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# ON INERTIAL TYPE ALGORITHMS WITH GENERALIZED CONTRACTION MAPPING FOR SOLVING MONOTONE VARIATIONAL INCLUSION PROBLEMS

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**Abstract.** In this article, we introduced two iterative processes consisting of an inertial term, forward-backward algorithm and generalized contraction for approximating the solution of monotone variational inclusion problem. The motivation for this work is to prove the strong convergence of inertial-type algorithms under some relaxed conditions because many of the existing results in this direction have only achieved weak convergence. We note that when the space is finite dimension, there is no disparity between weak and strong convergence, however this is not the case in infinite dimension. We provide some numerical examples to justify that inertial algorithms converge faster than non-inertial algorithms in terms of number of iterations and cpu time taken for the computation.
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## 1. Introduction

Let $H$ be a real Hilbert space. An operator $S : H \to 2^H$ is said to be monotone if for any $x, y \in H$, we have

$$\langle u - v, x - y\rangle \geq 0,$$

for $u \in Sx$ and $v \in Sy$. A monotone operator $S$ is said to be maximal if the graph of $S$, $Gr(S) := \{(x, u) \in H \times H : u \in Sx\}$ is not a subset of the graph of any other monotone operator. The problem of finding zeroes of sum of two monotone operators $S$ and $T$, namely, a solution to the inclusion problem

$$0 \in (S+T)x \tag{1.1}$$

continues to be a very attractive research area due to its vast applications in solving real-life problems that can be formulated in this form. For instance, a stationary solution to the initial value problem of the evolution equation

$$0 \in \frac{\partial u}{\partial t} + Fu, \; u(0) = u_0,$$

can be formulated as (1.1), where the governing maximal monotone operator $F$ is of the form $F := S + T$ (see [21]). Also in optimization problem, there is often needs to solve the minimization problem of the form

$$\min_{x \in H} \{f(x) + h(Bx)\}, \tag{1.2}$$

where $f, h : H \to \mathbb{R} \cup \{+\infty\}$ are proper lower semi-continuous convex functions and $B$ is a bounded linear operator on $H$. As a matter of fact, problem (1.2) is equivalent to (1.1) (assuming that $f$ and $h \circ B$ have a common point of continuity) with $S := \partial f$ and $T := B^* \circ \partial h \circ B$ where $B^*$ is the adjoint of $B$ and $\partial f$ is the subdifferential operator of $f$ in the sense of convex analysis. It is known that the minimization problem (1.2) and related optimization problems are widely used in image recovery, signal processing and machine learning (see, for instance [12]).

The classical Forward-Backward Splitting (FBS) algorithm for solving Problem (1.1) is given by the following recursion formula

$$x_{n+1} = \underbrace{(I + \lambda T)^{-1}}_{backward} \underbrace{(x_n - \lambda S x_n)}_{forward}, \quad \lambda > 0, n \in \mathbb{N}. \tag{1.3}$$

In the last several years, different modifications of te FBS algorithm have been considered by many authors, see for instance [8, 13, 20, 22] and reference therein. One of the popular methods used for accelerating the speed of convergence of iterative schemes is the multi-step method which can be viewed as the following discretization of the second-order dynamical system with friction:

$$\ddot{x}(t) + \gamma \dot{x}(t) + \nabla \varphi(x(t)) = 0,$$

where $\gamma > 0$ represents a friction parameter and $\varphi : H \to \mathbb{R}$ is a differentiable function. This can be formulated as a two-step heavy ball method, in which, given $x_n$ and $x_{n-1}$, the next point $x_{n+1}$ is determined via

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} + \gamma \frac{x_n - x_{n-1}}{h} + \nabla \varphi(x_n) = 0,$$

for $h > 0$, which results in an iterative algorithm of the form

$$x_{n+1} = x_n + \beta(x_n - x_{n-1}) - \alpha \nabla \varphi(x_n), \tag{1.4}$$

for each $n \geq 0$, where $\beta = 1 - \gamma h$ and $\alpha = h^2$. In 1964, Polyak [31] first used (1.4) to solve the optimization problem:

$$\min \varphi(x),$$

for all $x \in H$ and called it an *inertial type extrapolation algorithm.* In 1987, Polyak [30] also considered the relationship between the heavy ball method and the following conjugate gradient method

$$x_{n+1} = x_n + \beta_n(x_n - x_{n-1}) - \alpha_n \nabla\varphi(x_n), \tag{1.5}$$

for each $n \geq 0$, where $\alpha_n$ and $\beta_n$ can be choosen through different ways. It is obvious that the only difference between the heavy ball method (1.4) and (1.5) is the choice of the parameters.

From Polyak's work, as an acceleration process, the inertial extrapolation algorithms were widely studied by many researchers, see, for instance [1, 3, 19] and references therein. In [4], Alvarez and Attouch translated the idea of the heavy ball method to the setting of a general maximal monotone operator using the framework of the proximal point algorithm. The resulting algorithm which was called inertial proximal point algorithm is written as:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n T)^{-1} y_n, \ n \geq 1. \end{cases} \tag{1.6}$$

They showed that if $\{r_n\}$ is non-decreasing and $\alpha_n \in [0, 1)$ is such that

$$\sum_{n=0}^{\infty} \|x_n - x_{n-1}\|^2 < \infty, \tag{1.7}$$

then the algorithm (1.6) converges weakly to a zero of $T$.

In subsequent work, Moudafi and Oliny [28] introduced an additional single-valued and Lipschitz continuous operator $S$ into the inertial proximal point algorithm:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n T)^{-1}(y_n - r_n S x_n), \ n \geq 1. \end{cases} \tag{1.8}$$

They proved that the sequence generated by (1.8) converges weakly to a solution of Problem (1.1) provided that (1.8) satisfied the condition (1.7) used in [4]. Note that (1.8) does not take the form of the FBS algorithm since $S$ is still evaluated at the point of $\{x_n\}$.

Recently, Cholomjiak et al. [11] introduced the following inertial FBS algorithm for approximating solution of Problem (1.1) where $T : H \to 2^H$ is maximal monotone, $S : H \to H$ is $\alpha$-inverse strongly monotone and finding the fixed point of quasi-nonexpansive mapping $U$ in Hilbert spaces: Given $x_0, x_1 \in H$ compute

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = \alpha_n y_n + (1 - \alpha_n) U y_n, \\ x_{n+1} = \beta_n z_n + (1 - \beta_n) J_{r_n}^T (I - r_n S) z_n, \ n \geq 1, \end{cases}$$

where $J_{r_n}^T = (I + r_n T)^{-1}$, $\{r_n\} \subset (0, 2\alpha)$, $\{\theta_n\} \subset [0, \theta]$ for some $\theta \in [0, 1)$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$. They proved a weak convergence theorem, provided the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}|| < \infty$,
(ii) $0 < \liminf n \to \infty \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

(iii) $\limsup_{n\to\infty} \beta_n < 1$,

(iv) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\alpha$.

In this paper, we introduce some inertial FBS algorithms for approximating solution of Problem (1.1) without necessarily imposing condition (1.7). Using the concept of generalized contraction, we prove strong convergence results for the sequence generated by our algorithms under some mild conditions. It is worthy to note that strong convergence algorithms are more desirable than the weak convergence ones when solving optimization problems (see [7]). We also give some application and numerical example to illustrate the applicability of our proposed methods.

## 2. Preliminaries

In this section, we give some basic definitions and results which will be used in the sequel. We denote the strong convergence of $\{x_n\}$ to $a$ by $x_n \to a$ and the weak convergence of $\{x_n\}$ to $a$ by $x_n \rightharpoonup a$.

Let $H$ be a Hilbert space and $C$ be a nonempty, closed and convex subset of $H$. The metric projection $P_C : H \to C$ is defined, for each $x \in H$, as the unique element $P_C x \in C$ such that

$$\|x - P_C(x)\| = \min\{\|x - y\| : y \in C\}.$$

It is well known that $P_C$ satisfies the following properties:

(i) $\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2$, for every $x, y \in H$;

(ii) for $x \in H$ and $z \in C$, $z = P_C(x) \Leftrightarrow$

$$\langle x - z, z - y\rangle \geq 0, \ \forall y \in C; \tag{2.1}$$

(iii) for $x \in H$ and $y \in C$,

$$\|y - P_C(x)\|^2 + \|x - P_C(x)\|^2 \leq \|x - y\|^2. \tag{2.2}$$

Given $x, y \in H$, $y \neq 0$, let $Q = \{z \in H : \langle y, z - x\rangle \leq 0\}$. Then, for all $u \in H$, the projection $P_Q(u)$ is defined by

$$P_Q(u) = u - \max\left\{0, \frac{\langle y, u - x\rangle}{\|y\|^2}\right\} y, \tag{2.3}$$

which gives us an explicit formula for finding the projection of any point onto a half space.

