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EXISTENCE AND APPROXIMATION OF FIXED POINTS OF λ-HYBRID MAPPINGS IN COMPLETE CAT(0) SPACES

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Abstract. In this paper, we first state and establish some general existence theorems in Hadamard spaces. Then, we introduce class of λ-hybrid mappings in Hadamard spaces and prove existence of fixed point for such mappings. Finally, we establish convergence theorem of Mann's iterative process for λ-hybrid mappings.

Key Words and Phrases: Existence theorem, λ-hybrid mappings, nonexpansive mapping, fixed point, CAT(0) metric space.
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1. Introduction

Let (X, d) be a metric space. Berg and Nikolaev [5] introduced the concept of *quasilinearization* in metric spaces. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is the map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2} \left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d) \right), \qquad (a, b, c, d \in X). \tag{1.1}$$

It is clear that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ and

$$d^2(a, b) = d^2(a, x) + d^2(b, x) - 2\langle \overrightarrow{ax}, \overrightarrow{bx} \rangle \tag{1.2}$$

for all $a, b, c, d, x \in X$.
Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l]$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. The image of c is called a geodesic segment with endpoints x and y. The metric space X is geodesically connected if any two of its points can be joined by a geodesic segment. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesically connected metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of) and a geodesic segment between each pair of

vertices (the edges of). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(x_i, x_j) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$.

A metric space (X, d) is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane, that is, for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

For other equivalent definitions and basic properties, we refer the reader to standard texts such as [4, 7]. Complete CAT(0) spaces are often called Hadamard spaces. Let $x, y \in X$ and $t \in [0, 1]$. We write $tx \oplus (1-t)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = (1-t)d(x, y) \quad \text{and} \quad d(z, y) = td(x, y). \tag{1.3}$$

We also denote by $[x, y]$ the geodesic segment joining from x to y, that is,

$$[x, y] = \{tx \oplus (1-t)y : t \in [0, 1]\}.$$

A subset C of a CAT(0) space is convex if $[x, y] \subseteq C$ for all $x, y \in C$. The metric space X is said to satisfy the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leqslant d(a, b)d(c, d),$$

for all $a, b, c, d \in X$. It is known [5, Corollary 3] that a geodesically connected metric space is CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

The concept of Δ-convergence introduced by Lim [17] in 1976 was shown by Kirk and Panyanak [15] in CAT(0) spaces to be very similar to the weak convergence in Hilbert space setting. Next, we give the concept of Δ-convergence and collect some basic properties. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [10] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. We say that a subset C of X is Δ-closed if for every sequence $\{x_n\} \subset C$ that Δ-converges to x we have $x \in C$. In recent years, Δ-convergence of iterative sequences to fixed points and solutions of optimization problems find more attentions, see, e.g., [12, 1, 26, 20] and references therein. Uniqueness of asymptotic center implies that CAT(0) space X satisfies Opial's property, i.e., for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y). \tag{1.4}$$

We need the following lemmas in the sequel.

Lemma 1.1. [15] *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 1.2. [9] *If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C. In other words, every closed convex subset of a complete CAT(0) space is Δ-closed.*

Lemma 1.3. [13] *Let X be a complete CAT(0) space, $\{x_n\}$ be a bounded sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0$ for all $y \in X$.*

Lemma 1.4. [11, Lemma 2.5] *A geodesic space X is a CAT(0) space if and only if the following inequality*

$$d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y), \tag{1.5}$$

is satisfied for all $x, y, z \in X$ and $t \in [0, 1]$.

Let C be a nonempty closed convex subset of a complete CAT(0) space X. It is known that for any $x \in X$ there exists a unique point $u \in C$ such that

$$d(x, u) = \inf_{y\in C} d(x, y).$$

The mapping $P_C : X \to C$ defined by $P_C x = u$ is called the metric projection from X onto C. Dehghan and Rooin [8] obtained the following characterization of metric projection in CAT(0) metric spaces.

