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FIXED POINT THEOREMS IN THE STUDY OF OPERATOR EQUATIONS IN ORDERED BANACH SPACES AND THEIR APPLICATIONS

SALIMA MECHROUK

Faculty of Sciences, UMBB, Boumerdes, Algeria
E-mail: mechrouk@gmail.com and s.mechrouk@univ-boumerdes.dz

Abstract. We use fixed point index properties and the general minorant principle (see Theorem 7.B in [12]) to prove new fixed point theorems for operators leaving invariant a cone in a Banach space. Main ideas of this work are inspired from the work in [11]. The results obtained are used to prove existence of at least one positive solution to a $\phi-$laplacian boundary value problem.
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1. Introduction

In the study of nonlinear operators in ordered Banach spaces having an invariant cone, it is often convenient to make use of minorants, majorants and the special concept of the derivatives in order to establish the existence of non-zero fixed points. Krasnosel'skii has provided in [10] many other interesting fixed point theorems. These theorems and their generalization state that if such an operator is approximatively linear at 0 and $+\infty$, and the spectral radius of the linear approximations are oppositely located with respect to 1, then it has a fixed point. See [Theorems 1 and 2] in [2], [Theorem 7.4] by Amann [1], and [Theorem 4.9] by Krasnosel'skii [10], where the maps are supposed to be monotone. The spirit of hypotheses in this work aries in many results in the literature. Theorem 7.B in [12] state that if a positive mapping T has a linear minorant having an eigensubsolution, then T has eigensolutions.

The main goal of this paper is to study completely continuous maps in ordered Banach spaces having an invariant cone and give sufficient conditions on minorants and majorants which yield the existence of at least one non-zero fixed point. We are interested in giving some fixed point theorems that complete the results obtained in [12] motivated by the works cited above and inspired from the works in [3], [5], [6], [4] and [8] where the main assumptions are on the behavior of the operator at 0 and ∞. More precisely, we will assume that the mapping T has an asymptotically linear minorant h and has a right differentiable at zero majorant g or vice visa and existence of the fixed point is obtained under additional conditions about the positive spectrums of the derivatives. The proofs are based on the fixed point index theory, developed in [7] and [9].

The paper is organized as follows. Section 2 gives some preliminaries. Section 3 is devoted to prove new fixed point theorems for positive maps having approximative minorant and majorant at 0 and ∞ in specific classes of operators. Applications to the existence of solutions to a $\phi-$laplacian boundary value problem with mixed boundary conditions are presented in the last section.

2. Abstract Background

We will use extensively in this work cones and the fixed point index theory; so let us recall some facts related to these two tools. Let X be a real Banach space endowed with a norm $\|\cdot\|$, and let $L(X) = L(X, X)$ be the set of all linear continuous mappings from X into X. A nonempty closed convex subset C of X is said to be a cone if $(tC) \subset C$ for all $t \geq 0$ and $C \cap (-C) = \{0_X\}$. It is well known that a cone C induces a partial order in the Banach space X. We write for all $x, y \in X :\ x \preceq y$ if $y - x \in C,\ x \prec y$ if $y - x \in C,\ y \neq x$ and $x \npreceq y$ if $y - x \notin C$. Notations $\succeq,\ \succ$ and $\nsucceq$ denote respectively the reverse situations. We said that the cone C is solid if $int(C) \neq \emptyset$ and is said to be normal with a constant $n_C > 0$ if for all u, v in $C,\ u \preceq v$ implies $\|u\| \leq n_C \|v\|$.
Let C be a cone in X and let $L : X \to X$.

Definition 2.1. The mapping L is said to be positive if $L(C) \subset C$. In this case, a nonnegative constant μ is said to be a positive eigenvalue of L if there exists $u \in C \smallsetminus \{0_X\}$ such that $Lu = \mu u$.

Definition 2.2. Let A be a nonempty set and let B be an ordered set. A map $g : A \to B$ is said to be a majorant of the map $f : A \to B$ if $f(x) \leq g(x)$ for all $x \in X$. Minorant is defined by reversing the above inequality sign.

Definition 2.3. Let $L : X \to X$ be a positive mapping. L is said to be

i) increasing if for all $u, v \in X,\ u \preceq v$ implies $Lu \preceq Lv$,
ii) lower bounded on C if $L_C^- = \inf\{\|Lu\|,\ u \in C \text{ and } \|u\| = 1\} > 0$,
iii) upper bounded on C if $L_C^+ = \sup\{\|Lu\|,\ u \in C \text{ and } \|u\| = 1\} > 0$,
iv) u_0-bounded below on C, (for u_0 a fixed element of $C^\star$), if for every $u \in C^\star$, a positive number α and a natural number n can be found such that $L^n u \succeq \alpha\, u_0$,
v) u_0-bounded on C, (for u_0 a fixed element of $C^\star$), if for every $u \in C^\star$, a positive numbers α, β and a natural number n can be found such that $\alpha\, u_0 \preceq L^n u \preceq \beta\, u_0$,
vi) $C-$normal if for all $u, v \in C$; $u \leq v$ implies $\|Lu\| \leq \|Lv\|$.

Remark 2.4. Strongly positive operators are the simplest examples of u_0-positive operators where u_0 is an arbitrary interior element of the cone C.

Definition 2.5. Let $L_1, L_2 : X \to X$ be positive maps. We write $L_1 \preceq L_2$ if for all $x \in C,\ L_1 x \preceq L_2 x$.

Definition 2.6. A function $f : \Omega \subset X \to X$ is said to be bounded, if it maps bounded sets into bounded sets, and it is said to be completely continuous, if it is continuous and maps bounded sets into relatively compact sets. In general f is said to be compact if it is continuous and $f\left(\bar{\Omega}\right)$ is compact.

Definition 2.7.[12] A map $g : C \to X$ is said to be differentiable at $x_0 \in C$ along C if there exists $g'(x_0) \in L(X)$ such that

$$\lim_{h \in C, h \to 0} \frac{\|g(x_0 + h) - g(x_0) - g'(x_0)h\|}{\|h\|} = 0.$$

We said that $g'(x_0)$, is the derivative of g at x_0 along C, and it is uniquely determined.

The map g is said to be asymptotically linear along C if there exists $g'(\infty) \in L(X)$ such that

$$\lim_{x \in C, \|x\| \to +\infty} \frac{\|g(x) - g'(\infty)x\|}{\|x\|} = 0.$$

Again, $g'(\infty)$ is uniquely determined and is called the derivative at infinity along C.

Lemma 2.8.[10] *The derivative $g'(\nu)$, ($\nu = +\infty$, or $x_0 \in C$), with respect to a cone of the positive operator g is a linear positive operator.*

Detailed presentation of the differentiability with respect to a cone can be found in [10] and [12].