**Lemma 2.1.** [29] *Let $H$ be a real Hilbert space. Then for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Let $S : D(S) \to H$ be a nonlinear mapping defined on $D(S) \subset H$. A point $x \in H$ is called a fixed point of $S$ if $Sx = x$. We denote the set of all fixed points of $S$ by $F(S)$. Then $S$ is said to be

(i) a contraction if there exists $\alpha \in [0, 1)$ such that

$$\|Sx - Sy\| \leq \alpha \|x - y\|, \quad \forall x, y \in D(S). \tag{2.4}$$

If $\alpha = 1$, then $S$ is called a nonexpansive mapping;

(ii) quasi-nonexpansive if $F(S) \neq \emptyset$ and

$$\|Sx - p\| \leq \|x - p\|, \quad x \in D(S) \quad \text{and} \quad p \in F(S);$$

(iii) firmly nonexpansive if for all $x, y \in D(S)$, we have

$$\|Sx - Sy\|^2 \leq \langle Sx - Sy, x - y\rangle;$$

(iv) $\beta$-inverse strongly monotone (shortly $\beta$-ism) if there exists $\beta > 0$ such that

$$\langle x - y, Sx - Sy\rangle \geq \beta \|Sx - Sy\|^2, \quad \forall x, y \in D(S).$$

It is well known that the projection mapping $P_C$ is nonexpansive and 1-*ism*. Also, the mapping $S$ is nonexpansive if and only if $I - S$ is monotone, where $I$ is the identity operator on $H$. Also, if $T : H \to 2^H$ is maximal monotone operator, then the resolvent of $T$ denoted by $J_T$ is nonexpansive. The inverse strongly monotone (also referred to as coercive) operators have been widely used to solve practical problems in various fields, for instance, in traffic assignment problems, see [18] and references therein.

**Definition 2.2.** [2] Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a Hilbert space $H$. We define $s - Li_n C_n$ and $w - Ls_n C_n$ as follows:

(i) $x \in s - Li_n C_n$ if and only if there exists $x_n \subset C_n$, for all $n \in \mathbb{N}$, such that $x_n \to x$.

(ii) $y \in w - Ls_n C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and $y_i \subset C_{n_i}$, for all $i \in \mathbb{N}$, such that $y_i \rightharpoonup y$.

(iii) If $C_0$ satisfies

$$C_0 = s - Li_n C_n = w - Ls_n C_n,$$

it is said that $\{C_n\}$ converges to $C_0$ in the sense of Mosco [27] and we write $C_0 = M - \lim\limits_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap\limits_{n=1}^{\infty} C_n$ in the sense of Tsukada [34].

The following result is proved by Tsukada [34] for metric projections.

**Lemma 2.3.** [34] *Let $H$ be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of $H$. If $C_0 = M - \lim\limits_{n\to\infty} C_n$ exists and is nonempty, then for each $x \in H$, $\{P_{C_n}(x)\}$ converges strongly to $P_{C_0}(x)$, where $P_{C_n}$ and $P_{C_0}$ are the metric projections of $H$ onto $C_n$ and $C_0$ respectively.*

**Definition 2.4.** Let $(X, d)$ be a complete metric space. A mapping $f : X \to X$ is called a Meir-Keeler contraction [25] if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \varepsilon + \delta \text{ implies } d(f(x), f(y)) < \varepsilon, \tag{2.5}$$

for all $x, y \in X$.

In the following technical lemma, we give some properties of Meir-Keeler contraction mappings which will be useful throughout out this work.

**Lemma 2.5.** *The following properties hold.*

(1) [25] *A Meir-Keeler contraction defined on a complete metric space has a unique fixed point and is nonexpansive as well.*
(2) [33] *Let $f$ be a Meir-Keeler contraction on a convex subset $C$ of a Banach space $E$. Then for every $\varepsilon > 0$, there exists $r \in (0,1)$ such that*

$$\|x-y\| \geq \varepsilon \text{ implies } \|f(x)-f(y)\| \leq r\|x-y\|,$$

*for all $x, y \in C$.*
(3) [33] *Let $C$ be a convex subset of a Banach space $E$. Let $T$ be a nonexpansive mapping on $C$ and let $f$ be a Meir-Keeler contraction on $C$. Then*
  (i) *$T \circ f$ is a Meir-Keeler contraction on $C$;*
  (ii) *for each $\alpha \in (0,1)$, $(1-\alpha)T + \alpha f$ is a Meir-Keeler contraction on $C$.*

**Lemma 2.6.** [22] *Let $H$ be a real Hilbert space. Let $T : H \to 2^H$ be a maximal monotone operator and $S : H \to H$ be an $\alpha$-inverse strongly monotone mapping on $H$. Define $K_r := (I + rT)^{-1}(I - rS)$, where $r > 0$ and $I$ is the identity map, then we have*

$$F(K_r) = (S+T)^{-1}(0). \tag{2.6}$$

*Also, note that $K_r$ is nonexpansive and $F(K_r)$ is closed and convex.*

**Lemma 2.7.** [23] *Let $\{\alpha_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\delta_n)\alpha_n + \beta_n + \gamma_n, \qquad n \geq 1,$$

*where $\{\delta_n\}$ is a sequence in $(0,1)$ and $\{\beta_n\}$ is a real sequence. Assume that*

$$\sum_{n=0}^{\infty} \gamma_n < \infty.$$

*Then, the following results hold:*

(i) *If $\beta_n \leq \delta_n M$ for some $M \geq 0$, then $\{\alpha_n\}$ is a bounded sequence.*
(ii) *If $\sum_{n=0}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \dfrac{\beta_n}{\delta_n} \leq 0$, then $\lim_{n\to\infty} \alpha_n = 0$.*

## 3. Main results

The first result deals with a new relaxed hybrid algorithm with inertial term.

**Theorem 3.1.** *Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. Let $S : C \to H$ be a $\nu$-inverse strongly monotone operator and $T : H \to 2^H$ be a maximal monotone operator such that $\Gamma := (S+T)^{-1}(0) \neq \emptyset$. Let $f$ be a Meir-Keeler contraction on $C$ and $\{x_n\}$ be a sequence in $H$ defined as follows: Fix*

*$x_0, x_1 \in C = D_1 = Q_1$. For all $n \geq 1$, choose $\lambda > 0, \{\theta_n\} \subset [0,1)$, $\{\alpha_n\} \subset (0,1)$ such that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and $\gamma_n \in [\varepsilon_0, \frac{1}{2}]$ for some $\varepsilon_0 \in (0, \frac{1}{2}]$. Compute*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1-\alpha_n)y_n + \alpha_n(I+\lambda T)^{-1}(I-\lambda S)y_n, \\ D_n = \{z \in H : \langle y_n - z_n, z - y_n - \gamma_n(z_n - y_n)\rangle \leq 0\}, \\ Q_n = \{z \in H : \langle f(x_n) - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{D_n \cap Q_n} f(x_n) \quad \forall\ n \geq 1. \end{cases} \tag{3.1}$$

*Then $\{x_n\}$ converges strongly to $\bar{x}$. Moreover we have $\bar{x} = P_\Gamma \circ f(\bar{x})$, i.e., $\bar{x}$ is the unique fixed point of the Meir-Keeler contraction $P_\Gamma \circ f$.*

*Proof.* First note that Lemma 2 implies that $P_\Gamma \circ f$ is a Meir-Keeler contraction and has a unique fixed point $\bar{x} \in C$. Also, observe that the sets $D_n$ and $Q_n$ are half spaces, hence, the projection $P_{D_n \cap Q_n}$ can be calculated using the formula (2.3). Next, we divide the proof into several steps:

**Step I:** We first show that $\Gamma \subset D_n \cap Q_n$ and $x_{n+1}$ is well defined, for all $n \geq 1$. By Lemma 2, the solution set $\Gamma$ is closed and convex. From the definitions of $D_n$ and $Q_n$, we see that the sets are closed and convex. Fix $n \geq 1$. Set

$$C_n := \{z \in H : \|z - z_n\| \leq \|z - y_n\|\}.$$

Then

$$C_n = \left\{ z \in H : \left\langle y_n - z_n, z - y_n - \frac{1}{2}(z_n - y_n) \right\rangle \leq 0 \right\}.$$

Since $\gamma_n \in [\varepsilon_0, \frac{1}{2}]$, we have $C_n \subset D_n$. Let $p \in \Gamma$, then we have

$$\begin{aligned} \|y_n - p\| &= \|x_n - p + \theta_n(x_n - x_{n-1})\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{3.2}$$

Let $U_\lambda = (I+\lambda T)^{-1}(I-\lambda S)$, then

$$\begin{aligned} \|z_n - p\| &= \|(1-\alpha_n)y_n + \alpha_n U_\lambda y_n - p\| \\ &\leq (1-\alpha_n)\|y_n - p\| + \alpha_n\|U_\lambda y_n - p\| \\ &\leq \|y_n - p\|, \end{aligned} \tag{3.3}$$

since $U_\lambda$ is nonexpansive and $F(U_\lambda) = \Gamma$. Hence $p \in C_n$ which implies that $\Gamma \subset C_n \subset D_n$. Therefore, $\Gamma \subset D_n$ holds for all $n \geq 1$. Next, we prove by induction that $\Gamma \subset D_n \cap Q_n$ for all $n \geq 1$. For $n = 1$, we have $x_1 \in C = D_1 = Q_1$, then $\Gamma \subset D_1 \cap Q_1$. Fix $k > 1$ and assume that $\Gamma \subset D_k \cap Q_k$. From $x_{k+1} = P_{D_k \cap Q_k} f(x_k)$ and (2.1), we obtain

$$\langle f(x_k) - x_{k+1}, x_{k+1} - z\rangle \geq 0, \quad \forall\ z \in D_k \cap Q_k.$$

Since $\Gamma \subset D_k \cap Q_k$,

$$\langle f(x_k) - x_{k+1}, x_{k+1} - z\rangle \geq 0, \quad \forall\ z \in \Gamma.$$

This together with definition of $Q_{k+1}$ implies that $\Gamma \subset Q_{k+1}$ and so $\Gamma \subset D_{k+1} \cap Q_{k+1}$. Thus, by induction, we obtain $\Gamma \subset D_n \cap Q_n$ for all $n \geq 0$. Since $\Gamma \neq \emptyset$, $D_n \cap Q_n$ is nonempty and hence $x_{n+1}$ is well defined.

