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ON THE LOCAL CONVERGENCE OF HIGHER ORDER METHODS IN BANACH SPACES

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Abstract. We study the local convergence analysis of two higher-order methods using Hölder continuity condition on the first Fréchet derivative to solve nonlinear equations in Banach spaces. Hölder continuous first derivative is used to extend the applicability of the method on such problems for which Lipschitz condition fails. Also, this convergence analysis generalizes the local convergence analysis based on Lipschitz continuity condition. Our analysis provides the radius of convergence ball and error bounds along with the uniqueness of the solution. Numerical examples like Hammerstein integral equation and a system of nonlinear equations are solved to verify our theoretical results.

Key Words and Phrases: Banach space, local convergence, iterative methods, Hölder continuity condition.
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1. Introduction

The prime objective of the study presented in this manuscript is to find a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where $F : \Omega \subseteq X \to Y$ is a Fréchet differentiable operator and Ω is a convex subset of X. X and Y are Banach spaces. Taking into account that numerous problems in applied sciences and engineering such as the boundary value problems occur in Kinetic theory of gases, the integral equations related to radiative transfer theory, problems in optimization and many others can be solved by obtaining the solutions of nonlinear equations in the form (1.1), many efficient algorithms have been derived. In most cases, the solutions of these nonlinear equations can not be obtained in closed form. So, iterative schemes are frequently used to avoid such problems.

The second order convergent Newton's scheme is extensively used as a solver of (1.1), which can be expressed as:

$$x_{n+1} = x_n - [F'(x_n)]^{-1}F(x_n),\ n \geq 0. \tag{1.2}$$

Also, Some classical third-order algorithms include Chebyshev's, the Halley's and Super-Halley's schemes are produced by putting ($\alpha = 0$), ($\alpha = \frac{1}{2}$) and ($\alpha = 1$) respectively in

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}(1 - \alpha H_F(x_n))^{-1} H_F(x_n)\right) [F'(x_n)]^{-1} F(x_n), \tag{1.3}$$

where $H_F(x_n) = F'(x_n)^{-1} F''(x_n) F'(x_n)^{-1} F(x_n)$.
To overcome the computation of higher-order derivatives present in the traditional third-order schemes, many researchers have developed higher-order Newton-like methods [1, 3, 10, 23, 24, 15, 16, 17, 18, 21, 25, 26, 29, 30, 20] such as harmonic mean Newton's method, midpoint Newton's method and other variants.

Local and semi-local convergence analysis of iterative schemes has been studied by numerous researchers [2, 13, 6, 8, 7, 27, 9, 28, 14, 19, 11, 12, 22, 4, 5], and many important results have been derived. "The semi-local convergence analysis, which is based on the information around an initial guess gives us the necessary condition to ensure the convergence and the local convergence analysis, which is based on the information around a solution provides radii of convergence balls" [13]. The local convergence analysis of many varieties of the methods defined in (1.3) has been studied by numerous authors in [13, 6, 8, 7]. Also, the local convergence analysis of efficient iterative schemes (Jarratt-type, Weerakoon-type and Newton-like) is studied in Banach spaces in [27, 9, 28, 14, 19, 11, 12].

In this paper, we use the Hölder continuity condition only on the first derivative to enhance the applicability of two higher-order convergent methods by generalizing the local convergence analysis based on Lipschitz continuity condition.

In [15], Cordero et al. studied the modification of Weerakoon's method [30] with fifth-order convergence to solve nonlinear systems. The method is given as:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n) \\ z_n &= x_n - 2[F'(x_n) + F'(y_n)]^{-1} F(x_n) \\ x_{n+1} &= z_n - F'(y_n)^{-1} F(z_n) \end{aligned} \tag{1.4}$$

Also, they modified the fourth order method proposed by Cordero et al. [16] to obtain a sixth order convergent method for systems of nonlinear equations, which is given by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n) \\ z_n &= y_n - F'(x_n)^{-1} [2I - F'(y_n) F'(x_n)^{-1}] F(y_n) \\ x_{n+1} &= z_n - F'(y_n)^{-1} F(z_n) \end{aligned} \tag{1.5}$$

In these methods, the iteration function contains only the first-order derivative F' of F. But the convergence analysis is shown with the assumption on at least fifth-order derivative. Therefore, the applicability of these schemes is restricted for such problems where the higher-order derivatives are unbounded or unobtainable. In [27], the authors studied the local convergence of the method (1.4) applying Lipschitz condition on the first-order derivative to overcome such problem. However, there are numerous examples for which Lipschitz condition fails. For instance, consider the nonlinear integral equation [28] given by

$$F(x)(s) = x(s) - 3\int_0^1 G_1(s,t)x(t)^{\frac{5}{4}}dt,$$

where $x(s) \in C[0,1]$ and $G_1(s,t)$ is Green's function defined on $[0,1] \times [0,1]$ by

$$G_1(s,t) = \begin{cases} (1-s)t, & \text{if } t \leq s \\ s(1-t), & \text{if } s \leq t \end{cases}.$$

Then,

$$||F'(x) - F'(y)|| \leq \frac{15}{32}||x-y||^{\frac{1}{4}}$$

It is clear that Lipschitz condition does not hold for this problem. However, Hölder continuity condition holds on F' for $p = \frac{1}{4}$. In this paper, we provide the local convergence analysis of the methods (1.4) and (1.5) using hypotheses only on F' to avoid the use of higher-order derivatives. Particularly, the Hölder continuous first derivative is employed to extend the applicability of the method by generalizing the local convergence under Lipschitz condition.

Another advantage of this approach is that the radius, error bounds and uniqueness of the solution information is provided and is based on Hölder constants. This is in contrast to studies in [15, 16], where expensive Taylor expansions are used to show convergence and the above are not computed.