Let us recall some lemmas providing fixed point index computations. Let C be a cone in X. Let for $R > 0$, $C_R = C \cap B(0_X, R)$ where $B(0_X, R)$ is the open ball of radius R centered at 0_X, and let ∂C_R be its boundary and consider a compact mapping $f : \overline{C_R} \to C$.

Lemma 2.9. *If $fx \neq \lambda x$ for all $x \in \partial C_R$ and $\lambda \geq 1$ then $i(f, C_R, C) = 1$.*

Lemma 2.10. *If $fx \not\succeq x$ for all $x \in \partial C_R$ then $i(f, C_R, C) = 1$.*

Lemma 2.11. *If $fx \not\preceq x$ for all $x \in \partial C_R$ then $i(f, C_R, C) = 0$.*

Lemma 2.12. *If there exists $e \succ 0_X$ such that $x \neq fx + te$ for all $t \geq 0$ and all $u \in \partial C_R$ then $i(f, C_R, C) = 0$.*

A detailed presentation of the fixed point index theory can be found in [7], [9] and [12].

In all this section E is a real Banach space, K, P are two nontrivial cones in E with $P \subset K$ (it may happen that $K = P$) and $L(E)$ denote the set of all linear continuous self mapping on E endowed with the norm, $\|L\| = \sup\limits_{\|u\|=1} \|Lu\|$. Let $C^+(E)$ denote the subset of $L(E)$ consisting of all compact positive operators. Hereafter $\preceq$ is the order induced by the cone K on E and we set

$$L_K^P(E) = \{L \in L(E), \text{ and } L(K) \subset P\}, \text{ and}$$
$$C_K^P(E) = \left\{L \in L_K^P(E) : L \text{ is compact}\right\}.$$

Now, for $L \in L_K^P(E)$, we define the subsets

$$\Lambda_P^L = \{\lambda \geq 0 : \text{ there exists } u \in P \setminus \{0_E\} \text{ such that } Lu \preceq \lambda u\},$$

$$\Theta_P^L = \{\theta \geq 0 : \text{ there exists } u \in P \setminus \{0_E\} \text{ such that } Lu \succeq \theta u\}.$$

Remark 2.13. Note that

i) $0 \in \Theta_P^L$ and if $\theta \in \Theta_P^L$, then $[0, \theta] \subset \Theta_P^L$.

ii) If $\lambda \in \Lambda_P^L$ then $[\lambda, +\infty) \subset \Lambda_P^L$.

iii) $\Lambda_P^L \subset \Lambda_K^L$ and $\Theta_P^L \subset \Theta_K^L$.

iv) If μ is positive eigenvalue of L, then $\mu \in \Theta_P^L \cap \Lambda_P^L$.

In all this paper, we set for $L \in L_K^P(E)$,

$$\theta_P^L = \sup \Theta_P^L$$

and when Λ_P^L is nonempty

$$\lambda_P^L = \inf \Lambda_P^L.$$

If $L : E \to E$ is a bounded linear operator, then we define $r(L)$, its spectral radius by

$$r(L) = \lim_{n \to +\infty} \|L^n\|^{\frac{1}{n}}.$$

Lemmas 2 and 2 give sufficient conditions for the existence of λ_P^L and θ_P^L.

Lemma 2.14. [5] *Let $L \in L_K^P(E)$. Then the subset Λ_P^L is nonempty.*

Lemma 2.15. [5] *Assume that K is normal and $L \in L_K^P(E)$. Then the subset Θ_P^L is bounded from above by $r(L)$.*

Lemma 2.16.[5] *Assume that L is $K-$normal and $L \in L_K^P(E)$. Then the subset Θ_P^L is bounded from above by $r(L)$.* Observe that if $L \in C_K^P(E)$, then for all $R > 0$, the permanence property of the fixed point index implies that

$$i(L, P_R, P) = i(L, K_R, K).$$

Lemma 2.17.[5] *Let $L \in C_K^P(E)$ and let γ be a positive real number. For any $R > 0$, we have*

i) *$i(\gamma L, P_R, P) = 1$, if $\gamma\theta_P^L < 1$.*

ii) *$i(\gamma L, P_R, P) = 0$, if the subset Λ_P^L is nonempty and $\gamma\lambda_P^L > 1$.*

Theorem 2.18.[5] *Let $L \in C_K^P(E)$. Then, we have $\lambda_P^L \leq \theta_P^L$.*

For $L \in C_K^P(E)$, $\sigma_K(L)$ denote the set of all positive eigenvalues of L and

$$\sigma_L^- = \inf \sigma_L(N).$$

Remark 2.19. We deduce from iv) in Remark 2 and Theorem 2 that for all $L \in C_K^P(E)$, $\sigma_K(L) \subset \left[\lambda_P^L, \theta_P^L\right]$.

Lemma 2.20.[5] *Let L be a compact operator with $int(K) \neq \emptyset$ and $L(K \setminus \{0_X\}) \subset int(K)$, and either K is normal or L is $K-$normal. Then $\lambda_K^L = \theta_K^L = r(L)$ is the principal and unique positive eigenvalue of L.*

Remark 2.21.[5] Note that Lemma 2 affirms that 0 cannot be an eigenvalue of L, then for every subset $P \subset K$, with $L(K) \subset P$, we have $\lambda_P^L = \lambda_K^L$ and $\theta_P^L = \theta_K^L$.

Proposition 2.22. *Let ξ be a fixed point on K (ξ may be $+\infty$) and suppose that the operator $g : K \to K$ has a Fréchet derivative on ξ such that $g'(\xi) \in C_K^P(E)$. If $\lambda_P^{g'(\xi)} > 0$, then $\lambda_P^{g'(\xi)} = \sigma_{g'(\xi)}^-$.*

Proof. Let $(\lambda_n) \subset \left(\lambda_P^{g'(\xi)}, +\infty\right) \subset \Lambda_P^{g'(\xi)}$ be a decreasing sequence converging to $\lambda_P^{g'(\xi)}$ and $(\phi_n) \subset P \setminus \{0_E\}$ such that $g'(\xi)\phi_n \preceq \lambda_n\phi_n$. Consider for each integer $n \geq 1$, $P_n = \{u \in P : g'(\xi)u \preceq \lambda_n u\}$ and note that the linearity of $g'(\xi)$ makes of the set P_n convex and so, a cone in E which is different from the trivial one, since $\phi_n \in P_n$. We have also, $g'(\xi)(P_n) \subset P_n$ and so, consider the sets

$$\Lambda_n^{g'(\xi)} = \{\lambda \geq 0 : \text{ there exists } u \in P_n \setminus \{0_E\} \text{ such that } g'(\xi)u \preceq \lambda u\},$$

$$\Theta_n^{g'(\xi)} = \{\theta \geq 0 : \text{ there exists } u \in P_n \setminus \{0_E\} \text{ such that } g'(\xi)u \succeq \theta u\}$$

and the constants

$$\lambda_n^{g'(\xi)} = \inf \Lambda_n^{g'(\xi)} \text{ and } \theta_n^{g'(\xi)} = \sup \Theta_n^{g'(\xi)}.$$