Theorem 1.5. [8, Theorem 2.2] *Let C be a nonempty closed convex subset of a complete CAT(0) space X, $x \in X$ and $u \in C$. Then*

$$u = P_C x \quad \textit{if and only if} \quad \langle \overrightarrow{ux}, \overrightarrow{yu}\rangle \geq 0, \quad \textit{for all } y \in C.$$

Lemma 1.6. [1, Lemma 4.3] *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $\{z_n\}$ be a sequence in X such that*

$$d(z_{n+1}, z) \leq d(z_n, z),$$

for all $z \in C$ and $n \geq 1$. Then, $\{P_C z_n\}$ converges to some $u \in C$.

Let l^∞ be the Banach space of bounded real sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$ for all $f = (x_1, x_2, \ldots) \in l^\infty$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have

$$\mu(f) = \mu_n(x_n) = a.$$

For a proof of existence of a Banach limit and its other elementary properties, see [24].

2. Some existence theorems in Hadamard spaces

In this section, we first prove some fundamental existence theorems in Hadamard space. Then, using these results we obtain existence of fixed points for large class of mappings.

Let X be an Hadamard space. Recall a function $f : X \to \mathbb{R}$ is said to be lower semicontinuous if for every $\alpha \in \mathbb{R}$ the set

$$\{x \in X : f(x) \leq \alpha\},$$

is closed. If for each $x, y \in X$ and each $t \in [0, 1]$ the inequality

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds, we say that f is convex.

In a general topological space X, it is well known (see e.g., [21]) that if $f : X \to \mathbb{R}$ is lower semicontinuous then for every sequence $\{x_n\}$ in X that converges to x_0, we have

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

Note that both of lower semicontinuity of f and covergence of x_n are respect to the same topology. Some known topologies in Hadamard spaces studied by Kakavandi in [13], which are related to strong convergence, w-convergence and Δ-convergence. A natural question is: what happen if we weaken the convergence condition on the sequence? In the following lemma, we provide an appropriate answer to the question.

Lemma 2.1. [3, Lemma 3.2.3] *Let X be an Hadamard space and $f : X \to \mathbb{R}$ be a lower semicontinuous and convex function. If the sequence $\{x_n\}$ in X, Δ-converges to x_0, then*

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n). \tag{2.1}$$

The existing of minima for lower semicontinuous functions on complete metric spaces (possibly noncompact) has been studied by many authors from different points of view (see e.g. [6, 22]). Usually, there is at least an existence assumption in such results (see e.g. [6, (iii) in Section 5]). In the next theorem, there is no existence assumption. The existence of minima derived from structure of Hadamard spaces.

Theorem 2.2. *Let C be a nonempty closed convex subset of an Hadamard space X and o be an arbitrary and fixed element of X. Let $f : X \to \mathbb{R}$ be a lower semicontinuous and convex function such that $f(z_m) \to \infty$ as $d(z_m, o) \to \infty$. Then f takes its mimum, i.e., there exists $x_0 \in C$ such that*

$$f(x_0) = \min\{f(x) : x \in C\}.$$

Proof. Let $\{x_n\}$ be a minimizing sequence of f over C. This means that for the sequence $\{x_n\} \subset C$ we have

$$\lim_{n\to\infty} f(x_n) = \inf\{f(x) : x \in C\}.$$

From the assumption $f(z_m) \to \infty$ as $d(z_m, o) \to \infty$ we conclude that $\{x_n\}$ is bounded. Using Lemma 1.1 there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that Δ-converges to x_0. It follows from Lemma 1.2 that $x_0 \in C$. Note that C is closed and convex and so it is an Hadamard space. Hence, by Lemma 2.1 we have

$$f(x_0) \leq \liminf_{k\to\infty} f(x_{n_k}).$$

Therefore,

$$f(x_0) \leq \liminf_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = \inf\{f(x) : x \in C\} \leq f(x_0).$$

This completes the proof. □

To prove fixed point theorem in Hadamard spaces we need the following lemma in which we follow the techniques in [25].

