial's condition, $\emptyset \neq C \subset X$ closed bounded and convex, and $T : C \to C$ asymptotically nonexpansive with sequence $\{k_n\} \subset [1, \infty)$ for which*

$$\sum_{n=1}^{\infty} (k_n - 1) < \infty$$

and $\alpha_n \in [0, 1]$ is bounded away. Let $\{x_n\}$ be a sequence generated in (1.1)*. Then, the sequence $\{x_n\}$ converges weakly to some fixed point of T.*

In real Hilbert spaces, Schu [26] also proved that if T is a completely continuous and asymptotically nonexpansive self-mapping of a nonempty closed bounded and convex subset of a real Hilbert space H, then $\{x_n\}$ be a sequence generated in (1.1) converges strongly to a fixed point of T. The modified Mann iteration (1.1) has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space or Banach spaces by many authors, we refer the reader to [1, 3, 5, 6, 9, 13, 14, 15, 19, 23, 24, 27, 28, 29] and the references contained therein.

We know that $\|Tx - x\| = 0$ if and only if $Tx = x$ and $\|Tx_n - x_n\| \to 0$ holds when $F(T) \neq \emptyset$ (see, for example, [26, Thm. 1.4] [23, Lem. 4]) when T is asymptotically nonexpansive mapping. Therefore, a crucial step in proving the weak convergence of the sequence of iterates $\{x_n\}$ generated in (1.1) is to show that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

which is a property known as asymptotic regularity of T (please, see [22]). When T is a nonexpansive mapping, it has been it has been established in [7] recently that $\|x_n - Tx_n\|$ in Krasnoselski-Mann iteration,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad \forall n = 1, 2, \dots \tag{1.2}$$

converges to zero at a rate of $O(1/\sqrt{\sigma_n})$ (big-O), where

$$\sigma_n := \sum_{k=1}^{n} \alpha_k(1-\alpha_k), \ n \in \mathbb{N}.$$

Further convergence rate analysis for both exact and inexact Krasnoselski-Mann iterations for nonexpansive mappings have been established recently in [8, 18, 21]. For example, it has been shown in [18, Thm. 1] that $\|x_n - Tx_n\| = O(1/\sqrt{n})$.
Our aim in this paper is to establish the nonasymptotic $O(1/n)$ convergence rate result of modified Mann algorithm (1.1) in real Hilbert spaces. As far as we know, this is the first time a convergence rate result is established for modified Mann algorithm (1.1) for asymptotically nonexpansive mappings. The result in this paper can also be considered as an extension of convergence rate results obtained in [7, 8, 18, 21] from the class of nonexpansive mappings to the class of asymptotically nonexpansive mappings in real Hilbert spaces. Our method of proof is of independent interest.
The paper is therefore organized as follows: We first recall some basic definitions and results in Section 2. The convergence rate result of (1.1) is then investigated in Section 3. We conclude with some final remarks in Section 4.

2. Preliminaries

Here we state some basic lemma that will be used in our convergence theorems.

Lemma 2.1. *Let X be a real inner product space. Then the following statement hold:*

$$\|tx + sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - st\|x-y\|^2, \quad \forall x, y \in X, \forall s, t \in \mathbb{R}.$$

Lemma 2.2. (see [23]) *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+c_n)a_n + b_n, \ n \geq 1.$$

If $\sum_{n=1}^{\infty} c_n < \infty, \ \sum_{n=1}^{\infty} b_n < \infty$, *then*

(i) $\lim_{n\to\infty} a_n$ *exists.*

(ii) If in particular, $\liminf_{n\to\infty} a_n = 0$, *one has* $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we give the convergence rate result for modified Mann iteration (1.1) for asymptotically nonexpansive mappings.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping such that its set of fixed points $F(T)$ is nonempty and*

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty.$$

For any $x_1 \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ in C be generated by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \tag{3.1}$$

where $0 < a \leq \alpha_n \leq b < 1$, for some $a, b \in (0, 1)$. Then for any positive integer n, the following hold

(i) $\min_{1\leq j\leq n} \|x_j - T^j x_j\| = O(1/\sqrt{n})$

(ii) $\min_{1\leq j\leq n} \|x_{j+1} - x_j\| = O(1/\sqrt{n})$

(iii) $\min_{1\leq j\leq n} \|x_{j+1} - T^j x_{j+1}\| = O(1/\sqrt{n})$

(iv) $\min_{1\leq j\leq n} \|x_j - Tx_j\| = O(1/\sqrt{n})$.