**Step II:** Let $H_n = D_n \cap Q_n$, for all $n \geq 1$. We prove that $\{x_n\}$ converges to some point in $\bigcap\limits_{n=1}^{\infty} H_n$. Since $\Gamma \subset \bigcap\limits_{n=1}^{\infty} H_n$, we conclude that $\bigcap\limits_{n=1}^{\infty} H_n$ is a closed and convex nonempty subset. Using Lemma 2, we know that $P_{\bigcap\limits_{n=1}^{\infty} H_n} \circ f$ is a Meir-Keeler contraction on $C$ with a unique fixed point $\bar{x}$ which obviously belongs to $\bigcap\limits_{n=1}^{\infty} H_n$. Since $\{H_n\}$ is a nonincreasing sequence of nonempty closed convex subsets, it follows that $\Gamma \subset \bigcap\limits_{n=1}^{\infty} H_n = M - \lim\limits_{n\to\infty} H_n$. Setting $u_n = P_{H_n} \circ f(\bar{x})$ and applying Lemma 2, we conclude that

$$\lim_{n\to\infty} u_n = P_{\bigcap\limits_{n=1}^{\infty} H_n} \circ f(\bar{x}) = \bar{x}.$$

Now, we show that $\lim\limits_{n\to\infty} \|x_n - \bar{x}\| = 0$. Assume $d = \limsup\limits_{n\to\infty} \|x_n - \bar{x}\| > 0$. Since $f$ is a Meir-Keeler contraction, for any $\varepsilon \in (0, d)$, there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \|x_n - \bar{x}\| > \varepsilon + \delta, \tag{3.4}$$

and

$$\|x - y\| < \varepsilon + \delta \text{ implies } \|f(x) - f(y)\| < \varepsilon \tag{3.5}$$

for all $x, y \in C$. Since $u_n \to \bar{x}$, there exists $n_0 \in \mathbb{N}$ such that

$$\|u_n - \bar{x}\| < \delta, \ \forall n \geq n_0. \tag{3.6}$$

Assume there exists $n_1 \geq n_0$ such that

$$\|x_{n_1} - \bar{x}\| < \varepsilon + \delta.$$

From (3.5) and (3.6), we get

$$\begin{aligned}
\|x_{n_1+1} - \bar{x}\| &\leq \|x_{n_1+1} - u_{n_1+1}\| + \|u_{n_1+1} - \bar{x}\| \\
&= \|P_{H_{n_1+1}} \circ f(x_{n_1}) - P_{H_{n_1+1}} \circ f(\bar{x})\| + \|u_{n_1+1} - \bar{x}\| \\
&\leq \|f(x_{n_1}) - f(\bar{x})\| + \|u_{n_1+1} - \bar{x}\| \\
&< \varepsilon + \delta.
\end{aligned}$$

By induction, we can obtain $\|x_{n_1+m} - \bar{x}\| \leq \varepsilon + \delta$, for all $m \geq 1$, which implies that

$$\limsup_{n\to\infty} \|x_n - \bar{x}\| \leq \varepsilon + \delta.$$

This contradiction with (3.4) allows us to conclude that $\|x_n - \bar{x}\| \geq \varepsilon + \delta$, for all $n \geq n_0$. By Lemma 2, there exists $r \in (0,1)$ such that

$$\|f(x_n) - f(\bar{x})\| \leq r\|x_n - \bar{x}\|, \ \forall n \geq n_0.$$

Thus, we have

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &= \|P_{H_{n+1}} \circ f(x_n) - P_{H_{n+1}} \circ f(\bar{x})\| \\
&\leq \|f(x_n) - f(\bar{x})\| \\
&\leq r\|x_n - \bar{x}\|,
\end{aligned}$$

for every $n \geq n_0$, which implies

$$\limsup_{n\to\infty} \|x_{n+1} - \bar{x}\| = \limsup_{n\to\infty} \|x_{n+1} - u_{n+1}\| \leq r \ \limsup_{n\to\infty} \|x_n - \bar{x}\|.$$

Since $r < 1$, we conclude that $\lim\limits_{n\to\infty} \|x_n - \bar{x}\| = 0$, i.e., $\{x_n\}$ converges to $\bar{x}$ as claimed.
**Step III:** In order to finish the proof of Theorem 3, we prove that $\bar{x} = P_\Gamma \circ f(\bar{x})$. Note that since $\{x_n\}$ is convergent, it is bounded and $\lim\limits_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Moreover, we have

$$\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0. \tag{3.7}$$

Since $x_{n+1} \in D_n$ and by the definition of $D_n$, we have

$$\begin{aligned} \|U_\lambda y_n - y_n\| &= \frac{1}{\alpha_n}\|z_n - y_n\| \leq \frac{1}{\alpha_n}\Big(\|z_n - x_{n+1}\| + \|x_{n+1} - y_n\|\Big) \\ &\leq \frac{2}{\alpha_n}\|x_{n+1} - y_n\| \leq \frac{2}{\alpha_n}\Big(\|x_{n+1} - x_n\| + \|x_n - y_n\|\Big). \end{aligned}$$

The properties satisfied by $\{\alpha_n\}$ will imply $\lim\limits_{n\to\infty} \|U_\lambda y_n - y_n\| = 0$. Since $U_\lambda$ is nonexpansive, we get

$$\begin{aligned} \|U_\lambda \bar{x} - \bar{x}\| &\leq \|U_\lambda \bar{x} - U_\lambda x_n\| + \|U_\lambda x_n - x_n\| + \|x_n - \bar{x}\| \\ &\leq 2\,\|x_n - \bar{x}\| + \|U_\lambda x_n - x_n\| \\ &\leq 2\,\|x_n - \bar{x}\| + \|U_\lambda x_n - U_\lambda y_n\| + \|U_\lambda y_n - y_n\| + \|y_n - x_n\| \\ &\leq 2\,\|x_n - \bar{x}\| + 2\|x_n - y_n\| + \|U_\lambda y_n - y_n\|, \end{aligned}$$

for any $n \in \mathbb{N}$.
Using the above properties, we conclude that $\|U_\lambda \bar{x} - \bar{x}\| = 0$, i.e., $U_\lambda \bar{x} = \bar{x}$. In other words, we have $\bar{x} \in F(U_\lambda) = \Gamma$. Note that since $x_{n+1} = P_{D_n \cap Q_n} \circ f(x_n)$, we have $\langle f(x_n) - x_{n+1}, x_{n+1} - y\rangle \geq 0$, for any $y \in D_n \cap Q_n$. Using the fact $\Gamma \subset D_n \cap Q_n$, we get

$$\langle f(x_n) - x_{n+1}, x_{n+1} - y\rangle \geq 0,\ \forall y \in \Gamma.$$

Since $\{x_n\}$ converges to $\bar{x} \in \Gamma$, we get

$$\langle f(\bar{x}) - \bar{x}, \bar{x} - y\rangle \geq 0,\ \forall y \in \Gamma.$$

Thus $\bar{x} = P_\Gamma \circ f(\bar{x})$, which completes the proof of Theorem 3.

The next result deals with a Halpern-type algorithm with inertial term.
**Theorem 3.2.** *Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. Let $S : C \to H$ be a $\nu$-ism operator and $T : H \to 2^H$ be a maximal monotone operator such that $\Gamma := (S + T)^{-1}(0) \neq \emptyset$. Let $f$ be a Meir-Keeler contraction on $C$. Consider the sequence $\{x_n\}$ in $H$ defined as follows: Fix $x_0, x_1 \in C$ , choose $\lambda > 0$, $\{\theta_n\} \subset [0, \alpha]$ for some $\alpha \in (0, 1)$, $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}$ are sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \delta_n = 1$. Compute*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = (I + \lambda T)^{-1}(I - \lambda S)w_n, \\ x_{n+1} = \alpha_n x_n + \beta_n y_n + \delta_n f(x_n), \quad n \geq 1, \end{cases} \tag{3.8}$$

*Suppose the following conditions are satisfied:*

(C1) $\lim\limits_{n\to\infty} \frac{\theta_n}{\delta_n}\|x_n - x_{n-1}\| = 0$,
(C2) $\liminf\limits_{n\to\infty} \alpha_n\beta_n > 0$,
(C3) $\lim\limits_{n\to\infty} \delta_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \delta_n = +\infty$.