Lemma 2.3. *Let C be a nonempty closed convex subset of an Hadamard space X, $\{x_n\}$ be a bounded sequence in X and μ be a Banach limit. If $f : C \to \mathbb{R}$ is defined by*

$$f(x) = \mu_n d^2(x_n, x), \quad \forall x \in C,$$

then there exists a unique $x_0 \in C$ such that

$$f(x_0) = \min\{f(x) : x \in C\}.$$

Proof. Put $y, z \in C$ and $t \in [0, 1]$. Then by (1.5) we have

$$\begin{aligned} d^2(x_n, ty \oplus (1-t)z) &\leq td^2(x_n, y) + (1-t)d^2(x_n, z) - t(1-t)d^2(y, z) \\ &\leq td^2(x_n, y) + (1-t)d^2(x_n, z). \end{aligned}$$

Taking Banach limit μ from both sides we get

$$\begin{aligned} f(ty \oplus (1-t)z) &= \mu_n d^2(x_n, ty \oplus (1-t)z) \\ &\leq \mu_n t d^2(x_n, y) + \mu_n (1-t) d^2(x_n, z) \\ &= tf(y) + (1-t)f(z). \end{aligned}$$

This shows that f is convex function. Let $\{z_m\}$ be a sequence in C such that $z_m \to z$. For any $n, m \in \mathbb{N}$, we have

$$\begin{aligned} d^2(x_n, z_m) - d^2(x_n, z) &= |d(x_n, z_m) - d(x_n, z)| \left(d(x_n, z_m) + d(x_n, z)\right) \\ &\leq M_1 d(z_m, z), \end{aligned}$$

where $M_1 = \sup_{n,m\in\mathbb{N}}(d(x_n, z_m) + d(x_n, z))$. Therefore,

$$f(z_m) - f(z) \leq M_1 d(z_m, z).$$

Similarly, we have

$$f(z) - f(z_m) \leq M_1 d(z_m, z).$$

So, we have

$$|f(z_m) - f(z)| \leq M_1 d(z_m, z).$$

This implies that f is continuous. Suppose that o is an arbitrary and fixed element of X and $\{z_m\}$ is a sequence in C such that $d(z_m, o) \to \infty$. Then, by (1.2) we have

$$\begin{aligned} d^2(z_m, o) &= d^2(z_m, x_n) + d^2(o, x_n) + 2\langle \overrightarrow{z_m x_n}, \overrightarrow{x_n o} \rangle \\ &\leq d^2(z_m, x_n) + d^2(o, x_n) + 2(d(o, z_m) + d(o, x_n))d(o, x_n) \\ &\leq d^2(z_m, x_n) + M_2^2 + 2(d(o, z_m) + M_2)M_2, \end{aligned}$$

where $M_2 = \sup_{n\in\mathbb{N}} d(o, x_n)$. Therefore, we have

$$d(z_m, o)(d(z_m, o) - 2M_2) - 3M_2^2 \leq d^2(z_m, x_n).$$

So,

$$d(z_m, o)(d(z_m, o) - 2M_2) - 3M_2^2 \leq \mu_n d^2(z_m, x_n) = f(z_m).$$

It follows that $f(z_m) \to \infty$ as $d(z_m, o) \to \infty$. Using Theorem 2.2, there exists $x_0 \in C$ such that

$$f(x_0) = \min\{f(x) : x \in C\}.$$

To prove uniqueness, let x_0 and y_0 be elements in C such that $x_0 \neq y_0$ and

$$f(x_0) = f(y_0) = \min\{f(x) : x \in C\} = t.$$

Put an arbitrary $t_0 \in (0, 1)$. It follows from (1.5) that

$$d^2(x_n, t_0x_0 \oplus (1 - t_0)y_0) \leq t_0 d^2(x_n, x_0) + (1 - t_0)d^2(x_n, z_0) - t_0(1 - t_0)d^2(x_0, y_0).$$

Applying Banach limit μ, we have

$$\begin{aligned} f(t_0x_0 \oplus (1 - t_0)y_0) &\leq t_0 f(x_0) + (1 - t_0)f(y_0) - t_0(1 - t_0)d^2(x_0, y_0) \\ &= t - t_0(1 - t_0)d^2(x_0, y_0) \\ &< t. \end{aligned}$$

This is a contradiction. So, we have $x_0 = y_0$. □

The following theorem is a generalization of [25, Theorem 4.1].

Theorem 2.4. *Let C be a nonempty closed convex subset of an Hadamard space X and $T : C \to C$ be a mapping. Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded and*

$$\mu_n d^2(T^n x, Ty) \leq \mu_n d^2(T^n x, y), \quad \forall y \in C,$$

for some Banach limit μ. Then, T has a fixed point in C.