The rest portion of this paper is arranged as follows: The local convergence analysis of the methods (1.4) and (1.5) is placed in Section 2. Section 3 is devoted to demonstrating the applications of our theoretical outcomes on standard numerical examples. Conclusions are discussed in the last section.

2. Local convergence analysis

The local convergence analysis of the methods (1.4) and (1.5) is studied in this section. Let the open and closed balls in X are denoted as $B(c,\rho)$ and $\bar{B}(c,\rho)$ respectively with center c and radius $\rho > 0$. Suppose the parameters $p \in (0,1]$, $k_0 > 0$ and $k > 0$ be given with $k_0 \leq k$. Furthermore, let us assume the following hold for the Fréchet differentiable operator $F : \Omega \subseteq X \to Y$.

$$F(x^*) = 0,\ F'(x^*)^{-1} \in BL(Y,X), \tag{2.1}$$

$$||F'(x^*)^{-1}(F'(x) - F'(x^*))|| \leq k_0||x - x^*||^p,\ \forall x \in \Omega \tag{2.2}$$

and

$$||F'(x^*)^{-1}(F'(x) - F'(y))|| \leq k||x-y||^p,\ \forall x, y \in \Omega, \tag{2.3}$$

where $BL(Y,X)$ is the set of all bounded linear operators from Y to X.
In several studies [2, 13, 6, 9, 11], an additional condition assumed is

$$||F'(x^*)^{-1}F'(x)|| \leq M,\ \forall x \in B\left(x^*, \left(\frac{1}{k_0}\right)^{\frac{1}{p}}\right). \tag{2.4}$$

This assumption is not taken in our study. We use the following results to avoid this extra condition.

Lemma 2.1. *If F obeys (2.2)-(2.3) and $\bar{B}(x^*, R) \subseteq \Omega$, then $\forall x \in B(x^*, R)$, we get*

$$||F'(x^*)^{-1}F'(x)|| \leq 1 + k_0||x - x^*||^p \tag{2.5}$$

and

$$||F'(x^*)^{-1}F(x)|| \leq (1 + k_0||x - x^*||^p)||x - x^*||, \tag{2.6}$$

where R is defined in (2.12). The above results also hold true if we use R' (2.35) in place of R.

Proof. Applying (2.2), we obtain

$$||F'(x^*)^{-1}F'(x)|| \leq 1 + ||F'(x^*)^{-1}(F'(x) - F'(x^*))|| \leq 1 + k_0||x - x^*||^p.$$

For $\theta \in [0, 1]$,

$$||F'(x^*)^{-1}F'(x^* + \theta(x - x^*))|| \leq 1 + k_0\theta||x - x^*||^p \leq 1 + k_0||x - x^*||^p$$

The mean value theorem is used to obtain

$$\begin{aligned} ||F'(x^*)^{-1}F(x)|| &= ||F'(x^*)^{-1}(F(x) - F(x^*))|| \\ &\leq ||F'(x^*)^{-1}F'(x^* + \theta(x - x^*))(x - x^*)|| \\ &\leq (1 + k_0||x - x^*||^p)||x - x^*||. \end{aligned}$$

□

2.1. **Local convergence analysis of method (1.4).** To study the local convergence of the scheme (1.4), we introduce the function J_1 on the interval $[0, (\frac{1}{k_0})^{\frac{1}{p}})$ by

$$J_1(u) = \frac{ku^p}{(p+1)(1 - k_0u^p)} \tag{2.7}$$

and the parameter

$$R_1 = \left(\frac{p+1}{(p+1)k_0 + k}\right)^{\frac{1}{p}} < \left(\frac{1}{k_0}\right)^{\frac{1}{p}}.$$

Observe that $J_1(R_1) = 1$. Again, we define functions J_2 and K_2 on $[0, (\frac{1}{k_0})^{\frac{1}{p}})$ by

$$J_2(u) = \frac{k_0}{2}(1 + J_1(u)^p)u^p \tag{2.8}$$

and

$$K_2(u) = J_2(u) - 1.$$

Now, $K_2(0) = -1 < 0$ and $\lim\limits_{u \to ((\frac{1}{k_0})^{\frac{1}{p}})^-} K_2(u) = +\infty$. According to the intermediate value theorem, the interval $(0, (\frac{1}{k_0})^{\frac{1}{p}})$ contains the zeros of the function $K_2(u)$. Let the smallest zero of $K_2(u)$ in $(0, (\frac{1}{k_0})^{\frac{1}{p}})$ is R_2. Also, we introduce functions J_3 and K_3 on $[0, R_2)$ by

$$J_3(u) = \frac{k[\frac{2}{p+1} + J_1(u)^p]u^p}{2(1 - J_2(u))} \tag{2.9}$$

and

$$K_3(u) = J_3(u) - 1.$$

Now, $K_3(0) = -1 < 0$ and $\lim\limits_{u \to R_2^-} K_3(u) = +\infty$. The intermediate value theorem confirms that the interval $(0, R_2)$ contains the zeros of the function $K_3(u)$. Let the smallest zero of $K_3(u)$ in $(0, R_2)$ is R_3. Again, we define J_4 and K_4 on $[0, (\frac{1}{k_0})^{\frac{1}{p}})$ by

$$J_4(u) = k_0 J_1(u)^p u^p \tag{2.10}$$

and

$$K_4(u) = J_4(u) - 1.$$

Now, $K_4(0) = -1 < 0$ and $\lim\limits_{u \to ((\frac{1}{k_0})^{\frac{1}{p}})^-} K_4(u) = +\infty$. According to the intermediate value theorem, the interval $(0, (\frac{1}{k_0})^{\frac{1}{p}})$ contains the zeros of the function $K_4(u)$. Let the smallest zero of $K_4(u)$ in $(0, (\frac{1}{k_0})^{\frac{1}{p}})$ is R_4. Finally, let us define J_5 and K_5 on $[0, R_4)$ by