*Then the sequence* $\{x_n\}$ *converges strongly to the unique fixed point of the Meir-Keeler contraction* $P_\Gamma \circ f$.

*Proof.* First, we show that $\{x_n\}$ is bounded. Fix $\varepsilon > 0$ and $x^* \in \Gamma$. Since $f$ is a Meir-Keeler contraction mapping, there exists $\rho \in [0,1)$ such that

$$\varepsilon < \|x - y\| \quad \textit{implies} \quad \|f(x) - f(y)\| \le \rho\ \|x - y\|,$$

for any $x, y \in C$. From the assumption (C1), it is clear that

$$M = \sup_{n\ge 1}\ \left\{\|x_0 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\rho} + \frac{\beta_n}{1-\rho}\ \frac{\theta_n}{\delta_n}\ \|x_n - x_{n-1}\|\right\} < +\infty.$$

Put $U_\lambda = (I + \lambda T)^{-1}(I - \lambda S)$, then $U_\lambda$ is nonexpansive and $F(U_\lambda) = \Gamma$, which implies that $U_\lambda(x^*) = x^*$. By definition of the sequence $\{x_m\}$, we have

$$\begin{array}{rcl}\|w_m - x^*\| & = & \|x_m - x^* + \theta_m\ (x_m - x_{m-1})\| \\ & \le & \|x_m - x^*\| + \theta_m\|x_m - x_{m-1}\|,\end{array}$$

which implies

$$\begin{array}{rcl}\|y_m - x^*\| & = & \|U_\lambda(w_m) - x^*\| \\ & = & \|U_\lambda(w_m) - U_\lambda(x^*)\| \\ & \le & \|w_m - x^*\| \\ & \le & \|x_m - x^*\| + \theta_m\|x_m - x_{m-1}\|,\end{array}$$

for any $m \ge 1$. Thus

$$\begin{aligned}\|x_{m+1} - x^*\| &= \|\alpha_m x_m + \beta_m y_m + \delta_m f(x_m) - x^*\| \\ &= \|\alpha_m(x_m - x^*) + \beta_m(y_m - x^*) + \delta_m(f(x_m) - x^*)\| \\ &\le \alpha_m\|x_m - x^*\| + \beta_m\|y_m - x^*\| + \delta_m\|f(x_m) - x^*\| \\ &\le \alpha_m\|x_m - x^*\| + \beta_m(\|x_m - x^*\| + \theta_m\|x_m - x_{m-1}\|) \\ &+ \delta_m(\|f(x_m) - f(x^*)\| + \|f(x^*) - x^*\|) \\ &\le (\alpha_m + \beta_m)\|x_m - x^*\| + \beta_m\theta_m\ \|x_m - x_{m-1}\| \\ &+ \delta_m\|f(x_m) - f(x^*)\| + \delta_m\|f(x^*) - x^*\|\end{aligned}$$

for all $m \geq 1$. Fix $m \geq 1$. Then if $\|x_m - x^*\| \leq \varepsilon$ and since Meir-Keeler contraction mappings are nonexpansive, we get

$$\begin{aligned}
\|x_{m+1} - x^*\| &\leq (\alpha_m + \beta_m)\|x_m - x^*\| + \beta_m\theta_m \ \|x_m - x_{m-1}\| \\
&+ \delta_m\|x_m - x^*\| + \delta_m\|f(x^*) - x^*\| \\
&\leq (\alpha_m + \beta_m + \delta_m))\|x_m - x^*\| + \beta_m\theta_m \ \|x_m - x_{m-1}\| \\
&+ \delta_m\|f(x^*) - x^*\| \\
&= \|x_m - x^*\| + \beta_m\theta_m \ \|x_m - x_{m-1}\| + \delta_m\|f(x^*) - x^*\|.
\end{aligned}$$

Since

$$\begin{aligned}
&\beta_m\theta_m \ \|x_m - x_{m-1}\| + \delta_m\|f(x^*) - x^*\| \\
&\leq \delta_m(1-\rho) \ \left(\frac{\beta_m\theta_m}{\delta_m(1-\rho)}\|x_m - x_{m-1}\| + \frac{1}{1-\rho}\|f(x^*) - x^*\|\right) \\
&\leq \delta_m(1-\rho) \ M \\
&\leq M,
\end{aligned}$$

we get

$$\|x_{m+1} - x^*\| \leq \varepsilon + M.$$

Otherwise, assume $\|x_m - x^*\| > \varepsilon$. In this case, we have

$$\begin{aligned}
\|x_{m+1} - x^*\| &\leq (\alpha_m + \beta_m)\|x_m - x^*\| + \beta_m\theta_m \ \|x_m - x_{m-1}\| \\
&+ \delta_m\|f(x_m) - f(x^*)\| + \delta_m\|f(x^*) - x^*\| \\
&\leq (\alpha_m + \beta_m)\|x_m - x^*\| + \beta_m\theta_m \ \|x_m - x_{m-1}\| \\
&+ \delta_m\rho\|x_m - x^*\| + \delta_m\|f(x^*) - x^*\| \\
&\leq (1 - \delta_m + \delta_m\rho)\|x_m - x^*\| + \beta_m\theta_m \ \|x_m - x_{m-1}\| + \delta_m\|f(x^*) - x^*\| \\
&= (1 - \delta_m(1-\rho))\|x_m - x^*\| + \beta_m\theta_m \ \|x_m - x_{m-1}\| + \delta_m\|f(x^*) - x^*\| \\
&= (1 - \delta_m(1-\rho))\|x_m - x^*\| \\
&+ \delta_m(1-\rho)\left(\frac{\beta_m\theta_m}{\delta_m(1-\rho)} \ \|x_m - x_{m-1}\| + \frac{1}{1-\rho}\|f(x^*) - x^*\|\right) \\
&\leq (1 - \delta_m(1-\rho))\|x_m - x^*\| + \delta_m(1-\rho) \ M.
\end{aligned}$$

Finally, let us prove that

$$\|x_n - x^*\| \leq \varepsilon + M, \tag{Bo}$$

for any $n \in \mathbb{N}$. Clearly, if $\|x_n - x^*\| \leq \varepsilon$, then the inequality (Bo) holds for this $n$. Assume that for some $n \geq 1$ and $p \geq 0$, we have

$$\left\{\begin{array}{lcl}
\|x_{n-1} - x^*\| & \leq & \varepsilon \\
\|x_{n+i} - x^*\| & > & \varepsilon, \quad \text{for } i = 0, \cdots, p, \\
\|x_{n+p+1} - x^*\| & \leq & \varepsilon.
\end{array}\right.$$

Set $\bar{\delta}_j = \delta_j(1-\rho)$, for any $j \in \mathbb{N}$. Since $\|x_n - x^*\| > \varepsilon$, our previous calculations imply

$$\|x_{n+1} - x^*\| \leq (1 - \bar{\delta}_n)\|x_n - x^*\| + \bar{\delta}_n \ M,$$

which implies

$$\|x_{n+2}-x^*\| \leq (1-\bar{\delta}_{n+1})(1-\bar{\delta}_n)\|x_n-x^*\| + (1-\bar{\delta}_{n+1})\bar{\delta}_n\ M + \bar{\delta}_{n+1}\ M.$$

Since $\bar{\delta}_j = 1-(1-\bar{\delta}_j)$, for any $j \geq 0$, we get

$$\begin{aligned} (1-\bar{\delta}_{n+1})\bar{\delta}_n\ M + \bar{\delta}_{n+1}\ M &= (1-\bar{\delta}_{n+1})(1-(1-\bar{\delta}_n))\ M + \bar{\delta}_{n+1}\ M \\ &= -(1-\bar{\delta}_{n+1})(1-\bar{\delta}_n)\ M + (1-\bar{\delta}_{n+1})M + \bar{\delta}_{n+1}\ M \\ &= M - (1-\bar{\delta}_{n+1})(1-\bar{\delta}_n)\ M, \end{aligned}$$

which implies

$$\|x_{n+2}-x^*\| \leq (1-\bar{\delta}_{n+1})(1-\bar{\delta}_n)\|x_n-x^*\| + M(1-(1-\bar{\delta}_{n+1})(1-\bar{\delta}_n)).$$

Similar calculations will give

$$\|x_{n+3}-x^*\| \leq (1-\bar{\delta}_{n+2})(1-\bar{\delta}_{n+1})(1-\bar{\delta}_n)\|x_n-x^*\| + M(1-(1-\bar{\delta}_{n+2})(1-\bar{\delta}_{n+1})(1-\bar{\delta}_n)).$$