Proof. For a Banach limit μ on l^∞, we can define $f : C \to (-\infty, +\infty)$ as follows:

$$f(z) = \mu_n d^2(T^n x, z), \quad \forall z \in C.$$

From Lemma 2.3, there exists a unique $x_0 \in C$ such that

$$f(x_0) = \min\{f(x) : x \in C\}.$$

Hence,

$$f(Tx_0) = \mu_n d^2(T^n x, Tx_0) \leq \mu_n d^2(T^n x, x_0) = f(x_0).$$

Since $Tx_0 \in C$, it follows from uniqueness of x_0 that $Tx_0 = x_0$. This completes the proof. □

3. λ-HYBRID MAPPINGS AND FIXED POINT THEOREMS

In this section, we present definition of a broad class of mappings containing the class of nonexpansive mappings, nonspreading mappings and hybrid mappings in metric space.

Let C be a nonempty subset of a metric space X. Then the mapping $T : C \to X$ is said to be λ-*hybrid* if

$$d^2(Tx, Ty) \leq d^2(x, y) + 2(1 - \lambda)\langle \overrightarrow{xTx}, \overrightarrow{yTy} \rangle \tag{3.1}$$

or equivalently

$$2d^2(Tx, Ty) \leq d^2(x, Ty) + d^2(y, Tx) - 2\lambda\langle \overrightarrow{xTx}, \overrightarrow{yTy} \rangle \tag{3.2}$$

for all $x, y \in C$. This is a generalization of the concept of λ-hybrid mappings introduced by Aoyama et al. [2]. Let $T : C \to X$ be a λ-hybrid mapping.

- If $\lambda = 0$, then T is called nonspreading;
- if $\lambda = 1/2$, then T is called hybrid;
- if $\lambda = 1$, then T is called nonexpansive.

Also, a λ-hybrid mapping with a fixed point is quasi-nonexpansive.

Example 3.1. Consider $\mathbb{R}^2$ with the usual Euclidean norm $\|\cdot\|$. Let $X = \mathbb{R}^2$ be an $\mathbb{R}$-tree with the radial metric d_r, where $d_r(x, y) = d(x, y) = \|x - y\|$ if x and y are situated on a Euclidean straight line passing through the origin $\mathbf{0} = (0, 0)$ and

$$d_r(x, y) = d(x, \mathbf{0}) + d(y, \mathbf{0}) := \|x\| + \|y\|$$

otherwise (see [14] and [18, page 65]). Let $\lambda \in [0, 1)$,

$$\alpha = \frac{\lambda(1 - \lambda) + \sqrt{2(1 - \lambda)}}{1 - \lambda^2},$$

$A = \{x \in X : \|x\| \leq 1\}$, $B = \{x \in X : \|x\| \leq \alpha\}$, and define the mapping $T : X \to X$ as follows:

$$Tx = \begin{cases} 0 & (x \in B) \\ P_A(x) = x/\|x\| & (x \in X \backslash B). \end{cases}$$

We show that Then T is λ-hybrid. First, note that $\alpha > 1$. We may write the inequality (3.1) as

$$\lambda d_r^2(x, y) + (\lambda - 2)d_r^2(Tx, Ty) + (1 - \lambda)\left(d_r^2(x, Ty) + d_r^2(y, Tx)\right) \geq 0. \tag{3.3}$$

(i) In the case that $x, y \in B$ we have $d_r(Tx, Ty) = 0$ and so (3.3) clearly holds.

(ii) (a) In the case that $x, y \in X\backslash B$ are on a straight ray initiating from the origin, again we have $d_r(Tx, Ty) = 0$ and so (3.3) holds.