$$J_5(u) = \left(1 + \frac{1 + k_0 J_3(u)^p u^p}{1 - J_4(u)}\right) J_3(u) \tag{2.11}$$

and

$$K_5(u) = J_5(u) - 1.$$

Now, $K_5(0) = -1 < 0$ and $\lim\limits_{u \to R_4^-} K_5(u) = +\infty$. The intermediate value theorem confirms that the interval $(0, R_4)$ contains the zeros of the function $K_5(u)$. Let the smallest zero of $K_5(u)$ in $(0, R_4)$ is R_5. Let us choose

$$R = \min\{R_1, R_3, R_5\}. \tag{2.12}$$

Now, we have

$$0 \leq J_1(u) < 1, \tag{2.13}$$

$$0 \leq J_2(u) < 1, \tag{2.14}$$

$$0 \leq J_3(u) < 1, \tag{2.15}$$

$$0 \leq J_4(u) < 1, \tag{2.16}$$

and

$$0 \leq J_5(u) < 1 \tag{2.17}$$

for each $u \in [0, R)$.

Next, the local convergence analysis of the method (1.4) is presented in Theorem 2.1.

Theorem 2.1. *Let $F : \Omega \subseteq X \to Y$ be a Fréchet differentiable operator. Suppose $x^* \in \Omega$, F obeys (2.1)-(2.3) and*

$$\bar{B}(x^*, R) \subseteq \Omega, \tag{2.18}$$

where R is defined in (2.12). Starting from $x_0 \in B(x^*, R)$ the method (1.4) generates the sequence of iterates $\{x_n\}$ which is well defined, $\{x_n\}_{n \geq 0} \in B(x^*, R)$ and converges to the solution x^* of (1.1). Moreover, the following estimations hold $\forall n \geq 0$

$$||y_n - x^*|| \leq J_1(||x_n - x^*||)||x_n - x^*|| < ||x_n - x^*|| < R, \tag{2.19}$$

$$||z_n - x^*|| \leq J_3(||x_n - x^*||)||x_n - x^*|| < ||x_n - x^*|| < R \tag{2.20}$$

and

$$||x_{n+1} - x^*|| \leq J_5(||x_n - x^*||)||x_n - x^*|| < ||x_n - x^*|| < R, \tag{2.21}$$

where the functions J_1, J_3 and J_5 are given in (2.7), (2.9) and (2.11), respectively. Furthermore, the solution x^* of the equation $F(x) = 0$ is unique in $\bar{B}(x^*, \Delta) \cap \Omega$, where $\Delta \in [R, (\frac{p+1}{k_0})^{\frac{1}{p}})$.

Proof. Using the definition of R, the equation (2.2) and the assumption $x_0 \in B(x^*, R)$, we find

$$||F'(x^*)^{-1}(F'(x_0) - F'(x^*))|| \leq k_0||x_0 - x^*||^p < k_0R^p < 1. \tag{2.22}$$

Now, Banach Lemma on invertible operators [1, 10, 23, 25, 29] confirms that

$$F'(x_0)^{-1} \in BL(Y, X)$$

and

$$||F'(x_0)^{-1}F'(x^*)|| \leq \frac{1}{1 - k_0||x_0 - x^*||^p} < \frac{1}{1 - k_0R^p}. \tag{2.23}$$

Hence, it follows from the first step of the method (1.4) for $n = 0$ that y_0 is well defined. Again,

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ = -\left[F'(x_0)^{-1}F'(x^*)\right] &\left[\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)\, d\theta\right]. \end{aligned} \tag{2.24}$$

Using (2.3), (2.7), (2.13), (2.23) and (2.24), we find

$$\begin{aligned} &||y_0 - x^*|| \\ &\leq \left[||F'(x_0)^{-1}F'(x^*)||\right]\left[\left|\left|\int_0^1 F'(x^*)^{-1}(F'(x^*+\theta(x_0-x^*)) - F'(x_0))(x_0 - x^*)\, d\theta\right|\right|\right] \\ &\leq \frac{k||x_0 - x^*||^p}{(p+1)(1 - k_0||x_0 - x^*||^p)}||x_0 - x^*|| \\ &= J_1(||x_0 - x^*||)||x_0 - x^*|| < ||x_0 - x^*|| < R \end{aligned} \tag{2.25}$$

and this shows (2.19) for $n = 0$. Then we show $[F'(x_0) + F'(y_0)]^{-1} \in BL(Y, X)$. The equations (2.2), (2.8), (2.12), (2.14) and (2.25) are used to obtain

$$\begin{aligned} &||(2F'(x^*))^{-1}(F'(x_0) + F'(y_0) - 2F'(x^*))|| \\ &\leq \frac{1}{2}[||F'(x^*)^{-1}(F'(x_0) - F'(x^*))|| + ||F'(x^*)^{-1}(F'(y_0) - F'(x^*))||] \\ &\leq \frac{k_0}{2}[||x_0 - x^*||^p + ||y_0 - x^*||^p] \\ &\leq \frac{k_0}{2}[||x_0 - x^*||^p + J_1(||x_0 - x^*||^p||x_0 - x^*||^p] \\ &= \frac{k_0}{2}[1 + J_1(||x_0 - x^*||)^p]||x_0 - x^*||^p \\ &= J_2(||x_0 - x^*||) < J_2(R) < 1. \end{aligned}$$