Proof. Let $x^* \in F(T)$. Then

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|T^n x_n - x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n k_n\|x_n - x^*\| \\ &\leq k_n\|x_n - x^*\| \\ &= (1 + (k_n - 1))\|x_n - x^*\|. \end{aligned} \tag{3.2}$$

Observe that since

$$\lim_{n\to\infty}(k_n - 1) = 0,$$

there exists $N_0 \in \mathbb{N}$ such that $k_n - 1 < 1$ for all $n \geq N_0$. Then

$$(k_n - 1)^2 \leq k_n - 1, \forall n \geq N_0,$$

and this implies that

$$\sum_{n=N_0}^{\infty}(k_n - 1)^2 \leq \sum_{n=N_0}^{\infty}(k_n - 1) \leq \sum_{n=1}^{\infty}(k_n - 1) < \infty. \tag{3.3}$$

Now, since

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty,$$

we have that $\{x_n\}_{n=1}^{\infty}$ is bounded by Lemma 2.2 in (3.2). Furthermore, using Lemma 2.1 in (3.1), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|(1-\alpha_n)(x_n-x^*)+\alpha_n(T^nx_n-x^*)\|^2\\ &= (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|T^nx_n-x^*\|^2-\alpha_n(1-\alpha_n)\|x_n-T^nx_n\|^2\\ &\leq (1-\alpha_n)\|x_n-x^*\|^2+\alpha_nk_n^2\|x_n-x^*\|^2\\ &-\alpha_n(1-\alpha_n)\|x_n-T^nx_n\|^2\\ &\leq k_n^2\|x_n-x^*\|^2-\alpha_n(1-\alpha_n)\|x_n-T^nx_n\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}a(1-b)\|x_n-T^nx_n\|^2 &\leq \alpha_n(1-\alpha_n)\|x_n-T^nx_n\|^2\\ &\leq k_n^2\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2,\ \forall n\geq 1.\end{aligned}$$

So,

$$\begin{aligned}a(1-b)\sum_{j=1}^{n}\|x_j-T^jx_j\|^2 &\leq \sum_{j=1}^{n}\Big[k_j^2\|x_k-x^*\|^2-\|x_{k+1}-x^*\|^2\Big]\\ &= \sum_{j=2}^{n}(k_j^2-1)\|x_k-x^*\|^2+k_1^2\|x_1-x^*\|^2\\ &\quad -\|x_{n+1}-x^*\|^2\\ &\leq \sum_{j=2}^{n}(k_j^2-1)\|x_k-x^*\|^2+k_1^2\|x_1-x^*\|^2\\ &\leq \sum_{j=2}^{n}(k_j^2-1)M^*+k_1^2\|x_1-x^*\|^2,\end{aligned}$$

where $M^* := \sup\limits_{n\geq 1}\|x_n-x^*\|^2$. This implies that

$$\begin{aligned}\sum_{j=1}^{n}\|x_j-T^jx_j\|^2 &\leq \frac{1}{a(1-b)}\Big[\sum_{j=2}^{n}(k_j^2-1)M^*+k_1^2\|x_1-x^*\|^2\Big]\\ &= \frac{1}{a(1-b)}\Big[\sum_{j=2}^{N_0}(k_j^2-1)M^*+\sum_{j=N_0}^{n}(k_j^2-1)M^*\\ &\quad +k_1^2\|x_1-x^*\|^2\Big]\\ &\leq \frac{1}{a(1-b)}\Big[\sum_{j=2}^{N_0}(k_j^2-1)M^*+\sum_{j=N_0}^{\infty}(k_j^2-1)M^*\\ &\quad +k_1^2\|x_1-x^*\|^2\Big]. \qquad (3.4)\end{aligned}$$