(b) If $x, y \in X\backslash B$ are not on a straight ray initiating from the origin, then

$$\begin{aligned}&\lambda d_r^2(x,y) + (\lambda-2)d_r^2(Tx,Ty) + (1-\lambda)\left(d_r^2(x,Ty) + d_r^2(y,Tx)\right)\\ &= \lambda(\|x\|+\|y\|)^2 + 4(\lambda-2) + (1-\lambda)\left((\|x\|+1)^2 + (\|y\|+1)^2\right)\\ &= 2\lambda(\|x\|-1)(\|y\|-1) + \|x\|^2 + \|y|2 + 2\|x\| + 2\|y\| - 6\\ &\geq \|x\|^2 + \|y\|^2 + 2\|x| + 2\|y\| - 6\\ &\geq 0.\end{aligned}$$

(iii) (a) Let $x \in B$ and $y \in X\backslash B$ be on a straight ray initiating from the origin. Since in this case $d_r(x,y) = d(x,y) = \|x-y\|$ and quasilinearization $\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle$ coincides with the inner product $\langle a-b, c-d\rangle$ for all a,b,c,d on this ray, the conclusion follows from [2, Example 3.4].

(b) If $x \in B$ and $y \in X\backslash B$ are not on a straight ray initiating from the origin, then

$$\begin{aligned}&\lambda d_r^2(x,y) + (\lambda-2)d_r^2(Tx,Ty) + (1-\lambda)\left(d_r^2(x,Ty) + d_r^2(y,Tx)\right)\\ &= \lambda(\|x\|+\|y\|)^2 + (\lambda-2) + (1-\lambda)\left((\|x\|+1)^2 + \|y\|^2\right)\\ &= 2\lambda\|x\|\|y\|-1) + \|x\|^2 + \|y\|^2 + 2\|x\| - 1\\ &\geq \|x\|2 + \|y\|^2 + 2\|x\| - 1\\ &\geq 0.\end{aligned}$$

Note that T is not nonexpansive mapping.
In fact, if $x = (\alpha - 1/4, 0)$ and $y = (\alpha + 1/4, 0)$, then we have

$$d_r(Tx,Ty) = 1 > \frac{1}{2} = d_r(x,y).$$

The following theorem generalizes the known results from Hilbert spaces in [19, 23, 16].

Theorem 3.2. *Let X be an Hadamard space and $T : X \to X$ be a λ-hybrid mapping. Then T has a fixed point if and only if $\{T^n z\}$ is bounded for some $z \in X$.*

Proof. If $F(T) \neq \emptyset$, then $\{T^n z\} = \{z\}$ for all $z \in F(T)$. Therefore, $\{T^n z\}$ is bounded. To show the reverse, take $z \in X$ such that $\{T^n z\}$ is bounded. Since $T : X \to X$ is a λ-hybrid mapping, then we have

$$d^2(Tx,Ty) \leq d^2(x,y) + 2(1-\lambda)\langle \overrightarrow{xTx}, \overrightarrow{yTy}\rangle \tag{3.4}$$

for all $x, y \in X$. Let $n \in \mathbb{N}$. Replacing x by $T^n z$ in (3.4) gives us

$$\begin{aligned}d^2(T^{n+1}z, Ty) &\leq d^2(T^n z, y) + 2(1-\lambda)\langle \overrightarrow{T^n z T^{n+1} z}, \overrightarrow{yTy}\rangle\\ &= d^2(T^n z, y) + 2(1-\lambda)\left(\langle \overrightarrow{zT^{n+1}z}, \overrightarrow{yTy}\rangle - \langle \overrightarrow{zT^n z}, \overrightarrow{yTy}\rangle\right)\end{aligned}$$

for all $y \in X$. Let μ be a Banach limit. Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ to both sides of the above inequality. Then, we get

$$\mu_n d^2(T^{n+1}z, Ty) \leq \mu_n d^2(T^n z, y) + 2(1-\lambda)\left(\mu_n\langle \overrightarrow{zT^{n+1}z}, \overrightarrow{yTy}\rangle - \mu_n\langle \overrightarrow{zT^n z}, \overrightarrow{yTy}\rangle\right).$$

This together with shift property of Banach limit implies that

$$\mu_n d^2(T^n z, Ty) \le \mu_n d^2(T^n z, y) + 2(1-\lambda)\left(\mu_n \langle \overrightarrow{zT^n z}, \overrightarrow{yTy}\rangle - \mu_n \langle \overrightarrow{zT^n z}, \overrightarrow{yTy}\rangle\right).$$

So, we obtain

$$\mu_n d^2(T^n z, Ty) \le \mu_n d^2(T^n z, y)$$

for all $y \in X$. By Theorem 2.4, we have a fixed point in X.