Clearly, we have $0 < \lambda_P^{g'(\xi)} \leq \lambda_n^{g'(\xi)} \leq \theta_n^{g'(\xi)} \leq \lambda_n$ and $g'(\xi)$ admits for all $n \geq 1$ a positive eigenvalue μ_n associated with a normalized eigenvector $\psi_n \in P_n$ with

$$0 < \lambda_P^{g'(\xi)} \leq \lambda_n^{g'(\xi)} \leq \mu_n \leq \theta_n^{g'(\xi)} \leq \lambda_n.$$

Thus, we have $\lim \mu_n = \lambda_P^{g'(\xi)}$ and for each $n \geq 1$

$$\left(g'(\xi)\right)^2 \psi_n = \mu_n g'(\xi)\psi_n = \mu_n^2 \psi_n$$

and the compactness of $g'(\xi)$ leads to $\phi = \lim g'(\xi)\psi_n$ (up to a subsequence) satisfies $g'(\xi)\phi = \lambda_P^{g'(\xi)}\phi$ and $\|\phi\| = \lambda_P^{g'(\xi)} > 0$. At the end, Remark 2 leads to $\lambda_P^{g'(\xi)} = \sigma^-_{g'(\xi)}$.

Definition 2.23.[3] An operator $L \in C^+(E)$ is said to have the strongly index-jump property at $\nu > 0$ (SIJP for short) if $\lambda_P^L = \theta_P^L = \nu$.

3. Main results

Let $T : K \to K$ be a completely continuous mapping, the main goal of this section is to prove fixed point theorems for the mapping T.

Theorem 2.24. *Suppose that T has a right differentiable at zero majorant $g : K \to K$ such that $g(0) = 0$, $g'(0) \in C_K^P(E)$ is lower-bounded satisfying $\theta_P^{g'(0)} < 1$. Then T has at least one positive fixed point.*

Proof. Let us prove existence of $r > 0$ small enough, such that for all $t \in [0, 1]$, equation $tTu = u$ has no solution in ∂K_r. By the contrary suppose that for each integer $n \geq 1$ there exist $t_n \in [0, 1]$ and $u_n \in \partial K_{\frac{1}{n}}$ such that

$$u_n = t_n T u_n.$$

Note that $v_n = u_n / \|u_n\| \in \partial K_1$ and satisfies

$$v_n \preceq \frac{g(u_n)}{\|u_n\|}. \tag{3.1}$$

Thus, we have:

$$\frac{g(u_n)}{\|u_n\|} = \frac{g(u_n) - g'(0)(u_n)}{\|u_n\|} + \frac{g'(0)(u_n)}{\|u_n\|}. \tag{3.2}$$

We set

$$G_n(u_n) = \frac{g(u_n) - g'(0)(u_n)}{\|u_n\|}.$$

Clearly

$$v_n \preceq G_n(u_n) + g'(0)(v_n).$$

Using the fact that $g'(0)$ is increasing, we get:

$$g'(0)(v_n) \preceq g'(0)(G_n(u_n)) + g'(0)(g'(0)(v_n)). \tag{3.3}$$

Because of the compactness of $g'(0)$, there exists a subsequence (v_{n_k}) such that $\lim g'(0)v_{n_k} = v \in P$. In fact, we have that $v \succ 0_E$. Indeed, if $\lim g'(0)v_{n_k} = 0_E$, then because of the lower boundeness of $g'(0)$, we have:

$$\|v\| = \lim \|g'(0)v_{n_k}\| \geq N^-_{g,P} > 0.$$

Thus, letting $k \to \infty$ in (3.3), we obtain $v \preceq g'(0)v$ and $1 \in \Theta_P^{g'(0)}$. This contradicts the hypothesis $1 > \theta_P^{g'(0)}$ and proves existence of $r > 0$ small enough such that for all $t \in [0,1]$, equation $tTu = u$ has no solution in ∂K_r. For a such $r > 0$, we deduce from Lemma 2 that

$$i(T, K_r, K) = 1$$

and T has a positive fixed point u with $\|u\| < r$. This completes the proof.

Arguing as in the proof of Theorem 3, we obtain the following result.

Theorem 2.25. *Suppose that T has an asymptotically linear majorant $g : K \to K$ such that $g'(\infty) \in C_K^P(E)$ is lower-bounded satisfying $\theta_P^{g'(\infty)} < 1$. Then T has at least one positive fixed point.*

Corollary 2.26. *Let $u_0 \in K^\star$. Suppose that T has an asymptotically linear majorant $g : K \to K$ such that $g'(\infty) \in C_K^P(E)$ is u_0-bounded bellow satisfying $\theta_P^{g'(\infty)} < 1$. Then T has at least one positive fixed point.*

Theorem 2.27. *Suppose that T has an asymptotically linear majorant $g : K \to K$ such that $g'(\infty) \in C_K^P(E)$ satisfying $\theta_P^{g'(\infty)} < 1$ and K is a normal cone. Then T has at least one positive fixed point.*

Proof. Consider the function $H_\infty : [0,1] \times K \to K$ defined by

$$H_\infty(t,u) = (1-t)Tu + t\, g'(\infty)\,(u)$$

and let us prove existence of $R > 0$ large enough, such that for all $t \in [0,1]$, equation $H_\infty(t,u) = u$ has no solution in ∂K_R. By the contrary, suppose that for each integer $n \geq 1$, there exist $t_n \in [0,1]$ and $u_n \in \partial K_n$ such that

$$u_n = (1-t_n)Tu_n + t_n g'(\infty)u_n.$$

Note that $w_n = u_n / \|u_n\| \in \partial K_1$ and satisfies

$$w_n = (1-t_n)\,(Tu_n / \|u_n\|) + t_n g'(\infty)w_n. \tag{3.4}$$

Thus, the inequality

$$(Tu_n / \|u_n\|) \preceq G_n(u_n) + g'(\infty)(w_n). \tag{3.5}$$

holds, where

$$G_n(u_n) = \frac{g(u_n) - g'(\infty)(u_n)}{\|u_n\|}$$

combined with the normality of the cone K implies that the sequence $(Tu_n / \|u_n\|)$ is bounded. Because of the compactness of $g'(\infty)$, there exists a subsequence (w_{n_k}) such that $\lim g'(\infty)w_{n_k} = w \in P$. In fact, we have that $w \succ 0_E$.