When we reach $p$, we get

$$\|x_{n+p}-x^*\| \leq \prod_{k=0}^{p-1}(1-\bar{\delta}_{n+k})\|x_n-x^*\| + M\Big(1-\prod_{k=0}^{p-1}(1-\bar{\delta}_{n+k})\Big).$$

Since $\|x_{n-1}-x^*\| \leq \varepsilon$, our above calculations imply that $\|x_n-x^*\| \leq \varepsilon + M$ which implies

$$\begin{aligned} \|x_{n+p}-x^*\| &\leq (\varepsilon+M)\ \prod_{k=0}^{p-1}(1-\bar{\delta}_{n+k}) + M\Big(1-\prod_{k=0}^{p-1}(1-\bar{\delta}_{n+k})\Big) \\ &= \varepsilon\ \prod_{k=0}^{p-1}(1-\bar{\delta}_{n+k}) + M \\ &\leq \varepsilon + M. \end{aligned}$$

Therefore, we just proved that either $\|x_n-x^*\| \leq \varepsilon$ or $\|x_n-x^*\| \leq \varepsilon+M$, for any $n \in \mathbb{N}$. In other words, we proved that $\|x_n-x^*\| \leq \varepsilon+M$, for any $n \in \mathbb{N}$, as claimed. So $\{\|x_n-x^*\|\}$ is a bounded sequence which implies that $\{x_n\}$ is bounded. Consequently, $\{w_n\},\{y_n\}$ and $\{U_\lambda w_n\}$ are bounded. Furthermore, observe that

$$\begin{aligned} \|w_n-x^*\|^2 &= \|x_n-x^*+\theta_n(x_n-x_{n-1})\|^2 \\ &= \|x_n-x^*\|^2 + 2\theta_n\langle x_n-x^*, x_n-x_{n-1}\rangle + \theta_n^2\|x_n-x_{n-1}\|^2. \end{aligned} \tag{3.9}$$

Also we have

$$2\langle x_n-x^*, x_n-x_{n-1}\rangle = \|x_n-x^*\|^2 + \|x_n-x_{n-1}\|^2 - \|x_{n-1}-x^*\|^2. \tag{3.10}$$

By substituting (3.10) into (3.9), we have

$$\begin{aligned}\|w_n - x^*\|^2 &= (1+\theta_n)\ \|x_n - x^*\|^2 + (\theta_n + \theta_n^2)\ \|x_n - x_{n-1}\|^2 - \theta_n\ \|x_{n-1} - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 + \theta_n[\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2] + 2\theta_n\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - x^*\|^2 + \theta_n(\|x_n - x^*\| + \|x_{n-1} - x^*\|)\|x_n - x_{n-1}\| \\ &+ 2\theta_n\|x_n - x_{n-1}\|^2 \\ &= \|x_n - x^*\|^2 + \theta_n(\|x_n - x^*\| + \|x_{n-1} - x^*\| \\ &+ 2\|x_n - x_{n-1}\|)\|x_n - x_{n-1}\|. \end{aligned} \tag{3.11}$$

Then

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(x_n - x^*) + \beta_n(y_n - x^*) + \delta_n(f(x_n) - x^*)\|^2 \\ &\leq \|\alpha_n(x_n - x^*) + \beta_n(y_n - x^*)\|^2 + 2\delta_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\ &\leq \alpha_n\|x_n - x^*\|^2 + \beta_n\|y_n - x^*\|^2 + 2\delta_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\ &\leq \alpha_n\|x_n - x^*\|^2 + \beta_n\|w_n - x^*\|^2 + 2\delta_n\langle f(x_n) - p, x_{n+1} - x^*\rangle \\ &\leq \alpha_n\|x_n - x^*\|^2 + \beta_n\Big(\|x_n - x^*\|^2 + \theta_n(\|x_n - x^*\| + \|x_{n-1} - x^*\| \\ &+ 2\|x_n - x_{n-1}\|)\|x_n - x_{n-1}\|\Big) + 2\delta_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\ &= (1-\delta_n)\|x_n - x^*\|^2 + \beta_n\theta_n\Big(\|x_n - x^*\| + \|x_{n-1} - x^*\| \\ &+ 2\|x_n - x_{n-1}\|\Big)\|x_n - x_{n-1}\| + 2\delta_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle. \end{aligned} \tag{3.12}$$

Also, by Lemma 2, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|x_n - x^*\|^2 + \beta_n\|y_n - x^*\|^2 + \delta_n\|f(x_n) - x^*\|^2 - \alpha_n\beta_n\|y_n - x_n\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + \beta_n\|w_n - x^*\|^2 + \delta_n\|f(x_n) - x^*\|^2 - \alpha_n\beta_n\|y_n - x_n\|^2. \end{aligned}$$

Hence by (3.11), we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \quad &\leq \quad \alpha_n\|x_n - x^*\|^2 + \beta_n\Big(\|x_n - x^*\|^2 + \theta_n(\|x_n - x^*\| + \|x_{n-1} - x^*\| \\ &\qquad +2\|x_n - x_{n-1}\|)\|x_n - x_{n-1}\|\Big) - \alpha_n\beta_n\|y_n - x_n\|^2 \\ &= \quad (1-\delta_n)\|x_n - x^*\|^2 + \beta_n\theta_n\Big(\|x_n - x^*\| + \|x_{n-1} - x^*\| \\ &\qquad +2\|x_n - x_{n-1}\|\Big)\|x_n - x_{n-1}\| - \alpha_n\beta_n\|y_n - x_n\|^2. \end{aligned} \tag{3.13}$$

Now, let $P_n = \|x_n - x^*\|^2$, for all $n \geq 1$. First assume there exists $N \in \mathbb{N}$ such that $P_{n+1} \leq P_n$, for all $n \geq N$. In this case $\{P_n\}$ is convergent and $P_n - P_{n+1} \to 0$, as $n \to \infty$. Since $\{x_n\}$ is bounded, it is easy to see from condition (C1) that

$$\theta_n\|x_n - x_{n-1}\| \to 0.$$

Hence from (3.13), we have

$$\begin{aligned} \alpha_n\beta_n\|y_n - x_n\|^2 &\leq (1-\delta_n)\|x_n - x^*\|^2 + \beta_n\theta_n\Big(\|x_n - x^*\| + \|x_{n-1} - x^*\| \\ &\quad +2\|x_n - x_{n-1}\|\Big)\|x_n - x_{n-1}\| - \|x_{n+1} - x^*\|^2 \\ &= P_n - P_{n+1} + \delta_n\|x_n - x^*\|^2 + \beta_n\theta_n\Big(\|x_n - x^*\| + \|x_{n-1} - x^*\| \\ &\quad +2\|x_n - x_{n-1}\|\Big)\|x_n - x_{n-1}\|, \end{aligned}$$

for all $n \geq 1$. Using $\delta_n \to 0$ and condition (C2), we get

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.14}$$

Furthermore,

$$\lim_{n\to\infty} \|w_n - x_n\| = \lim_{n\to\infty} \theta_n\|x_n - x_{n-1}\| = 0,$$

thus

$$\|y_n - w_n\| \leq \|y_n - x_n\| + \|x_n - w_n\| \to 0, \quad n \to \infty. \tag{3.15}$$

Now, let us re-write $x_{n+1}$ as $x_{n+1} = \alpha_n x_n + (1-\alpha_n)v_n$, where

$$v_n = \frac{\beta_n}{1-\alpha_n} y_n + \frac{\delta_n}{1-\alpha_n} f(x_n),$$

which implies

$$\|v_n - x_n\| \leq \frac{\beta_n}{1-\alpha_n}\|y_n - x_n\| + \frac{\delta_n}{1-\alpha_n}\|f(x_n) - x_n\|,$$

for all $n \geq 1$. Using $\delta_n \to 0$, condition (C2) and (3.14), we get

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{3.16}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q^*$ in $H$. Also, since $\|w_{n_i} - x_{n_i}\| \to 0$, $n \to \infty$, it implies that $w_{n_i} \rightharpoonup q^*$. Furthermore, since $U_\lambda$ is nonexpansive, then by the demiclosedness of the nonexpansive mapping and (3.15), we have $q^* \in F(U_\lambda)$. Therefore by Lemma 2, it follows that $q^* \in (S+T)^{-1}(0)$. Next, we show that $\limsup_{n\to\infty}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \leq 0$, where $x^* = P_\Gamma f(x^*)$. Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle = \lim_{i\to\infty}\langle f(x^*) - x^*, x_{n_i+1} - x^*\rangle.$$

Then from (2.2), we have

$$\begin{aligned} \limsup_{n\to\infty}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle &= \lim_{i\to\infty}\langle f(x^*) - x^*, x_{n_i+1} - x^*\rangle \\ &= \langle u - x^*, q^* - x^*\rangle \leq 0. \end{aligned} \tag{3.17}$$