Now, we obtain $[F'(x_0)+F'(y_0)]^{-1} \in BL(Y,X)$ using Banach Lemma on invertible operators. Also,

$$||[F'(x_0)+F'(y_0)]^{-1}F'(x^*)|| \leq \frac{1}{2(1-J_2(||x_0-x^*||))}. \tag{2.26}$$

Now, it follows from the second step of the method (1.4) for $n=0$ that z_0 is well defined. Using the definition of R, (2.3), (2.9), (2.15), (2.25) and (2.26), we get

$$\begin{aligned}
||z_0-x^*|| \leq & \big(||[F'(x_0)+F'(y_0)]^{-1}F'(x^*)||\big) \\
& \times \left(\left|\left|\int_0^1 F'(x^*)^{-1}\left(F'(x_0)-F'(x^*+\theta(x_0-x^*))\right)(x_0-x^*)\ d\theta\right|\right|\right. \\
& + \left.\left|\left|\int_0^1 F'(x^*)^{-1}\left(F'(y_0)-F'(x^*+\theta(x_0-x^*))\right)(x_0-x^*)\ d\theta\right|\right|\right) \\
& \leq \frac{\frac{k}{p+1}||x_0-x^*||^{p+1}+k\int_0^1(||y_0-x^*-\theta(x_0-x^*)||^p)\ d\theta||x_0-x^*||}{2(1-J_2(||x_0-x^*||))} \\
& \leq \frac{\frac{k}{p+1}||x_0-x^*||^{p+1}+k(||y_0-x^*||^p+\frac{||x_0-x^*||^p}{p+1})||x_0-x^*||}{2(1-J_2(||x_0-x^*||))} \\
& \leq \frac{\frac{k}{p+1}||x_0-x^*||^{p+1}+k[J_1(||x_0-x^*||)^p||x_0-x^*||^p+\frac{||x_0-x^*||^p}{p+1}]||x_0-x^*||}{2(1-J_2(||x_0-x^*||))} \\
& \leq \frac{(\frac{2k}{p+1}||x_0-x^*||^p+kJ_1(||x_0-x^*||)^p||x_0-x^*||^p)||x_0-x^*||}{2(1-J_2(||x_0-x^*||))} \\
& = \frac{[(\frac{2k}{p+1}+kJ_1(||x_0-x^*||)^p)||x_0-x^*||^p]||x_0-x^*||}{2(1-J_2(||x_0-x^*||))} \\
& = \frac{k[(\frac{2}{p+1}+J_1(||x_0-x^*||)^p)||x_0-x^*||^p]||x_0-x^*||}{2(1-J_2(||x_0-x^*||))} \\
& = J_3(||x_0-x^*||)||x_0-x^*|| < ||x_0-x^*|| < R.
\end{aligned} \tag{2.27}$$

Hence, we establish (2.20) for $n=0$. Again,

$$\begin{aligned}
||F'(x^*)^{-1}(F'(y_0)-F'(x^*))|| \leq k_0||y_0-x^*||^p &< k_0J_1(||x_0-x^*||)^p||x_0-x^*||^p \\
&= J_4(||x_0-x^*||) < 1.
\end{aligned} \tag{2.28}$$

So, $F'(y_0)^{-1} \in BL(Y,X)$ with

$$||F'(y_0)^{-1}F'(x^*)|| \leq \frac{1}{1-J_4(||x_0-x^*||)}. \tag{2.29}$$

Now, it follows from the last step of the method (1.4) for $n = 0$ that x_1 is well defined. Finally, we use (2.6), (2.11), (2.12), (2.17), (2.25), (2.27) and (2.29) to get

$$\begin{aligned}
||x_1 - x^*|| &\leq ||z_0 - x^*|| + ||F'(y_0)^{-1}F(z_0)|| \\
&\leq ||z_0 - x^*|| + ||F'(y_0)^{-1}F'(x^*)||\ ||F'(x^*)^{-1}F(z_0)|| \\
&\leq ||z_0 - x^*|| + \frac{(1 + k_0||z_0 - x^*||^p)||z_0 - x^*||}{1 - J_4(||x_0 - x^*||)} \\
&= \left(1 + \frac{(1 + k_0||z_0 - x^*||^p)}{1 - J_4(||x_0 - x^*||)}\right)||z_0 - x^*|| \\
&\leq \left(1 + \frac{(1 + k_0 J_3(||x_0 - x^*||)^p||x_0 - x^*||^p)}{1 - J_4(||x_0 - x^*||)}\right) J_3(||x_0 - x^*||)||x_0 - x^*|| \\
&= J_5(||x_0 - x^*||)||x_0 - x^*|| < ||x_0 - x^*|| < R. \qquad (2.30)
\end{aligned}$$

Thus, we show the estimate (2.21) for $n = 0$. We get the estimates (2.19)-(2.21) by substituting x_n, y_n, z_n and x_{n+1} in place of x_0, y_0, z_0 and x_1 respectively in the previous estimations. Using the fact $||x_{n+1} - x^*|| \leq J_5(R)||x_n - x^*|| < R$, we derive that $x_{n+1} \in B(x^*, R)$ and $\lim_{n\to\infty} x_n = x^*$. Now, we want to show the uniqueness of the solution x^*. Suppose there exist another solution y^* ($\neq x^*$) of $F(x) = 0$ in $B(x^*, \Delta) \cap \Omega$. Consider $Q = \int_0^1 F'(y^* + \theta(x^* - y^*))\ d\theta$. From equation (2.2), we get

$$\begin{aligned}
||F'(x^*)^{-1}(Q - F'(x^*))|| &\leq \int_0^1 k_0||y^* + \theta(x^* - y^*) - x^*||^p\ d\theta \\
&\leq \frac{k_0}{p+1}||x^* - y^*||^p \\
&\leq \frac{k_0\Delta^p}{p+1} < 1.
\end{aligned}$$