Indeed, if $\lim g'(\infty)w_{n_k} = 0_E$, then inequality (3.5) and the normality of the cone K imply $\lim (Tu_n / \|u_n\|) = 0_E$. This together with (3.4) leads to $\lim w_{n_k} = 0_E$, contradicting $\|w_{n_k}\| = 1$.

Therefore, letting $k \to \infty$ in

$$\begin{aligned} g'(\infty)w_{n_k} &= g'(\infty)\left((1-t_{n_k})\,(Tu_{n_k} / \|u_{n_k}\|) + t_{n_k} g'(\infty)w_{n_k}\right) \\ &\preceq g'(\infty)(G_{n_k}(u_{n_k})) + g'(\infty)\left(g'(\infty)w_{n_k}\right) \end{aligned}$$

we have $w \preceq g'(\infty)w$ and $1 \in \Theta_P^{g'(\infty)}$, contradicting the hypothesis $\theta_P^{g'(\infty)} < 1$ in Theorem 3 and proves existence of $R > 0$ large enough such that for all $t \in [0,1]$, equation $H_2(t,u)=u$ has no solution in ∂K_R. For such a radius $R>0$, homotopy and permanence properties of the fixed point index and Lemma 2 lead to

$$\begin{aligned} i(T,K_R,K)=i(H_\infty(0,\cdot),K_R,K) &= i(H_\infty(1,\cdot),K_R,K)\\ &= i(g'(\infty),K_R,K)\\ &= i(g'(\infty),P_R,P)\\ &= 1. \end{aligned}$$

This completes the proof.

Remark 2.28. Note that Remark 2 and the hypothesis $\theta_P^{g'(\infty)}<1$ in the last theorem imply that $g'(\infty)$ does not possess a positive eigenvector to an eigenvalue greater than or equal to 1 and is a generalization of Theorem 4.8 given by Krasnosel'kii in [10] where the maps and the norm are supposed to be monotone.

Corollary 2.29. *Suppose that K is a normal cone and $g:K\to K$ is an asymptotically linear operator such that $g'(\infty)\in C_K^P(E)$ satisfies $\theta_P^{g'(\infty)}<1$. Suppose that T is a right differentiable on K such that $T'_+\preceq g'(\infty)$ on K. Then T has at least one positive nontrivial fixed point.*

In the fact, for all $u\in K$, we have

$$\left(T-g'(\infty)\right)'(u)=T'_+(u)+\left[g'(\infty)\right]'(u)=T'_+(u)-g'(\infty)(u)\preceq 0.$$

This implies that operator $T-g'(\infty)$ is non-increasing on K and it follows from above that for all $u\in K$, $\left(T-g'(\infty)\right)(u)\preceq T(0)+g'(\infty)(0)=T(0)$ leading to $T\preceq T(0)+g'(\infty)$ on K which implies that $L=T(0)+g'(\infty)$ is an asymptotically linear majorant of T such that $L'(\infty)=g'(\infty)$ and $\theta_P^{L'(\infty)}=\theta_P^{g'(\infty)}<1$. We deduce from Theorem 3 that T has a fixed point on K.

Arguing as in the proof of Theorem 3, we obtain the following Theorem.

Theorem 2.30. *Suppose that T has a right differentiable at zero majorant $g:K\to K$ such that $g(0)=0$, $g'(0)\in C_K^P(E)$ satisfying $\theta_P^{g'(0)}<1$ and K is a normal cone. Then T has at least one positive fixed point.*

Remark 2.31. Suppose, in addition, that K is solid and that $g'(0)$ is strongly positive, then the condition $\theta_P^{g'(0)}<1$ of the above theorem can be replaced by $r\left(g'(0)\right)<1$.

Theorem 2.32. *Suppose that the cone K is a normal one with $int(K)\neq\emptyset$ and T has an asymptotically linear minorant $h:K\to K$ such that $h'(\infty)\in C_K^P(E)$ is strongly positive. Suppose that T has a right differentiable at zero majorant $g:K\to K$ such that $g(0)=0$, and $g'(0)\in C_K^P(E)$ satisfying $\theta_P^{g'(0)}<1<r\left(h'(\infty)\right)$. Then T has at least one positive nontrivial fixed point.*

Proof. We have to prove existence of $0<r<R$ such that

$$i(T,K_r,K)=1 \text{ and } i(T,K_R,K)=0.$$

In such a situation, additivity and solution properties of the fixed point index imply that

$$i(T,K_R\setminus\overline{K_r},K)=i(T,K_R,K)-i(T,K_r,K)=-1$$

and T has a positive fixed point u with $r<\|u\|<R$.

Let $e > 0$ and let us prove existence of $R > 0$ big enough, such that for all $t \in \mathbb{R}^+$, equation $Tu + te = u$ has no solution in ∂K_R. By the contrary suppose that for each integer $n \geq 1$, there exist $t_n \in \mathbb{R}^+$ and $u_n \in \partial K_n$ such that

$$u_n = Tu_n + t_n e.$$

Note that $v_n = u_n / \|u_n\| \in \partial K_1$ and satisfies the inequality:

$$v_n \succeq (Tu_n / \|u_n\|) \succeq \frac{h(u_n)}{\|u_n\|}. \tag{3.6}$$

Thus, we have:

$$\frac{h(u_n)}{\|u_n\|} = \frac{h(u_n) - h'(\infty)(u_n)}{\|u_n\|} + \frac{h'(\infty)(u_n)}{\|u_n\|}. \tag{3.7}$$

We set

$$H_n(u_n) = \frac{h(u_n) - h'(\infty)(u_n)}{\|u_n\|}.$$

Then

$$v_n \succeq H_n(u_n) + h'(\infty)(v_n). \tag{3.8}$$

Using the fact that $h'(\infty)$ is increasing, we get:

$$h'(\infty)(v_n) \succeq h'(\infty)(H_n(u_n)) + h'(\infty)(h'(\infty)(v_n)). \tag{3.9}$$

Because of the compactness of $h'(\infty)$ there exists a subsequence (v_{n_k}) such that $\lim h'(\infty)v_{n_k} = v \in P$. Since $h'(\infty)$ is strongly positive on K and $\|v_{n_k}\| = 1$, then $h'(\infty)v_{n_k} \in int\,(K)$. We deduce from Lemma 3.7 in [12] that there exists $r_0 > 0$ small enough such that $h'(\infty)v_{n_k} \succeq r_0 v_{n_k}$, we obtain $v \succ 0_E$. Thus, letting $k \to \infty$ in (3.9), we obtain $v \succeq h'(\infty)v$ and $1 \in \Lambda_P^{h'(\infty)}$. This contradicts the hypothesis $1 < \lambda_P^{h'(\infty)} = r\,(h'(\infty))$ and proves existence of $R > 0$ large enough such that for all $t \in \mathbb{R}^+$, equation $Tu + te = u$ has no solution in ∂K_R. For a such $R > 0$, we deduce from Lemma 2 that

$$i(T, K_R, K) = 0.$$

Let us prove existence of $r > 0$ small enough, such that for all $t \in [0, 1]$, equation $tTu = u$ has no solution in ∂K_r. By the contrary, suppose that for each integer $n \succeq 1$, there exist $t_n \in [0, 1]$ and $u_n \in \partial K_{\frac{1}{n}}$ such that

$$u_n = t_n T u_n.$$

Note that $w_n = u_n / \|u_n\| \in \partial K_1$ and satisfies

$$w_n \preceq \frac{g(u_n)}{\|u_n\|}. \tag{3.10}$$

Thus, we have:

$$\frac{g(u_n)}{\|u_n\|} = \frac{g(u_n) - g'(0)(u_n)}{\|u_n\|} + \frac{g'(0)(u_n)}{\|u_n\|}. \tag{3.11}$$