We now show that $\{x_n\}$ converges strongly to $x^*$. From (3.12), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le (1-\delta_n)\|x_n-x^*\|^2+\beta_n\theta_n\Big(\|x_n-x^*\|+\|x_{n-1}-x^*\| \\
&\quad +2\|x_n-x_{n-1}\|\Big)\|x_n-x_{n-1}\|+2\delta_n\langle f(x_n)-x^*,x_{n+1}-x^*\rangle \\
&\le (1-\delta_n)\|x_n-x^*\|^2 \\
&\quad +\beta_n\theta_n\Big(\|x_n-x^*\|+\|x_{n-1}-x^*\|+2\|x_n-x_{n-1}\|\Big)\|x_n-x_{n-1}\| \\
&\quad +2\delta_n\|f(x_n)-f(x^*)\|\|x_{n+1}-x^*\|+2\delta_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle \\
&\le (1-\delta_n)\|x_n-x^*\|^2 \\
&\quad +\beta_n\theta_n\Big(\|x_n-x^*\|+\|x_{n-1}-x^*\|+2\|x_n-x_{n-1}\|\Big)\|x_n-x_{n-1}\| \\
&\quad +\delta_n\rho(\|(x_n)-x^*\|^2+\|x_{n+1}-x^*\|^2)+2\delta_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle \\
&= (1-\delta_n(1-\rho))\|x_n-x^*\|^2 \\
&\quad +\beta_n\theta_n\Big(\|x_n-x^*\|+\|x_{n-1}-x^*\|+2\|x_n-x_{n-1}\|\Big)\|x_n-x_{n-1}\| \\
&\quad +\delta_n\rho\|x_{n+1}-x^*\|^2+2\delta_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \frac{(1-\delta_n(1-\rho))}{1-\delta_n\rho}\|x_n-x^*\|^2 \\
&\quad +\frac{\beta_n\theta_n}{1-\delta_n\rho}\Big(\|x_n-x^*\|+\|x_{n-1}-x^*\|+2\|x_n-x_{n-1}\|\Big) \\
&\quad \times\|x_n-x_{n-1}\|+\frac{2\delta_n}{1-\delta_n\rho}\langle f(x^*)-x^*,x_{n+1}-x^*\rangle \\
&= \left(1-\frac{\delta_n(1-2\rho)}{1-\delta_n\rho}\right)\|x_n-x^*\|^2 \\
&\quad +\frac{\beta_n\theta_n}{1-\delta_n\rho}\Big(\|x_n-x^*\|+\|x_{n-1}-x^*\|+2\|x_n-x_{n-1}\|\Big) \\
&\quad \times\|x_n-x_{n-1}\|+\frac{\delta_n(1-2\rho)}{1-\delta_n\rho}\times\frac{2\langle f(x^*)-x^*,x_{n+1}-x^*\rangle}{1-2\rho}. \\
&= (1-a_n)\|x_n-x^*\|^2+a_nb_n+c_n,
\end{aligned}\tag{3.18}$$

where

$$a_n=\frac{\delta_n(1-2\rho)}{1-\delta_n\rho},\quad b_n=\frac{2\langle f(x^*)-x^*,x_{n+1}-x^*\rangle}{1-2\rho},$$

and

$$c_n=\frac{\beta_n\theta_n}{1-\delta_n\rho}\Big(\|x_n-x^*\|+\|x_{n-1}-x^*\|+2\|x_n-x_{n-1}\|\Big)\|x_n-x_{n-1}\|.$$

Since $\sum_{n=0}^{\infty}\delta_n = \infty$, then $\sum_{n=0}^{\infty} a_n = \infty$ and from (3.17), $\limsup_{n\to\infty} b_n \leq 0$, then using Lemma 2(ii) and (3.18), we have $\lim_{n\to\infty}\|x_n - x^*\|^2 = 0$. Hence $\{x_n\}$ converges strongly to $x^*$.

Otherwise, assume there exists a subsequence $\{P_{n_i}\}$ of $\{P_n\}$ such that $P_{n_i} \leq P_{n_i+1}$ for all $i \in \mathbb{N}$. There exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $P_{m_k} \leq P_{m_k+1}$, for all $k \in \mathbb{N}$.

Following a similar argument as before, we obtain $\|y_{m_k} - x_{m_k}\| \to 0$, $\|w_{m_k} - x_{m_k}\| \to 0$, $\|y_{m_k} - w_{m_k}\| \to 0$ and $\|x_{m_k+1} - x_{m_k}\| \to 0$ as $k \to \infty$. Also, we get

$$\limsup_{k\to\infty}\ \langle f(x^*) - x^*, x_{m_k+1} - x^*\rangle \leq 0. \tag{3.19}$$

Similarly as in (3.18), we have

$$\begin{aligned}
\|x_{m_k+1} - x^*\|^2 \leq{}& \left(1 - \frac{\delta_{m_k}(1-2\rho)}{1-\delta_{m_k}\rho}\right)\|x_{m_k} - x^*\|^2 \\
&+ \frac{\beta_{m_k}\theta_{m_k}}{1-\delta_{m_k}\rho}\Big(\|x_{m_k} - x^*\| + \|x_{m_k-1} - x^*\| \\
&+ 2\|x_{m_k} - x_{m_k-1}\|\Big)\|x_{m_k} - x_{m_k-1}\| \\
&+ \frac{\delta_{m_k}(1-2\rho)}{1-\delta_{m_k}\rho} \times \frac{2\langle f(x^*) - x^*, x_{m_k+1} - x^*\rangle}{1-2\rho}.
\end{aligned} \tag{3.20}$$

Since $P_{m_k} \leq P_{m_k+1}$, then from (3.20), we have that

$$\begin{aligned}
0 &\leq \|x_{m_k+1} - x^*\|^2 - \|x_{m_k} - x^*\|^2 \\
&\leq \left(1 - \frac{\delta_{m_k}(1-2\rho)}{1-\delta_{m_k}\rho}\right)\|x_{m_k} - x^*\|^2 \\
&+ \frac{\beta_{m_k}\theta_{m_k}}{1-\delta_{m_k}\rho}\Big(\|x_{m_k} - x^*\| + \|x_{m_k-1} - x^*\| \\
&+ 2\|x_{m_k} - x_{m_k-1}\|\Big)\|x_{m_k} - x_{m_k-1}\| \\
&+ \frac{\delta_{m_k}(1-2\rho)}{1-\delta_{m_k}\rho} \times \frac{2\langle f(x^*) - x^*, x_{m_k+1} - x^*\rangle}{1-2\rho} - \|x_{m_k} - x^*\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{\delta_{m_k}(1-2\rho)}{1-\delta_{m_k}\rho}\|x_{m_k} - x^*\| \\
\leq{}& \frac{\beta_{m_k}\theta_{m_k}}{1-\delta_{m_k}\rho}\Big(\|x_{m_k} - x^*\| + \|x_{m_k-1} - x^*\| + 2\|x_{m_k} - x_{m_k-1}\|\Big)\|x_{m_k} - x_{m_k-1}\| \\
+{}& \frac{\delta_{m_k}(1-2\rho)}{1-\delta_{m_k}\rho} \times \frac{2\langle f(x^*) - x^*, x_{m_k+1} - x^*\rangle}{1-2\rho}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|x_{m_k}-x^*\| \\
\le\ & \frac{\theta_{m_k}}{\delta_{m_k}}\left(\frac{\beta_{m_k}}{1-2\rho}\right)\Big(\|x_{m_k}-x^*\|+\|x_{m_k-1}-x^*\|+2\|x_{m_k}-x_{m_k-1}\|\Big)\|x_{m_k}-x_{m_k-1}\| \\
+\ & \frac{2\langle f(x^*)-x^*, x_{m_k+1}-x^*\rangle}{1-2\rho}.
\end{aligned} \tag{3.21}$$

From condition (C1) and (3.19), we obtain $\|x_{m_k}-x^*\| \to 0$, as $k\to\infty$.
As a consequence, we obtain

$$\|x_{m_k+1}-x^*\| \le \|x_{m_k+1}-x_{m_k}\|+\|x_{m_k}-x^*\| \to 0,$$

as $n\to\infty$. Also, we have $P_n \le P_{m_k+1}$ and thus

$$P_n = \|x_n-x^*\|^2 \le \|x_{m_k+1}-x^*\|^2 \to 0, \tag{3.22}$$

as $n\to\infty$. This implies that $\{x_n\}$ converges strongly to $x^*$. This completes the proof.

If we replace $f(x_n)$ by a fixed $u$ in the last algorithm, we have the classical Halpern-type algorithm (see [17]) and the result in Theorem 3 still holds. It was shown in [32] that Halpern-type convergence theorems imply viscosity approximation convergence theorem for weak contraction.

**Remark 3.3.** In above algorithm, the condition $(C1)$ may sound bizarre because one assumes that the sequences $\{\theta_n\}$ and $\{\delta_n\}$ are given before we started generating the sequence $\{x_n\}$. In fact, the computational algorithm associated to this algorithm allows to construct the sequence $\{x_n\}$ and the sequences $\{\theta_n\}$ and $\{\delta_n\}$ at the same time to secure that

$$\lim_{n\to\infty}\frac{\theta_n}{\delta_n}\|x_n-x_{n-1}\| = 0.$$

As a consequence of our results, we give the following results which follows directly from our Theorem 3 and Theorem 3.