Applying Banach Lemma, we find $Q^{-1} \in BL(Y, X)$.
Now, Using the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, it is concluded that $x^* = y^*$. This completes the proof. □

2.2. **Local convergence analysis of method (1.5).** For the local convergence analysis of the method (1.5), we introduce the function H_1 on the interval $[0, (\frac{1}{k_0})^{\frac{1}{p}})$ by

$$H_1(v) = \frac{kv^p}{(p+1)(1 - k_0v^p)} \qquad (2.31)$$

and the parameter

$$R_1' = \left(\frac{p+1}{(p+1)k_0 + k}\right)^{\frac{1}{p}} < \left(\frac{1}{k_0}\right)^{\frac{1}{p}}.$$

Observe that $H_1(R_1') = 1$. Again, we define functions H_2 and G_2 on $[0, (\frac{1}{k_0})^{\frac{1}{p}})$ by

$$H_2(v) = \left(1 + \frac{1 + k_0H_1(v)^pv^p}{1 - k_0v^p} + \frac{(1 + k_0H_1(v)^pv^p)^2}{(1 - k_0v^p)^2}\right)H_1(v) \qquad (2.32)$$

and

$$G_2(v) = H_2(v) - 1.$$

Now, $G_2(0) = -1 < 0$ and $\lim_{v \to ((\frac{1}{k_0})^{\frac{1}{p}})^-} G_2(v) = +\infty$. According to the intermediate value theorem, the interval $(0, (\frac{1}{k_0})^{\frac{1}{p}})$ contains the zeros of the function $G_2(v)$. Let the smallest zero of $G_2(v)$ in $(0, (\frac{1}{k_0})^{\frac{1}{p}})$ is R_2'. Also, we introduce functions H_3 and G_3 on $[0, (\frac{1}{k_0})^{\frac{1}{p}})$ by

$$H_3(v) = k_0 H_1(v)^p v^p \tag{2.33}$$

and

$$G_3(v) = H_3(v) - 1.$$

Now, $G_3(0) = -1 < 0$ and $\lim_{v \to ((\frac{1}{k_0})^{\frac{1}{p}})^-} G_3(v) = +\infty$. The intermediate value theorem confirms that the interval $(0, (\frac{1}{k_0})^{\frac{1}{p}})$ contains the zeros of the function $G_3(v)$. Let the smallest zero of $G_3(v)$ in $(0, (\frac{1}{k_0})^{\frac{1}{p}})$ is R_3'. Finally, we define H_4 and G_4 on $[0, R_3')$ by

$$H_4(v) = \left(1 + \frac{1 + k_0 H_2(v)^p v^p}{1 - H_3(v)}\right) H_2(v) \tag{2.34}$$

and

$$G_4(v) = H_4(v) - 1.$$

Now, $G_4(0) = -1 < 0$ and $\lim_{v \to R_3'^-} K_4(v) = +\infty$. So, the interval $(0, R_3')$ contains the zeros of the function $G_4(v)$. Let the smallest zero of $G_4(v)$ in $(0, R_3')$ is R_4'.
Let us choose

$$R' = \min\{R_1', R_2', R_4'\}. \tag{2.35}$$

Now, we have

$$0 \le H_1(v) < 1, \tag{2.36}$$

$$0 \le H_2(v) < 1, \tag{2.37}$$

$$0 \le H_3(v) < 1 \tag{2.38}$$

and

$$0 \le H_4(v) < 1, \tag{2.39}$$

for each $v \in [0, R')$. Next, the local convergence analysis of the method (1.5) is presented in Theorem 2.2.

Theorem 2.2. *Let $F : \Omega \subseteq X \to Y$ be a Fréchet differentiable operator. Suppose $x^* \in \Omega$, F obeys (2.1)-(2.3) and*

$$\bar{B}(x^*, R') \subseteq \Omega, \tag{2.40}$$

where R' is defined in (2.35). Starting from $x_0 \in B(x^*, R')$ the method (1.5) generates the sequence of iterates $\{x_n\}$ which is well defined, $\{x_n\}_{n \ge 0} \in B(x^*, R')$ and converges to the solution x^* of (1.1). Moreover, the following estimations hold $\forall n \ge 0$

$$||y_n - x^*|| \le H_1(||x_n - x^*||)||x_n - x^*|| < ||x_n - x^*|| < R', \tag{2.41}$$

$$||z_n - x^*|| \le H_2(||x_n - x^*||)||x_n - x^*|| < ||x_n - x^*|| < R' \tag{2.42}$$

and

$$||x_{n+1} - x^*|| \leq H_4(||x_n - x^*||)||x_n - x^*|| < ||x_n - x^*|| < R', \tag{2.43}$$

where the functions H_1, H_2 and H_4 are given in (2.31), (2.32) and (2.34), respectively. Furthermore, the solution x^* of the equation $F(x) = 0$ is unique in $\bar{B}(x^*, \Delta') \cap \Omega$, where $\Delta' \in [R', (\frac{p+1}{k_0})^{\frac{1}{p}})$.