We set

$$G_n(u_n) = \frac{g(u_n) - g'(0)(u_n)}{\|u_n\|}.$$

Clearly

$$w_n \preceq G_n(u_n) + g'(0)(w_n). \tag{3.12}$$

Using the fact that $g'(0)$ is increasing, we get:

$$g'(0)(w_n) \preceq g'(0)(G_n(u_n)) + g'(0)(g'(0)(w_n)). \tag{3.13}$$

Because of the compactness of $g'(0)$ there exists a subsequence (w_{n_k}) such that $\lim g'(0)w_{n_k} = w \in P$. In fact, we have that $v \succ 0_E$. Indeed, if $\lim g'(0)w_{n_k} = 0_E$, then because of the compactness of $g'(\infty)$ there exists a subsequence (w_{n_k}) such that $\lim g'(\infty)w_{n_k} = w \in P$. In fact, we have that $w \succ 0_E$. Indeed, if $\lim g'(\infty)w_{n_k} = 0_E$, then inequality (3.12) and the normality of the cone K lead to $\lim w_{n_k} = 0_E$, contradicting $\|w_{n_k}\| = 1$. Therefore, letting $k \to \infty$ in (3.13), we have $w \succeq g'(\infty)w$. This contradicts the hypothesis $\theta_P^{g'(0)} < 1$ and proves existence of $r > 0$ small enough such that for all $t \in [0,1]$, equation $tTu = u$ has no solution in ∂K_r. For a such $r > 0$, we deduce from Lemma 2 that

$$i(T, K_r, K) = 1.$$

This completes the proof.

Arguing as in the proof of Theorem 3, we obtain the following result.

Theorem 2.33. *Suppose that the cone K is normal and solid and T has a right differentiable at zero minorant $h : K \to K$ such that $h(0) = 0$ and $h'(0) \in C_K^P(E)$ is strongly positive. Suppose that T has an asymptotically linear majorant $g : K \to K$ and $g'(\infty) \in C_K^P(E)$ satisfying $\theta_P^{g'(\infty)} < 1 < r\left(h'(0)\right)$. Then T has at least one positive nontrivial fixed point.*

Remark 2.34. If the cone K is solid and for every u of $K^{\star}$, a natural number n can be found such that $\left[h'(0)\right]^n u$ is an interior element of the cone, then the operator $h'(0)$ is strongly positive.

Theorem 2.35. *Suppose that K is solid and T has an asymptotically linear minorant $h : K \to K$ such that $h'(\infty) \in C_K^P(E)$ is strongly positive. Suppose that T has a right differentiable at zero and lower bounded majorant $g : K \to K$, such that $g(0) = 0$ and $g'(0) \in C_K^P(E)$ satisfying $\theta_P^{g'(0)} < 1 < \lambda_P^{h'(\infty)}$. Then T has at least one positive nontrivial fixed point.*

Proof. We have to prove existence of $0 < r < R$ such that

$$i(T, K_r, K) = 1 \text{ and } i(T, K_R, K) = 0.$$

In such a situation, additivity and solution properties of the fixed point index imply that

$$i(T, K_R \setminus \overline{K_r}, K) = i(T, K_R, K) - i(T, K_r, K) = -1$$

and T has a positive fixed point u with $r < \|u\| < R$.

Let $e > 0$ and let us prove existence of $R > 0$ big enough, such that for all $t \in \mathbb{R}^+$, equation $Tu + te = u$ has no solution in ∂K_R. By the contrary, suppose that for each integer $n \geq 1$, there exist $t_n \in \mathbb{R}^+$ and $u_n \in \partial K_n$ such that

$$u_n = Tu_n + t_n e.$$

Note that $w_n = u_n / \|u_n\| \in \partial K_1$ and satisfies the inequality:

$$w_n \succeq (Tu_n / \|u_n\|) \succeq \frac{h(u_n)}{\|u_n\|}. \tag{3.14}$$

Thus, we have
$$\frac{h(u_n)}{\|u_n\|} = \frac{h(u_n) - h'(\infty)(u_n)}{\|u_n\|} + \frac{h'(\infty)(u_n)}{\|u_n\|}. \tag{3.15}$$
We set
$$H_n(u_n) = \frac{h(u_n) - h'(\infty)(u_n)}{\|u_n\|}.$$
Then
$$w_n \succeq H_n(u_n) + h'(\infty)(w_n). \tag{3.16}$$
Using the fact that $h'(\infty)$ is increasing, we get
$$h'(\infty)(w_n) \succeq h'(\infty)(H_n(u_n)) + h'(\infty)(h'(\infty)(w_n)). \tag{3.17}$$
Because of the compactness of the operator $h'(\infty)$, there exists a subsequence (w_{n_k}) such that $\lim h'(\infty)w_{n_k} = w \in P$. Since $h'(\infty)$ is strongly positive on K and $\|w_{n_k}\| = 1$, then $h'(\infty)w_{n_k} \in int\,(K)$; we deduce from Lemma 3.7 in [12] that there exists $r_0 > 0$ small enough such that $h'(\infty)w_{n_k} \succeq r_0 w_{n_k}$ and we obtain $w \succ 0_E$. Therefore, letting $k \to \infty$ in (3.17), we have $w \succeq h'(\infty)w$. This contradicts the hypothesis $1 < \lambda_P^{h'(\infty)}$ and proves existence of $R > 0$ large enough such that for all $t \in \mathbb{R}^+$, equation $Tu + te = u$ has no solution in ∂K_R. For a such $R > 0$, we deduce from Lemma 2 that
$$i(T, K_R, K) = 0.$$
Let us prove existence of $r > 0$ small enough, such that for all $t \in [0,1]$, equation $tTu = u$ has no solution in ∂K_r. By the contrary suppose that for each integer $n \geq 1$, there exist $t_n \in [0,1]$ and $u_n \in \partial K_{\frac{1}{n}}$ such that
$$u_n = t_n T u_n.$$
Note that $v_n = u_n/\|u_n\| \in \partial K_1$ and satisfies
$$v_n \preceq \frac{g(u_n)}{\|u_n\|}. \tag{3.18}$$
Thus, we have:
$$\frac{g(u_n)}{\|u_n\|} = \frac{g(u_n) - g'(0)(u_n)}{\|u_n\|} + \frac{g'(0)(u_n)}{\|u_n\|}. \tag{3.19}$$
We set
$$G_n(u_n) = \frac{g(u_n) - g'(0)(u_n)}{\|u_n\|}.$$
Clearly
$$v_n \preceq G_n(u_n) + g'(0)(v_n).$$
Using the fact that $g'(0)$ is increasing, we get:
$$g'(0)(v_n) \preceq g'(0)(G_n(u_n)) + g'(0)(g'(0)(v_n)). \tag{3.20}$$
Because of the compactness of $g'(0)$, there exists a subsequence (v_{n_k}) such that $\lim g'(0)v_{n_k} = v \in P$. In fact, we have that $v \succ 0_E$. Indeed, if $\lim g'(0)v_{n_k} = 0_E$, then because of the lower boundeness of $g'(0)$, we have $\|v\| = \lim \|g'(0)v_{n_k}\| \geq N^-_{g,P} > 0$. Thus, letting $k \to \infty$ in (3.20), we obtain $v \preceq g'(0)v$ and $1 \in \Theta_P^{g'(0)}$. This contradicts the hypothesis $1 > \theta_P^{g'(0)}$ and proves existence of $r > 0$ small enough such that for all