4. Δ-convergence theorem

In this section, we prove a Δ-convergence theorem of Mann's type for λ-hybrid mappings in Hadamard spaces. Before proving the theorem, we need the following lemma.

Lemma 4.1. *Let X be an Hadamard space, C be a nonempty closed convex subset of X, $\lambda \in \mathbb{R}$ and $T : C \to X$ be a λ-hybrid mapping. Then, $I - T$ is demiclosed, i.e., $\{x_n\}$ Δ-converges to z and $d(x_n, Tx_n) \to 0$ imply $z \in F(T)$.*

Proof. Since $T : C \to X$ is a λ-hybrid mapping, we have

$$d^2(Tx, Ty) \le d^2(x, y) + 2(1-\lambda)\langle \overrightarrow{xTx}, \overrightarrow{yTy}\rangle \tag{4.1}$$

for all $x, y \in C$. Suppose $\{x_n\}$ Δ-converges to z and $d(x_n, Tx_n) \to 0$. It is easily seen that the sequences $\{x_n\}$ and $\{Tx_n\}$ are bounded. Also, using the Cauchy-Schwarz inequality, we obtain

$$\lim_{n\to\infty} \langle \overrightarrow{x_n Tx_n}, \overrightarrow{ab}\rangle = 0, \tag{4.2}$$

$$\lim_{n\to\infty} \langle \overrightarrow{x_n Tx_n}, \overrightarrow{x_n b}\rangle = 0, \tag{4.3}$$

for all $a, b \in X$. Replacing x and y respectively by x_n and z in (4.1) we get

$$d^2(Tx_n, Tz) \le d^2(x_n, z) + 2(1-\lambda)\langle \overrightarrow{x_n Tx_n}, \overrightarrow{zTz}\rangle.$$

This inequality together with (1.2) implies that

$$d^2(Tx_n, x_n) + d^2(Tz, x_n) - 2\langle \overrightarrow{x_n Tx_n}, \overrightarrow{x_n Tz}\rangle \le d^2(x_n, z) + 2(1-\lambda)\langle \overrightarrow{x_n Tx_n}, \overrightarrow{zTz}\rangle.$$

Since the sequences $\{x_n\}$ and $\{Tx_n\}$ are bounded, we can apply a Banach limit μ to both sides of the above inequality. Then, we have

$$\mu_n\left(d^2(Tx_n, x_n) + d^2(Tz, x_n) - 2\langle \overrightarrow{x_n Tx_n}, \overrightarrow{x_n Tz}\rangle\right)$$
$$\le \mu_n\left(d^2(x_n, z) + 2(1-\lambda)\langle \overrightarrow{x_n Tx_n}, \overrightarrow{zTz}\rangle\right).$$

It follows from linearity of Banach limit and (4.2) and (4.3) that

$$\mu_n d^2(x_n, Tz) \le \mu_n d^2(x_n, z).$$

Again, from (1.2), we obtain

$$\mu_n d^2(x_n, z) + \mu_n d^2(Tz, z) - 2\mu_n \langle \overrightarrow{zx_n}, \overrightarrow{zTz}\rangle \le \mu_n d^2(x_n, z).$$

It follows from Lemma 1.3 that

$$\mu_n d^2(Tz,z) \le 2\mu_n\langle \overrightarrow{zx_n}, \overrightarrow{zTz}\rangle \le 2\limsup_{n\to\infty}\langle \overrightarrow{zx_n}, \overrightarrow{zTz}\rangle \le 0.$$

Therefore, $d^2(Tz,z) \le 0$ and so $Tz = z$. This implies that $I - T$ is demiclosed. □

Theorem 4.2. *Let X be an Hadamard space, $T : X \to X$ be a λ-hybrid mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in X$ and*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Tx_n, \qquad n \ge 1. \tag{4.4}$$