Proof. Using the definition of R', the equation (2.2) and the assumption

$$x_0 \in B(x^*, R'),$$

we find

$$||F'(x^*)^{-1}(F'(x_0) - F'(x^*))|| \leq k_0||x_0 - x^*||^p < k_0 R'^p < 1.$$

Now, Banach Lemma on invertible operators [1, 10, 23, 25, 29] confirms that

$$F'(x_0)^{-1} \in BL(Y, X)$$

and

$$||F'(x_0)^{-1}F'(x^*)|| \leq \frac{1}{1 - k_0||x_0 - x^*||^p} < \frac{1}{1 - k_0 R'^p}. \tag{2.44}$$

Hence, it follows from the first step of the method (1.5) for $n = 0$ that y_0 is well defined. Again,

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= -\left[F'(x_0)^{-1}F'(x^*)\right]\left[\int_0^1 F'(x^*)^{-1}(F'(x^*+\theta(x_0-x^*)) - F'(x_0))(x_0-x^*)\, d\theta\right]. \end{aligned} \tag{2.45}$$

Using (2.31), (2.35), (2.36), (2.44) and (2.45) we find

$$\begin{aligned} &||y_0 - x^*|| \\ &\leq \left[||F'(x_0)^{-1}F'(x^*)||\right]\left[\left|\left|\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0-x^*)\, d\theta\right|\right|\right] \\ &\leq \frac{k||x_0 - x^*||^p}{(p+1)(1 - k_0||x_0 - x^*||^p)}||x_0 - x^*|| \\ &= H_1(||x_0 - x^*||)||x_0 - x^*|| < ||x_0 - x^*|| < R' \end{aligned} \tag{2.46}$$

and this shows (2.41) for $n = 0$. Since $F'(x_0)^{-1} \in BL(Y, X)$ so, z_0 is well defined. Using the definition of R', (2.6), (2.32), (2.37), (2.44) and (2.46), we get

$$\begin{aligned}
&||z_0 - x^*|| \\
&\leq ||y_0 - x^*|| + 2||F'(x_0)^{-1}F(y_0)|| + ||F'(x_0)^{-1}F'(y_0)||\ ||F'(x_0)^{-1}F(y_0)|| \\
&\leq ||y_0 - x^*|| + ||F'(x_0)^{-1}F'(x^*)||\ ||F'(x^*)^{-1}F(y_0)|| \\
&+ ||F'(x_0)^{-1}F'(x^*)||\ ||F'(x^*)^{-1}F'(y_0)||\ ||F'(x_0)^{-1}F'(x^*)||\ ||F'(x^*)^{-1}F(y_0)|| \\
&\leq ||y_0 - x^*|| + \frac{(1 + k_0||y_0 - x^*||^p)||y_0 - x^*||}{1 - k_0||x_0 - x^*||^p} \\
&+ \frac{(1 + k_0||y_0 - x^*||^p)}{1 - k_0||x_0 - x^*||^p}\frac{(1 + k_0||y_0 - x^*||^p)||y_0 - x^*||}{1 - k_0||x_0 - x^*||^p} \\
&\leq \left(1 + \frac{(1 + k_0||y_0 - x^*||^p)}{1 - k_0||x_0 - x^*||^p} + \frac{(1 + k_0||y_0 - x^*||^p)}{1 - k_0||x_0 - x^*||^p}\frac{(1 + k_0||y_0 - x^*||^p)}{1 - k_0||x_0 - x^*||^p}\right)||y_0 - x^*|| \\
&\leq \left(1 + \frac{(1 + k_0H_1(||x_0 - x^*||)^p||x_0 - x^*||^p)}{1 - k_0||x_0 - x^*||^p}\right)H_1(||x_0 - x^*||)||x_0 - x^*|| \\
&+ \left(\frac{(1 + k_0H_1(||x_0 - x^*||)^p||x_0 - x^*||^p)^2}{(1 - k_0||x_0 - x^*||^p)^2}\right)H_1(||x_0 - x^*||)||x_0 - x^*|| \\
&= H_2(||x_0 - x^*||)||x_0 - x^*|| < ||x_0 - x^*|| < R'. \qquad (2.47)
\end{aligned}$$

Hence, we establish (2.42) for $n = 0$. Again,

$$\begin{aligned}
||F'(x^*)^{-1}(F'(y_0) - F'(x^*))|| &\leq k_0||y_0 - x^*||^p < k_0H_1(||x_0 - x^*||)^p||x_0 - x^*||^p \\
&= H_3(||x_0 - x^*||) < 1. \qquad (2.48)
\end{aligned}$$

So, $F'(y_0)^{-1} \in BL(Y, X)$ with

$$||F'(y_0)^{-1}F'(x^*)|| \leq \frac{1}{1 - H_3(||x_0 - x^*||)}. \qquad (2.49)$$

Now, it follows from the last step of the method (1.5) for $n = 0$ that x_1 is well define. Finally, we use (2.6), (2.34), (2.35), (2.39), (2.47) and (2.49) to get

$$\begin{aligned}
||x_1 - x^*|| &\leq ||z_0 - x^*|| + ||F'(y_0)^{-1}F(z_0)|| \\
&\leq ||z_0 - x^*|| + ||F'(y_0)^{-1}F'(x^*)||\ ||F'(x^*)^{-1}F(z_0)|| \\
&\leq ||z_0 - x^*|| + \frac{(1 + k_0||z_0 - x^*||^p)||z_0 - x^*||}{1 - H_3(||x_0 - x^*||)} \\
&= \left(1 + \frac{(1 + k_0||z_0 - x^*||^p)}{1 - H_3(||x_0 - x^*||)}\right)||z_0 - x^*|| \\
&\leq \left(1 + \frac{(1 + k_0H_2(||x_0 - x^*||)^p||x_0 - x^*||^p)}{1 - H_3(||x_0 - x^*||)}\right)H_2(||x_0 - x^*||)||x_0 - x^*|| \\
&= H_4(||x_0 - x^*||)||x_0 - x^*|| < ||x_0 - x^*|| < R'. \qquad (2.50)
\end{aligned}$$