$t \in [0,1]$, equation $tTu = u$ has no solution in ∂K_r. For a such $r > 0$, we deduce from Lemma 2 that

$$i(T, K_r, K) = 1.$$

This completes the proof.

Remark 2.36. If in addition, the cone K is normal, then the condition $\lambda_P^{h'(\infty)}$ can be replaced by $r\left(h'(\infty)\right)$.

Arguing as in the proof of Theorem 3, we obtain the following results.

Theorem 2.37. *Suppose that K is a solid cone and T has a right differentiable at zero minorant $h : K \to K$ such that $h(0) = 0$, $h'(0) \in C_K^P(E)$ is strongly positive. Suppose that T has an asymptotically linear and lower bounded majorant $g : K \to K$ and $g'(\infty) \in C_K^P(E)$ satisfying $\theta_P^{g'(\infty)} < 1 < \lambda_P^{h'(0)}$. Then T has at least one positive nontrivial fixed point.*

Arguing as in the above proofs, we obtain the following results.

Theorem 2.38. *Assume that the cone K is solid and suppose that T has a right differentiable at zero minorant $h : K \to K$ such that $h(0) = 0$ and $h'(0) \in C_K^P(E)$ is strongly positive. Suppose that T has an asymptotically linear majorant $g : K \to K$ and $g'(\infty) \in C_K^P(E)$ is strongly positive satisfying $r\left(g'(\infty)\right) < 1 < r\left(h'(0)\right)$. Then, T has at least one positive nontrivial fixed point.*

Theorem 2.39. *Assume that K is a normal cone in E. Suppose the cone K is solid and T has a right differentiable at zero majorant $g : K \to K$ such that $g(0) = 0$ and $g'(0) \in C_K^P(E)$ is strongly positive. Suppose that T has an asymptotically linear minorant $h : K \to K$ and $h'(\infty) \in C_K^P(E)$ is strongly positive satisfying $r\left(g'(0)\right) < 1 < r\left(h'(\infty)\right)$. Then T has at least one positive nontrivial fixed point.*

Corollary 2.40. *Assume that K is a normal cone in E. Suppose the cone K is solid and T has a right differentiable at zero majorant $g : K \to K$ such that $g(0) = 0$ and $g'(0) \in C_K^P(E)$ has a SIJP at $r\left(g'(0)\right)$. Suppose that T has an asymptotically linear minorant $h : K \to K$, $h'(\infty) \in C_K^P(E)$ is strongly positive and satisfying $r\left(g'(0)\right) < 1 < r\left(h'(\infty)\right)$. Then T has at least one positive nontrivial fixed point.*

4. $\phi-$Laplacian bvp with mixed boundary conditions

In this section, we present applications of Theorem 3 and Theorem 3 to a $\phi-$Laplacian bvp with mixed boundary conditions. In all this section, E is the Banach space of all continuous functions defined on $[0,1]$ equipped with its sup-norm denoted $\|\cdot\|$, C is the normal cone of nonnegative functions in E and $u \in E$ is said to be positive if $u \in C^*$, $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $a : [0,1] \to \mathbb{R}^+$ is continuous and does not vanishes identically on any subinterval of $[0,1]$ and ϕ is an odd increasing homeomorphism of $\mathbb{R}$. Throughout we assume that

$$\begin{aligned} &\exists \alpha, \beta \in \mathbb{R} \text{ with } 0 < \alpha < \beta \text{ such that} \\ &t^{\beta}\phi(x) \preceq \phi(tx) \leq t^{\alpha}\phi(x) \text{ for all } x \geq 0 \text{ and } t \in (0,1). \end{aligned} \tag{4.1}$$

In what follows, ψ is the inverse function of ϕ and we have from (4.1)

$$t^{\frac{1}{\alpha}}\psi(x) \leq \psi(tx) \leq t^{\frac{1}{\beta}}\psi(x) \text{ for all } x \geq 0 \text{ and } t \in (0,1). \tag{4.2}$$

Let ψ^+,ψ^- be the function defined on $\mathbb{R}^+$ by

$$\psi^+(x) = \begin{cases} x^{\frac{1}{\beta}} \text{ if } x \leq 1 \\ x^{\frac{1}{\alpha}} \text{ if } x \geq 1 \end{cases}, \quad \psi^-(x) = \begin{cases} x^{\frac{1}{\alpha}} \text{ if } x \leq 1 \\ x^{\frac{1}{\beta}} \text{ if } x \geq 1. \end{cases}$$

It follows from (4.2) that for all $t \geq 0$ and $x \geq 0$

$$\psi^-(t)\,\psi(x) \leq \psi(tx) \leq \psi^+(t)\,\psi(x). \tag{4.3}$$

Let $F : C \to C$ be the operator defined for $u \in C$ by

$$Fu(x) = \psi(f(x, u(x))) \text{ for all } x \in [0,1].$$

It is easy to see that F is continuous and bounded (maps bounded sets into bounded sets).
We set

$$f^0 = \limsup_{u \to 0} \left(\max_{t \in [0,1]} \frac{\psi(f(t,u))}{u} \right), \quad f^\infty = \limsup_{u \to +\infty} \left(\max_{t \in [0,1]} \frac{\psi(f(t,u))}{u} \right).$$