**Corollary 3.4.** *Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. Let $V: C\to C$ be a nonexpansive mapping such that the set of fixed points of $V$ is not empty, i.e., $F(V)\neq\emptyset$. Let $f$ be a Meir-Keeler contraction on $C$ and $\{x_n\}$ be a sequence in $H$ defined as follows:*

$$\begin{cases}
x_0, x_1\in H, \gamma_n\in[\varepsilon,\tfrac12] \text{ for some } \varepsilon\in(0,\tfrac12]. \\
y_n = x_n+\theta_n(x_n-x_{n-1}), \\
z_n = (1-\alpha_n)y_n+\alpha_n V y_n, \\
D_n = \{z\in H : \langle y_n-z_n, z-y_n-\gamma_n(z_n-y_n)\rangle \le 0\}, \\
Q_n = \{z\in H : \langle f(x_n)-x_n, x_n-z\rangle \ge 0\}, \\
x_{n+1} = P_{D_n\cap Q_n} f(x_n), \ \forall\ n\ge 1,
\end{cases} \tag{3.23}$$

*where $\{\theta_n\}\subset[0,1)$, and $\{\alpha_n\}\subset(0,1)$ such that $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1$. Then $\{x_n\}$ converges strongly to $\bar{x}=P_{F(V)}f(\bar{x})$.*

**Corollary 3.5.** *Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. Let $V : C \to C$ be a nonexpansive mapping such that the set of fixed points of $V$ is not empty, i.e., $F(V) \neq \emptyset$. Let $f$ be a Meir-Keeler contraction on $C$ and $\{x_n\}$ be a sequence in $H$ defined as follows:*

$$\begin{cases} x_0, x_1 \in C, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = V w_n, \\ x_{n+1} = \alpha_n x_n + \beta_n y_n + \delta_n f(x_n), \quad n \geq 1, \end{cases} \tag{3.24}$$

*where $\{\theta_n\} \subset [0, \alpha]$ for some $\alpha \in (0,1)$, and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \delta_n = 1$. Suppose the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \frac{\theta_n}{\delta_n} \|x_n - x_{n-1}\| = 0$,
(C2) $\liminf_{n\to\infty} \alpha_n \beta_n > 0$,
(C3) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = +\infty$.

*Then, the sequence $\{x_n\}$ generated by the last algorithm converges strongly to a point $x^*$, where $x^* = P_{F(V)} f(x^*)$.*

## 4. Application and numerical example

4.1. **Application to split feasibility problem.** Let $H_1$ and $H_2$ be real Hilbert spaces, $C$ and $Q$ be nonempty closed and convex subsets of $H_1$ and $H_2$ respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. The Split Feasibility Problem (SFP) is to find a point $\hat{x}$ which satisfies the condition

$$\hat{x} \in C \quad \text{and} \quad A\hat{x} \in Q. \tag{4.1}$$

We denote the solution set of SFP by $\Gamma$. The SFP was first introduced by Censor and Elfving [9] in finite-dimensional Hilbert spaces and has received much attention from many authors due to its various applications in signal processing. Several iterative methods have been developed for solving the SFP and its related optimization problems (see for example, [3, 19]) and references therein.

To solve the SFP (4.1), it is important to study the following Convexly Constrained Minimization Problem (CCMP):

$$\min_{x\in C} f(x) \quad \text{where} \quad f(x) = \frac{1}{2}\|Ax - P_Q(Ax)\|^2. \tag{4.2}$$

It is noted in [22] that the SFP (4.1) and the CCMP (4.2) are not fully equivalent because every solution to the SFP (4.1) is evidently a minimizer of the CCMP (4.2), however a solution to the CCMP (4.2) does not necessarily satisfy the SFP (4.1). Further, if the solution set of the SFP (4.1) is nonempty, then it follows from Lemma 4.2 in [35] that $\Gamma = C \cap (\nabla f)^{-1} Q \neq \emptyset$.

Recall that the indicator function on $C$ is the function $i_C$, defined as

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases} \tag{4.3}$$

It is well known that the resolvent of $i_C$ is the metric projection $P_C$. Now setting $S(x) = \frac{1}{2}\|Ax - P_Q(Ax)\|^2$ and $T(x) = i_C(x)$ in our Theorem 3 and 3, we obtain the following two strong convergence results for approximating the solution of SFP (4.1) in Hilbert spaces.

**Theorem 4.1.** *Let $H_1$ and $H_2$ be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator and $A^*$ be the adjoint of $A$. Let $C$ and $Q$ be nonempty closed convex subsets of $H_1$ and $H_2$ respectively. Suppose $\Gamma = C \cap A^{-1}(Q) \neq \emptyset$ and $\lambda \in (0, \frac{2}{\|A\|})$. Let $f$ be a Meir-Keeler contraction on $C$ and $\{x_n\}$ be a sequence in $H$ defined as follows:*

$$\begin{cases} x_0, x_1 \in H, \gamma_n \in [\varepsilon, \frac{1}{2}] \text{ for some } \varepsilon \in (0, \frac{1}{2}]. \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1-\alpha_n)y_n + \alpha_n P_C(I - \lambda A^*(I - P_Q)A)y_n, \\ D_n = \{z \in H : \langle y_n - z_n, z - y_n - \gamma_n(z_n - y_n)\rangle \leq 0\}, \\ Q_n = \{z \in H : \langle f(x_n) - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{D_n \cap Q_n} f(x_n), \ \forall\ n \geq 1, \end{cases} \tag{4.4}$$

*where $\lambda > 0$ and $\{\theta_n\} \subset [0, 1)$, $\{\alpha_n\} \subset (0, 1)$ such that*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

*Then $\{x_n\}$ converges strongly to $\bar{x} = P_\Gamma f(\bar{x})$.*

**Theorem 4.2.** *Let $H_1$ and $H_2$ be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator and $A^*$ be the adjoint of $A$. Let $C$ and $Q$ be nonempty closed convex subsets of $H_1$ and $H_2$ respectively. Suppose $\Gamma = C \cap A^{-1}(Q) \neq \emptyset$ and $\lambda \in (0, \frac{2}{\|A\|})$. Let $f$ be a Meir-Keeler contraction on $C$ and $\{x_n\}$ be a sequence in $H$ defined as follows:*

$$\begin{cases} x_0, x_1 \in C, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(I - \lambda A^*(I - P_Q)A)w_n, \\ x_{n+1} = \alpha_n x_n + \beta_n y_n + \delta_n f(x_n), \quad n \geq 1, \end{cases} \tag{4.5}$$

*where $\{\theta_n\} \subset [0, \alpha]$ for some $\alpha \in (0, 1)$, $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}$ are sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \delta_n = 1$ and $\lambda > 0$. Suppose the following conditions are satisfied:*

- (C1) $\lim_{n\to\infty} \frac{\theta_n}{\delta_n}\|x_n - x_{n-1}\| = 0$,
- (C2) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$,
- (C3) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = +\infty$.

*Then, the sequence $\{x_n\}$ generated by Algorithm 4.5 converges strongly to a point $x^*$, where $x^* = P_\Gamma f(x^*)$.*

4.2. **Numerical experiments.** In this subsection, we provide some numerical examples to show the relevance of our results.

We now present the following numerical examples which show that our Algorithms 3.1 and 3.8 performs better in terms of number of iteration and time of convergence than the non-inertial algorithms (i.e., by taking $\theta_n = 0$ in each algorithms). All computation are carried out using Matlab 2014b on a HP personal computer with 4gb RAM. The stopping criterion used for both test is $\frac{\|x_{n+1}-x_n\|}{\|x_2-x_1\|} < 10^{-4}$.