Thus, we show the estimate (2.43) for $n = 0$. We get the estimates (2.41)-(2.43) by substituting x_n, y_n, z_n and x_{n+1} in place of x_0, y_0, z_0 and x_1 respectively in the previous estimations. Using the fact $||x_{n+1} - x^*|| \leq H_4(R')||x_n - x^*|| < R'$, we derive that $x_{n+1} \in B(x^*, R')$ and $\lim_{n\to\infty} x_n = x^*$. Now, we want to show the uniqueness of the solution x^*. Suppose there exist another solution y^* ($\neq x^*$) of $F(x) = 0$ in $B(x^*, \Delta') \cap \Omega$. Consider $T = \int_0^1 F'(y^* + \theta(x^* - y^*))\, d\theta$. From equation (2.2), we get

$$\begin{aligned} ||F'(x^*)^{-1}(T - F'(x^*))|| &\leq \int_0^1 k_0 ||y^* + \theta(x^* - y^*) - x^*||^p \, d\theta \\ &\leq \frac{k_0}{p+1} ||x^* - y^*||^p \\ &\leq \frac{k_0 \Delta'^p}{p+1} < 1. \end{aligned}$$

Applying Banach Lemma, we find $T^{-1} \in BL(Y, X)$. Now, using the identity

$$0 = F(x^*) - F(y^*) = T(x^* - y^*),$$

it is concluded that $x^* = y^*$. This ends the proof. □

3. Numerical examples

In this section, numerical examples are provided to validate the theoretical results. We consider the Examples (1, 2 and 3) from the research paper of Argyros and George [6]. The examples 4 and 5 are selected from [28].

Example 3.1. Define F on $\Omega = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

We have $x^* = 1$. Also, $p = 1$ and $k_0 = k = 146.6629$. The values of R and R' are determined using the definitions of "J" and "H" functions respectively.

Table 1. Radii of convergence balls for Example 3.1

Method (1.4)	*Method (1.5)*
$R_1 = 0.004545$	$R'_1 = 0.004545$
$R_3 = 0.003994$	$R'_2 = 0.002012$
$R_5 = 0.002917$	$R'_4 = 0.001331$
$R = 0.002917$	$R' = 0.001331$

Example 3.2. Let us define F on $\Omega = [1, 3]$ by

$$F(x) = \frac{2}{3} x^{\frac{3}{2}} - x$$

We have $x^* = \frac{9}{4}$. Also, we have $p = 0.5$, $k_0 = 1$ and $k = 2$. R and R' are computed using "J" and "H" functions respectively.

Table 2. Radii of convergence balls for Example 3.2

Method (1.4)	*Method (1.5)*
$R_1 = 0.183673$	$R'_1 = 0.183673$
$R_3 = 0.107877$	$R'_2 = 0.025567$
$R_5 = 0.038326$	$R'_4 = 0.008795$
$R = 0.038326$	$R' = 0.008795$

Example 3.3. Let F is defined on $\bar{B}(0,1)$ for $(x_1, x_2, x_3)^t$ by

$$F(x) = (e^{x_1} - 1, \frac{e-1}{2}x_2^2 + x_2, x_3)^t$$

We have $x^* = (0,0,0)^t$. Also, we have $p = 1$, $k_0 = e - 1$ and $k = e$. We determine the values of R and R' using "J" and "H" functions respectively.

Table 3. Radii of convergence balls for Example 3.3

Method (1.4)	*Method (1.5)*
$R_1 = 0.324947$	$R'_1 = 0.324947$
$R_3 = 0.268633$	$R'_2 = 0.133649$
$R_2 = 0.184350$	$R'_4 = 0.083613$
$R = 0.184350$	$R' = 0.083613$

Example 3.4. Consider the nonlinear Hammerstein type integral equation given by

$$F(x)(s) = x(s) - 5\int_0^1 stx(t)^{\frac{3}{2}}dt,$$

where $x(s) \in C[0,1]$. We have $x^* = 0$. Also, $p = 0.5$ and $k_0 = k = \frac{15}{4}$. Using the definitions of "J" and "H" functions the values of R and R' are computed.

Table 4. Radii of convergence balls for Example 3.4

Method (1.4)	*Method (1.5)*
$R_1 = 0.025599$	$R'_1 = 0.025599$
$R_3 = 0.017992$	$R'_2 = 0.004240$
$R_5 = 0.007313$	$R'_4 = 0.001689$
$R = 0.007313$	$R' = 0.001689$

Example 3.5. Consider the nonlinear integral equation given by

$$F(x)(s) = x(s) - 3\int_0^1 G_1(s,t)x(t)^{\frac{5}{4}}dt,$$

where $x(s) \in C[0,1]$ and $G_1(s,t)$ is Green's function. We have $x^* = 0$. Also, $p = 0.25$ and $k_0 = k = \frac{15}{32}$. Using the definitions of "J" and "H" functions the values of R and R' are determined.

Table 5. Radii of convergence balls for Example 3.5

Method (1.4)	*Method (1.5)*
$R_1 = 1.973080$	$R'_1 = 1.973080$
$R_3 = 0.879329$	$R'_2 = 0.043572$
$R_5 = 0.101198$	$R'_4 = 0.006069$
$R = 0.101198$	$R' = 0.006069$

4. Conclusions

We studied the local convergence analysis of two higher-order methods to find a locally unique solution of a nonlinear equation in Banach spaces. The Hölder continuity condition on the first derivative is used to enhance the applicability of these methods. This study helps in solving those problems for which Lipschitz condition fails without applying higher-order derivative. Lastly, the theoretical outcomes are tested on standard numerical examples like Hammerstein equation and system of nonlinear equations.

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References

[1] I.K. Argyros, *Convergence and Application of Newton-type Iterations*, Springer, 2008.