Using the observation (3.3) in (3.4), we get

$$\sum_{j=1}^{n} \|x_j - T^j x_j\|^2 \leq M_1, \tag{3.5}$$

where M_1 is a positive integer such that

$$M_1 \geq \frac{1}{a(1-b)}\Big[\sum_{j=2}^{N_0}(k_j^2-1)M^* + \sum_{j=N_0}^{\infty}(k_j^2-1)M^* + k_1^2\|x_1 - x^*\|^2\Big].$$

From (3.5), we have

$$\min_{1\leq j\leq n} \|x_j - T^j x_j\|^2 \leq \frac{M_1}{n}$$

and this implies that

$$\min_{1\leq j\leq n} \|x_j - T^j x_j\| \leq \sqrt{\frac{M_1}{n}}. \tag{3.6}$$

Thus

$$\min_{1\leq j\leq n} \|x_j - T^j x_j\| = O(1/\sqrt{n}),$$

which establishes (i).
Furthermore, from (3.1), we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|(1-\alpha_n)x_n + \alpha_n T^n x_n - x_n\| \\ &\leq \alpha_n \|T^n x_n - x_n\| \leq b\|T^n x_n - x_n\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \min_{1\leq j\leq n} \|x_{j+1} - x_j\| &\leq b \min_{1\leq j\leq n} \|T^j x_j - x_j\| \\ &\leq b\sqrt{\frac{M_1}{n}}. \end{aligned} \tag{3.7}$$

Thus,

$$\min_{1\leq j\leq n} \|x_{j+1} - x_j\| = O(1/\sqrt{n}).$$

This establishes (ii).

$$\begin{aligned} \|T^n x_{n+1} - x_{n+1}\| &= \|T^n x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| \\ &= \|(T^n x_{n+1} - T^n x_n) + (T^n x_n - x_n) + \alpha_n(x_n - T^n x_n)\| \\ &\leq k_n\|x_{n+1} - x_n\| + (1+\alpha_n)\|T^n x_n - x_n\| \\ &\leq M_2\|x_{n+1} - x_n\| + (1+b)\|T^n x_n - x_n\|, \end{aligned}$$

where $M_2 := \sup_{n\geq 1} k_n$. So, by (3.6) and (3.7), we get

$$\begin{aligned}
\min_{1\leq j\leq n}\|x_{j+1}-T^j x_{j+1}\| &\leq M_2 \min_{1\leq j\leq n}\|x_{j+1}-x_j\| \\
&\quad +(1+b)\min_{1\leq j\leq n}\|x_j-T^j x_j\| \\
&\leq M_2 b\sqrt{\frac{M_1}{n}}+(1+b)b\sqrt{\frac{M_1}{n}} \\
&= (M_2+1+b)b\sqrt{\frac{M_1}{n}}. \qquad (3.8)
\end{aligned}$$

Thus,

$$\min_{1\leq j\leq n}\|x_{j+1}-T^j x_{j+1}\| = O(1/\sqrt{n}).$$

This establishes (iii).
Now,

$$\begin{aligned}
\|x_{n+1}-Tx_{n+1}\| &\leq \|x_{n+1}-T^{n+1}x_{n+1}\|+\|T^{n+1}x_{n+1}-Tx_{n+1}\| \\
&\leq \|x_{n+1}-T^{n+1}x_{n+1}\|+k_1\|x_{n+1}-T^n x_{n+1}\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\min_{1\leq j\leq n}\|x_{j+1}-Tx_{j+1}\| &\leq \min_{1\leq j\leq n}\|x_{j+1}-T^{j+1}x_{j+1}\|+k_1\min_{1\leq j\leq n}\|x_{j+1}-T^j x_{j+1}\| \\
&\leq \sqrt{\frac{M_1}{n}}+(M_2+1+b)b\sqrt{\frac{M_1}{n}}. \qquad (3.9)
\end{aligned}$$