Consider the $\phi-$Laplacian bvp

$$\begin{cases} -(\phi(u'))'(t) = a(t)\, f(t, u(t)), \quad t \in (0,1) \\ u(0) = u'(1) = 0, \end{cases} \tag{4.4}$$

and let the following linear eigenvalue problem

$$\begin{cases} -u''(t) = \mu\, a(t)\, u(t), \quad t \in (0,1) \\ u(0) = u'(1) = 0. \end{cases} \tag{4.5}$$

Let

$$Y = \left\{ u \in C^1([0,1]),\, u(0) = u'(1) = 0 \right\},$$

equipped with C^1-norm denoted $\|\cdot\|_1$ (for $u \in Y, \|u\|_1 = \max(\|u\|, \|u'\|)$). From Ascoli-Arzéla Theorem, the embedding $i_Y : Y \to E$ is compact.
Let A, $N : E \to E$ be the operators defined for $h \in E$ by

$$Nh(x) = \int_0^x \psi\left(\int_t^1 a(s)\phi(h(s))\, ds \right) dt, \text{ for all } x \in [0,1],$$

$$Ah(x) = \int_0^x \left(\int_t^1 a(s) h(s)\, ds \right) dt, \text{ for all } x \in [0,1],$$

and A_Y, $N_Y : Y \to Y$ be the restrictions of A and N to Y.
It is easy to see that

$$P = \{ u \in C, u(x) \geq x\, \|u\|,\ \forall x \in [0,1] \}$$

is a cone in E and $C_Y = C \cap Y = i_Y^{-1}(C)$ is a cone in Y.
The proof of the following Lemma is easy, so we omit it.
Lemma 3.1. *We have that*

1) *A is increasing on C,*
2) *A is lower bounded on P and $\|Au\| \geq M^- \|u\|$ for all $u \in P$ where*

$$M^- = \|a\| \int_0^1 \left(\int_t^1 s\, \frac{a(s)}{\|a\|} ds \right) dt.$$

3) *$A(C) \subset P$.*
4) *N is increasing on C.*
5) *N is upper bounded on C and $\|Nu\| \leq M_\phi^+ \|u\|$ for all $u \in C$ where*

$$M_\phi^+ = \psi^+ (\|a\|) \int_0^1 \left(\int_t^1 \frac{a(s)}{\|a\|} ds \right)^{\frac{1}{\beta}} dt,$$

6) *N is lower bounded on P and $\|Nu\| \geq M_\phi^- \|u\|$ for all $u \in P$ where*

$$M_\phi^- = \psi^- (\|a\|) \int_0^1 \left(\int_t^1 \frac{a(s)s^\beta}{\|a\|} ds \right)^{\frac{1}{\alpha}} dt.$$

7) *A and N are completely continuous operators.*

Observe that u is a positive solution of (4.4) if and only if u is a fixed point of the completely continuous operator $T = NF$.
In view of Lemma 2, let us prove that $A(C^*) \subset 0 \subset C_Y$, where

$$O = \{u \in Y, u(x) > 0, \ \forall x \in (0,1] \text{ and } u'(0) > 0\}$$

is an open set in Y.
Let $O^c = F_1 \cup F_2$ where

$$\begin{aligned} F_1 &= \{u \in X : \text{there exists } x \in]0,1], \ u(x) \leq 0\} \text{ and} \\ F_2 &= \{u \in X : u'(0) \leq 0\}. \end{aligned}$$

It is clear that F_2 is a closed set in X, so let $(u_n) \subset F_1$ tending to u in X and $(x_n) \subset]0,1]$ tending to $\overline{x} \in [0,1]$ with $u_n(x_n) \leq 0$. We distinguish the following cases:

- $\overline{x} \in]0,1]$; in a such situation $u(\overline{x}) = \lim u_n(x_n) \leq 0$ and $u \in F_1$,
- $\overline{x} = 0$; in this case we obtain

$$u'(0) = \lim \frac{u_n(x_n)}{x_n} \leq 0 \text{ and } u \in F_2.$$

Now, let us show $A_Y(C_Y^*) \subset O$. We deduce that for all $h \in C_Y^*$,

$$\|Ah\| = Ah(1) > 0 \text{ and } Ah(x) \geq x \|Ah\| > 0, \ \forall x \in (0,1]$$

and since $Ah(0) = 0$, then $(Ah)'(0) \geq 0$.
If $(Ah)'(0) = 0$, then after two integration we get

$$Ah(x) = -\int_0^x \left(\int_0^t a(s)h(s)ds \right) dt \leq 0$$

which is impossible. So we have $(Ah)'(0) \geq 0$ and $Ah \in O$. That is,

$$A(C_Y{}^\star) \subset O \subset int(C_Y).$$

To complete the proof, we show that A is a C_Y-normal operator.
Let $u_1, u_2 \in C_Y$ with $u_1 \leq u_2$, $v_1 = Au_1$ and $v_2 = Au_2$.

Clearly that $\|v_1\|_Y = \max(\|v_1\|, \|v_1'\|) = \max(v_1(1), v_1'(0))$, which implies

$$\begin{aligned} \|v_1\|_X = v_1(1) &= \int_0^1 \left(\int_t^1 a(s)u_1(s)\,ds \right) \\ &\le \int_0^1 \left(\int_t^1 a(s)u_2(s)\,ds \right) \\ &= v_2(1) \\ &= \|v_2\|_Y \end{aligned}$$

and

$$\begin{aligned} \|v_1\|_Y = v_1'(0) &= \int_0^1 a(s)u_1(s)\,ds \\ &\le \int_0^1 a(s)u_2(s)\,ds \\ &= v_2'(0) \\ &= \|v_2\|_Y . \end{aligned}$$

At the end, Lemma 2 guaranties the existence of a unique positive eigenvalue of A and we have $\mu^{-1} = \lambda_{C_Y} = \theta_{C_Y}$. Since 0 is not an eigenvalue of A and $A(C) \subset C_Y$, it follows from Remark 2 that

$$\mu^{-1} = \lambda_C = \theta_C.$$

We get that A admits a unique positive eigenvalue $\lambda_\star = \lambda_C^A = \theta_C^A$ and $(\lambda_\star)^{-1} = \mu$. Now, we discuss the existence of at least one positive solution to the boundary value problem (4.4).