Then the sequence $\{x_n\}$ is Δ-convergent to an element $q \in F(T)$, where

$$q = \lim_{n\to\infty} P_{F(T)} x_n.$$

Proof. Let $z \in F(T)$. Since every λ-hybrid mapping is quasi-nonexpansive, we have

$$\begin{aligned} d(x_{n+1},z) &= d(\alpha_n x_n \oplus (1-\alpha_n)Tx_n, z) \\ &\le \alpha_n d(x_n,z) + (1-\alpha_n)d(Tx_n,z) \\ &\le \alpha_n d(x_n,z) + (1-\alpha_n)d(x_n,z) \\ &= d(x_n,z). \end{aligned}$$

That is, the sequence $\{d(x_n,z)\}$ is decreasing and so $\lim_{n\to\infty} d(x_n,z)$ exists. So, we have that $\{x_n\}$ is bounded. It follows from (1.5) that

$$\begin{aligned} d^2(x_{n+1},z) &= d^2(\alpha_n x_n \oplus (1-\alpha_n)Tx_n, z) \\ &\le \alpha_n d^2(x_n,z) + (1-\alpha_n)d^2(Tx_n,z) - \alpha_n(1-\alpha_n)d^2(Tx_n,x_n) \\ &\le \alpha_n d^2(x_n,z) + (1-\alpha_n)d^2(x_n,z) - \alpha_n(1-\alpha_n)d^2(Tx_n,x_n) \\ &= d^2(x_n,z) - \alpha_n(1-\alpha_n)d^2(Tx_n,x_n). \end{aligned}$$

So, we have

$$\alpha_n(1-\alpha_n)d^2(Tx_n,x_n) \le d^2(x_n,z) - d^2(x_{n+1},z).$$

Since $\lim_{n\to\infty} d(x_n,z)$ exists and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have $\lim_{n\to\infty} d(Tx_n,x_n) = 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that Δ-converges to q. By Lemma 4.1, we obtain $q \in F(T)$. Let $\{x_{m_i}\}$ and $\{x_{k_i}\}$ be two subsequences of $\{x_n\}$ such that Δ-converges to q_1 and q_2. To complete the proof, we show $q_1 = q_2$. We know $q_1, q_2 \in F(T)$ and hence $\lim_{n\to\infty} d(x_n,q_1)$ and $\lim_{n\to\infty} d(x_n,q_2)$ exist. If $q_1 \neq q_2$, then from (1.4) we conclued that

$$\begin{aligned} \lim_{n\to\infty} d(x_n,q_1) &= \limsup_{i\to\infty} d(x_{m_i},q_1) < \limsup_{i\to\infty} d(x_{m_i},q_2) \\ &= \lim_{n\to\infty} d(x_n,q_2) = \limsup_{i\to\infty} d(x_{k_i},q_2) \\ &< \limsup_{i\to\infty} d(x_{k_i},q_1) = \lim_{n\to\infty} d(x_n,q_1), \end{aligned}$$

which is a contradiction. Hence, $q_1 = q_2$. Thus $\{x_n\}$ Δ-converges to q.
Put $u_n = P_{F(T)}x_n$. We show that $q = \lim_{n\to\infty} u_n$. Since $q \in F(T)$, it follows from Theorem 1.5 that

$$\langle \overrightarrow{u_n x_n}, \overrightarrow{q u_n} \rangle \geq 0.$$

By Lemma 1.6, $\{u_n\}$ converges strongly to some $u \in F(T)$. Also,

$$\begin{aligned} 0 &\leq \langle \overrightarrow{u_n x_n}, \overrightarrow{q u_n} \rangle \\ &= \langle \overrightarrow{u_n q}, \overrightarrow{q u_n} \rangle + \langle \overrightarrow{q x_n}, \overrightarrow{q u} \rangle + \langle \overrightarrow{q x_n}, \overrightarrow{u u_n} \rangle \\ &\leq \langle \overrightarrow{u_n q}, \overrightarrow{q u_n} \rangle + \langle \overrightarrow{q x_n}, \overrightarrow{q u} \rangle + d(q, x_n) d(u, u_n). \end{aligned}$$

Taking $\limsup_{n\to\infty}$, using Lemma 1.3 and the fact that $\{x_n\}$ Δ-converges to q and $u_n \to u$, we obtain

$$0 \leq \langle \overrightarrow{qu}, \overrightarrow{uq} \rangle = -d^2(q, u),$$

which gives us $q = u$ and the proof is complete. □

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