Therefore,

$$\min_{1\leq j\leq n}\|x_{j+1}-Tx_{j+1}\| = O(1/\sqrt{n}).$$

$$\begin{aligned}
\|x_n-Tx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-Tx_{n+1}\| \\
&\quad +\|Tx_{n+1}-Tx_n\| \\
&\leq (1+k_1)\|x_n-x_{n+1}\|+\|x_{n+1}-Tx_{n+1}\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\min_{1\leq j\leq n}\|x_j-Tx_j\| &\leq \min_{1\leq j\leq n}(1+k_1)\|x_j-x_{j+1}\|+\min_{1\leq j\leq n}\|x_{j+1}-Tx_{j+1}\| \\
&\leq (1+k_1)b\sqrt{\frac{M_1}{n}}+\sqrt{\frac{M_1}{n}} \\
&\quad +k_1(M_2+1+b)b\sqrt{\frac{M_1}{n}} \\
&= \frac{M_3}{\sqrt{n}}, \qquad (3.10)
\end{aligned}$$

where

$$M_3 := (1+k_1)b\sqrt{M_1}+\sqrt{M_1}+k_1(M_2+1+b)b\sqrt{M_1}.$$

Therefore,

$$\min_{1\leq j\leq n}\|x_j-Tx_j\| = O(1/\sqrt{n}).$$

This completes (iv) and hence the proof.

We give the following remarks about Theorem 3.1.

Remark 3.2. (a) It is known that for the modified Mann iteration (3.1), the quantity $\|Tx_n - x_n\|$ is not monotonically nonincreasing with n. Therefore, we are not able to remove the " $\min\limits_{1\leq j\leq n}$ " in our results in Theorem 3.1. Nonetheless, either with or without the " $\min\limits_{1\leq j\leq n}$ " a nonasymptotic $O(1/n)$ convergence rate would imply that an ϵ-accuracy solution, in the sense that $\|x_n - Tx_n\| \leq \epsilon$, is obtainable within no more than $O(1/\epsilon)$ iterations.
(b) If T in Theorem 3.1 is nonexpansive, then the modified Mann iteration (3.1) reduces to (1.2). Consequently, $\{\|x_n - Tx_n\|\}$ is monotone non-increasing and one can obtain the little-o rate of convergence of T, i.e.,

$$\|x_n - Tx_n\| = o(1/\sqrt{n}).$$

In this case, our result reduces to the result in [8, Thm. 1].
(c) For the case when T is a nonexpansive mapping, the convergence of inexact version of Krasnoselskii-Mann iteration

$$x_{n+1} = (1-\alpha_{n+1})x_n + \alpha_{n+1}(Tx_n + e_{n+1}),$$

where e_{n+1} can be interpreted as an error in the computation of Tx_n, or as a perturbation of the iteration, alongside its rate of convergence in Banach spaces have recently been considered by Bravo et al. in [2]. One of our future projects is to extend the results of Bravo et al. in [2] from the class of nonexpansive mappings to the class of asymptotically nonexpansive mappings.
(d) As a passing comment, we observe that the nonasymptotic $O(1/n)$ convergence rate result obtained in Theorem 3.1 still holds for a more general class of asymptotically quasi-nonexpansive mappings, i.e.,

$$\|T^n x - x^*\| \leq k_n\|x - x^*\|, \ \forall x \in C, x^* \in F(T), n \geq 1.$$

4. Final remarks

This paper presents a nonasymptotic convergence rate result for a modified Mann iteration for approximation of fixed points of asymptotically nonexpansive mappings. Our result extend the convergence rate results in [7, 8, 18, 21] from the class of nonexpansive mappings to the class of asymptotically nonexpansive mappings in real Hilbert spaces. Part of our future research is to obtain convergence rate result for a Krasnoselskii-Mann type iteration for a countable family of Lipschitzian mappings.

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