[2] I.K. Argyros, Y.J. Cho, S. George, *Local convergence for some third order iterative methods under weak conditions*, J. Korean Math. Soc., **53**(4)(2016), 781-793.

[3] I.K. Argyros, Y.J. Cho, S. Hilout, *Numerical Methods for Equations and its Applications*, Taylor & Francis, CRC Press, New York, 2012.

[4] I.K. Argyros, Y.J. Cho, S.K. Khattri, *On a new semilocal convergence analysis for the Jarratt method*, J. Inequal. Appl., **194**(2013), 16 pages.

[5] I.K. Argyros, Y.J. Cho, H.M. Ren, *Convergence of Halley's method for operators with the bounded second derivative in Banach spaces*, J. Inequal. Appl., **260**(2013), 12 pages.

[6] I.K. Argyros, S. George, *Local convergence of deformed Halley method in Banach space under Hölder continuity conditions*, J. Nonlinear Sc. Appl., **8**(2015), 246-254.

[7] I.K. Argyros, S. George, *Local convergence of modified Halley-like methods with less computation of inversion*, Novi. Sad. J. Math., **45**(2015), 47-58.

[8] I.K. Argyros, S. George, Á.A. Magreñán, Á. A, (2015), *Local convergence for multi-point-parametric Chebyshev-Halley-type methods of higher convergence order*, J. Comput. Appl. Math., **282**(2015), 215-224.

[9] I.K. Argyros, D. González, *Local convergence for an improved Jarratt-type method in Banach space*, Int. J. Interact. Multimed. Artif. Intell. (Special Issue on Teaching Mathematics Using New and Classic Tools), **3**(2015), 20-25.

[10] I.K. Argyros, S. Hilout, *Computational Methods in Nonlinear Analysis*, World Scientific Publ. House, New Jersey, USA, 2013.

[11] I.K. Argyros, S. Hilout, *On the local convergence of fast two-step Newton-like methods for solving nonlinear equations*, J. Comput. Appl. Math., **245**(2013), 1-9.

[12] I.K. Argyros, S.K. Khattri, *Local convergence for a family of third order methods in Banach spaces*, J. Math., **46**(2015), 53-62.

[13] I.K. Argyros, Á.A. Magreñán, *A study on the local convergence and the dynamics of Chebyshev-Halley-type methods free from second derivative*, Numer. Algor., **71**(2015), no. 1, 1-23.

[14] A. Cordero, J.A. Ezquerro, M.A. Hernandez-Veron, *On the local convergence of a fifth-order iterative method in Banach spaces*, Appl. Math. Comput., **251**(2014), 396-403.

[15] A. Cordero, J.L. Hueso, E. Martínez, J.R. Toregrossa, *Increasing the convergence order of an iterative method for nonlinear systems*, Appl. Math. Lett., **25**(2012), 2369-2374.

[16] A. Cordero, E. Martínez, J.R. Toregrossa, *Iterative methods of order four and five for systems of nonlinear equations*, J. Comput. Appl. Math., **231**(2012), 541-551.

[17] M.V. Kanwar, V.K. Kukreja, S. Singh, *On some third-order iterative methods for solving nonlinear equations*, Appl. Math. Comput., **171**(2005), no. 1, 272-280.

[18] J. Kou, Y. Li, X. Wang, *A composite fourth-order iterative method for solving non-linear equations*, Appl. Math. Comput., **184**(2007), 71-475.

[19] E. Martínez, S. Singh, J.L. Hueso, D.K. Gupta, D. K., (2016), *Enlarging the convergence domain in local convergence studies for iterative methods in Banach spaces*, Appl. Math. Comput., **281**(2016), 252-265.

[20] H.P.S. Nishani, S. Weerakoon, T.G.I. Fernando, M. Liyanag, *Weerakoon-Fernando method with accelerated third-order convergence for systems of nonlinear equations*, IJMMNO, **8**(3)(2018), 287-304.

[21] A.Y. Özban, *Some new variants of Newton's method*, Appl. Math. Lett., **17**(6)(2004), 677-682.

[22] S.K. Parhi, D.K., Gupta, *Semilocal convergence of a Stirling-like method in Banach spaces*, Int. J. Comput. Methods, **7**(2)(2010), 215-228.

[23] M.S. Petković, B. Neta, L. Petković, D. Džunić, *Multipoint Methods for Solving Nonlinear Equations*, Elsevier, 2013.
[24] F.A. Potra, V. Ptak, *Nondiscrete Induction and Iterative Processes*, Research Notes in Mathematics, Pitman Publ., Boston, MA, 103, 1984.
[25] L.B. Rall, *Computational Solution of Nonlinear Operator Equations*, Robert E. Krieger, New York, 1979.
[26] H. Ren, Q. Wu, W. Bi, *New variants of Jarratt method with sixth-order convergence*, Numer. Algor., **52**(4)(2009), 585-603.
[27] D. Sharma, S.K. Parhi, *On the local convergence of modified Weerakoon's method in Banach spaces*, J. Anal., (2019), DOI: https//doi.org/10.1007/s41478-019-00216-x.
[28] S. Singh, D.K. Gupta, R.P. Badoni, E. Martínez, J.L. Hueso, *Local convergence of a parameter based iteration with Hölder continuous derivative in Banach spaces*, Calcolo, **54**(2017), no. 2, 527-539.
[29] J.F. Traub, *Iterative Methods for Solution of Equations*, Prentice-Hall, Englewood Cliffs, 1964.
[30] S. Weerakoon, T.G.I. Fernando, *A variant of Newton's method with accelerated third-order convergence*, Appl. Math. Lett., **13**(8)(2000), 87-93.

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