Theorem 3.2. *Suppose that ϕ satisfies*

$$\lim_{x\to+\infty} \frac{\phi(x)}{x} = 1 , \tag{4.6}$$

then if

$$f^\infty \max\left(M_\phi^+, \lambda_*\right) < 1 \tag{4.7}$$

holds true, Problem (4.4) *admits a positive solution.*

Proof. It follows from conditions (4.6), that

$$\lim_{x\to+\infty} \frac{\psi(x)}{x} = 1.$$

Then

$$Nu = Au + \circ(\|u\|) \ \text{ at } \infty.$$

Moreover $f^\infty \max\left(M_\phi^+, \lambda_*\right) < 1$, implies that there exists $\varepsilon > 0$ small enough and positive constant c_1 such that

$$F(u) \le (f^\infty + \varepsilon)\,u + c_1, \quad \text{for all } \ u \in C^*.$$

Leading to

$$Tu \le N\left((f^\infty + \varepsilon)\,u + c_1\right), \quad \text{for all } \ u \in C^*.$$

We introduce the following notations

$$\begin{aligned} \gamma &= f^{\infty} + \varepsilon, \\ g(u) &= N(\gamma u + c_1). \end{aligned}$$

It is clear that

$$g(u) = \gamma A u + \circ(\|u\|) \quad \text{at} \quad \infty.$$

We have

$$\begin{aligned} \|g(h) - g'(\infty)(h)\| &= \|N(\gamma h + c_1) - \gamma A(h)\| \\ &= \|A(\gamma h + c_1) + \|\gamma h + c_1\| \varepsilon(h) - \gamma A(h)\| \\ &= \|\gamma A(h) + A(c_1) + \|\gamma h + c_1\| \varepsilon(h) - \gamma A(h)\| \\ &\leq \|A(c_1)\| + \|\gamma h + c_1\| \|\varepsilon(h)\| \\ &\leq c_1 M + \|\gamma h + c_1\| \|\varepsilon(h)\|, \end{aligned}$$

where

$$M = \int_0^1 a(s)ds \text{ and } \lim_{h\to\infty} \varepsilon(h) = 0.$$

Since

$$\frac{\|g(h) - g'(\infty)h\|}{\|h\|} \leq \frac{c_1 M}{\|h\|} + \left(\gamma + \frac{c_1}{\|h\|}\right) \|\varepsilon(h)\|,$$

then g is asymptotically linear along C and we have

$$g'(\infty)(u) = \gamma\, Au \text{ and } \theta_P^{g'(\infty)} = \gamma\, \lambda_* \leq \gamma \, \max\left(M_\phi^+, \lambda_*\right) < 1.$$

Applying Theorems 3, we deduce existence of a positive solution for Problem (4.4).

Theorem 3.3. *Suppose that ϕ satisfies*

$$\lim_{x\to 0} \frac{\phi(x)}{x} = 1 \ , \tag{4.8}$$

then if

$$f^0 \max\left(M_\phi^+, \lambda_*\right) < 1 \tag{4.9}$$

holds true, Problem (4.4) *admits a positive solution.*

Proof. It follows from condition (4.8) that

$$\lim_{x\to 0} \frac{\psi(x)}{x} = 1.$$

Then

$$Nu = Au + \circ(\|u\|) \text{ at } 0.$$

Moreover $f^0 \max\left(M_\phi^+, \lambda_*\right) < 1$, implies that there exists $\varepsilon > 0$ small enough and positive constants c_1 such that

$$F(u) \leq \left(f^0 + \varepsilon\right) u + Gu, \quad \text{for all } \ u \in C^*,$$

where $Gu(t) = \max\left\{\psi\left(f(t, u(t)) - f^0 u(t), 0\right\}\right.$.

Leading to

$$Tu \leq N\left(\left(f^0 + \varepsilon\right) u + Gu\right), \quad \text{for all } \ u \in C^*.$$

We introduce the following notations

$$\begin{aligned} \gamma &= f^0 + \varepsilon, \\ g(u) &= N(\gamma u + Gu). \end{aligned}$$

It is clear that

$$g(u) = \gamma A u + \circ(\|u\|) \quad \text{at} \quad 0.$$

We have

$$\begin{aligned} \|g(h) - g'(0)(h)\| &= \|N(\gamma h + Gh) - \gamma A(h)\| \\ &= \|A(\gamma h + Gh) + \|\gamma h + Gh\|\varepsilon(h) - \gamma A(h)\| \\ &= \|\gamma A(h) + A(Gh) + \|\gamma h + Gh\|\varepsilon(h) - \gamma A(h)\| \\ &= \|A(Gh)\| + \|\gamma h + Gh\|\|\varepsilon(h)\| \\ &\le c\|Gh\| + \|\gamma h + Gh\|\|\varepsilon(h)\|, \end{aligned}$$

where $\lim_{h\to 0} \varepsilon(h) = 0$.
Since

$$\frac{\|g(h) - g'(0)h\|}{\|h\|} \le c\frac{\|Gh\|}{\|h\|} + \left(\gamma + \frac{\|Gh\|}{\|h\|}\right)\|\varepsilon(h)\|,$$

then there exists the right derivative of g along P at 0 and that $g'(0)(u) = \gamma A u$ with

$$\theta_P^{g'(0)} = \gamma\lambda_* \le \gamma \max\left(M_\phi^+, \lambda_*\right) < 1.$$

Applying Theorem 3, we deduce existence of a positive solution for Problem (4.4).

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References

[1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces,* SIAM Rev., **18**(1976), 620-709.

[2] H. Amann, *Fixed points of asymptoticaly linear maps in ordered Banach spaces*, J. Funct. Anal., **14**(1973), 162-171.

[3] N. Benkaci-Ali, A. Benmezai, S. Ntouyas, *Eigenvalue criteria for existence of positive solutions to singular third-order BVPs via the index-jump property*, Comm. Appl. Nonl. Anal., **20**(2013), no. 2, 55-74.

[4] A. Benmezai, B. Bouchenb, J. Henderson, S. Mechrouk, *The index-jump property for 1-homogeneous positive maps and fixed point theorems in cones,* Nonlinear Funct. Anal., (2017), Article ID 6, 1-33.

[5] A. Benmezai, J.R. Graef, L. Kong, *Positive solutions for the abstract Hammerstein equations and applications*, Commun. Math. Anal., **16**(2014), no. 1, 47-65.

[6] A. Benmezai, S. Mechrouk, *Positive solutions for the nonlinear abstract Hammerstein equation and application to ϕ-Laplacian BVPs*, NoDEA, **20**(2013), 489-510.

[7] K. Deimling, *Nonlinear Functional Analysis,* Springer-Verlag, 1985.

[8] M.S. El Khannoussi, A. Zertiti, *Topological methods in the study of positive solutions for operator equations in ordered Banach spaces*, Electronic Journal of Differential Equations, **2016**(2016), no. 171, 1-13.

[9] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[10] M.A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff, Gröningen, 1964.

[11] S. Mechrouk, *Fixed point theorems in cones and application,* Mediterr. J. Math., **59**(2020), 1-20.

[12] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Vol. I, *Fixed Point Theorems*, Springer Verlag, New York 1986.

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