**Example 4.3.** Consider the variational inequality problem of finding a point $x^* \in C$ such that

$$\langle Sx^*, y - x^*\rangle \geq 0, \quad \forall\, y \in C, \tag{4.6}$$

where $S : C \to H$ is a monotone operator. It is well known that Problem (4.6) is equivalent to the following inclusion problem:

$$\text{find } x^* \in C \quad \text{such that} \quad 0 \in (S + N_C)x^*, \tag{4.7}$$

where $N_C$ is the normal cone of $C$.
Now, let $H = \mathbb{R}^m$ with the standard topology and $S : \mathbb{R}^m \to \mathbb{R}^m$ be defined by

$$Sx = Mx + q, \tag{4.8}$$

where

$$M = NN^T + K + D, \tag{4.9}$$

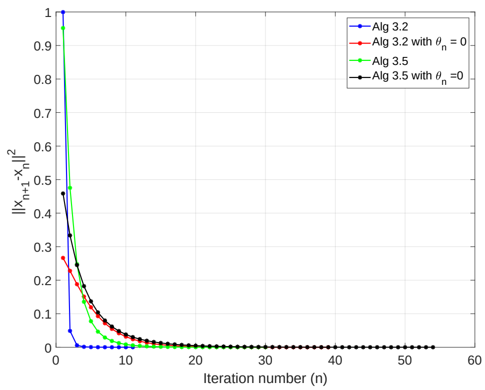
where $N$ is a $m \times m$ matrix, $K$ is a $m \times m$ skew-symmetric matrix, $D$ is a $m \times m$ diagonal matrix, whose diagonal entries are non-negative so that $M$ is positive definite and $q$ is a vector in $\mathbb{R}^m$. In this example, we consider the feasible set $C \subset \mathbb{R}^m$ as the closed and convex polyhedron which is defined as $C = \{x \in \mathbb{R}^m : Qx \leq b\}$, where $Q$ is a $l \times m$ matrix and $b$ is a non-negative vector. Since $S$ is monotone, we can apply our Algorithms 3.1 and 3.8 to solve problem (4.7) (in this case, the resolvent of $N_C$ is the metric projection operator $P_C$).
Taking

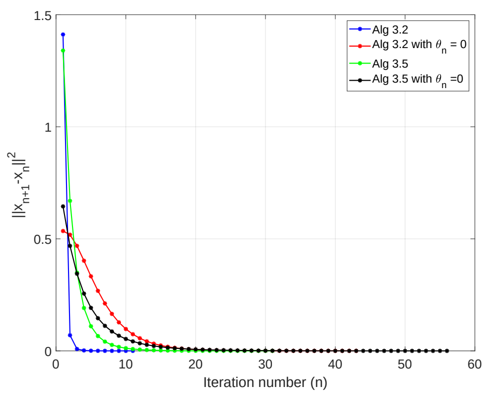
$$\theta_n = \frac{1}{(100n+1)^2},\ \delta_n = \frac{1}{\sqrt{100n+1}},\ \beta_n = \frac{n+1}{3(n+1)},\ \alpha_n = 1 - \delta_n - \beta_n,\ \lambda = 0.05,$$

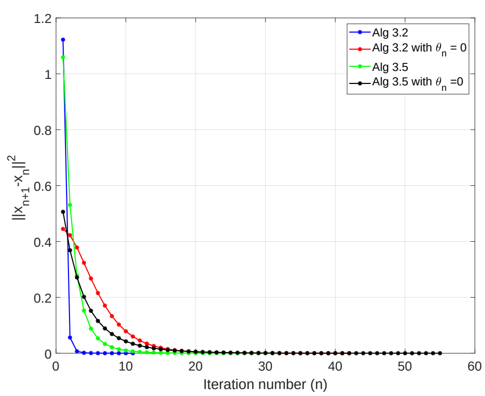
the matrices $N, K, D$, the vector $q$ and the initial points $x_0, x_1$ are generated randomly and the projection $P_C$ is computed using optimization tool box in Matlab. We test our Algorithms 3.1 and 3.8 for the cases when $m = 10, 20, 30, 50$ and compare the output with the non-inertial algorithms, i.e., by taking $\theta_n = 0$ in each algorithm. The numerical results are seen in Table 1 and Figure 1.

Table 1. Table showing computation results for Example 4.3.

| | | Alg 3.1 | Alg 3.1 with $\theta_n = 0$ | Alg 3.8 | Alg 3.8 with $\theta_n = 0$ |
|---|---|---|---|---|---|
| $m = 10$ | No. of Iter. | 11 | 39 | 30 | 54 |
| | CPU time (sec) | 1.2199 | 6.3814 | 3.0654 | 5.7250 |
| $m = 20$ | No. of Iter. | 11 | 42 | 31 | 55 |
| | CPU time (sec) | 1.4301 | 8.7918 | 4.0307 | 7.1364 |
| $m = 30$ | No. of Iter. | 11 | 42 | 31 | 55 |
| | CPU time (sec) | 2.3917 | 11.5262 | 3.4360 | 6.2248 |
| $m = 50$ | No. of Iter. | 12 | 44 | 32 | 58 |
| | CPU time (sec) | 2.9431 | 9.4373 | 5.2682 | 14.0695 |


Alg 3.2
Alg 3.2 with θn = 0
Alg 3.5
Alg 3.5 with θn =0
||xn+1-xn||2
Iteration number (n)


Alg 3.2
Alg 3.2 with θn = 0
Alg 3.5
Alg 3.5 with θn =0
||xn+1-xn||2
Iteration number (n)


Alg 3.2
Alg 3.2 with θn = 0
Alg 3.5
Alg 3.5 with θn =0
||xn+1-xn||2
Iteration number (n)

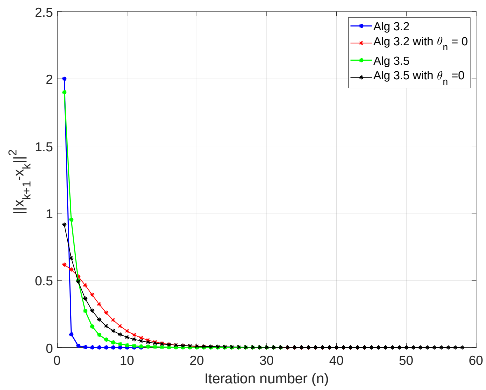



Figure 1. Example 4.3, Top Left: $m = 10$; Top Right: $m = 20$; Bottom Left: $m = 30$; Bottom Right: $m = 50$.

**Example 4.4.** Let $S : H \to \mathbb{R}$ be a convex and differentiable function and consider the problem:

$$\min_{x \in C}\{S(x)\}. \tag{4.10}$$

This is equivalent to finding a point $x^* \in C$ such that (see [16])

$$0 \in (\nabla S + N_C), \tag{4.11}$$

where $N_C$ is the normal cone of $C$. It is clear that $\nabla S$ is $k$-inverse strongly monotone. Let $T = \partial_C$, the indicator function of $C$,then, $T$ is maximal monotone) and the proximal operator with respect to $T$, $prox_{\lambda T} = P_C$, where $P_C$ is defined by

$$P_C(x) = \begin{cases} b - \frac{\langle a,x\rangle}{\|a\|^2}a + x, & \langle a, x\rangle > b, \\ x, & \langle a, x\rangle \leq b, \end{cases} \tag{4.12}$$

where $0 \neq a \in L^2([0, 1])$ and $b \in \mathbb{R}$. Hence, we can apply our Algorithms 3.1 and 3.8 to solve problem (4.11).

We choose $H = L^2([0, 1])$ with the inner product given by

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt.$$

Let us take

$$C = \{x(t) \in L^2([0, 1]) : \langle x(t), 3t^2\rangle = 1\},$$

and

$$S(x(t)) = \int_0^1 x(t)dt,\ \delta_n = \frac{1}{5n+1},\ \beta_n = \frac{2n}{3n+2},\ \alpha_n = 1 - \delta_n - \beta_n \text{ and } \lambda = 1.$$

We take

$$\theta_n = \begin{cases} \min\left\{0.5, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & if\ x_n \neq x_{n-1}, \\ 0.5 \quad if\ x_n = x_{n-1}, \end{cases} \tag{4.13}$$

where $\varepsilon_n = \frac{1}{(5n+1)^2}$. It is easy to see that condition (C1) (C2) and (C3) are satisfied. We test our Algorithms 3.1 and 3.8 with various initial values given below and compare the output with the non-inertial algorithms.

Case I: $x_1 = 3\exp(-2t)\cos(3t), \quad x_0 = t^3 + \cos(4t)$;

Case II: $x_1 = 5t\sin(2\pi t), \quad x_0 = \exp(3t) + \cos(-2t)$;

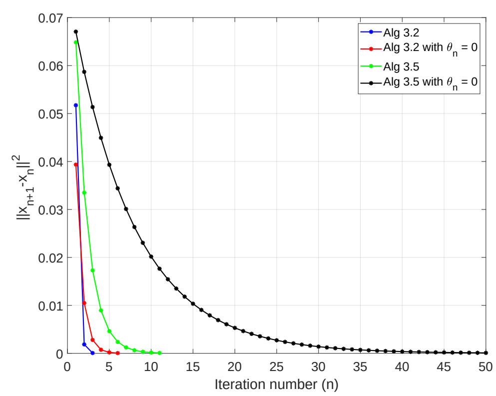
Case III: $x_1 = (2\cos(7t) + 3\sin(5t))/5, \quad x_0 = 3(t^3 + \exp(-3t))$;

Case IV: $x_1 = 5\pi t\exp(-2t), \quad x_0 = 3\cos(2\pi t)^2$.

The numerical results can be seen in Table 2 and Figures 2.

Table 2. Table showing computation results for Example 4.4.

| | | Alg 3.1 | Alg 3.1 with $\theta_n = 0$ | Alg 3.8 | Alg 3.8 with $\theta_n = 0$ |
|---|---|---|---|---|---|
| Case I | No. of Iter. | 3 | 6 | 13 | 50 |
| | CPU time (sec) | 0.6622 | 1.1389 | 0.7734 | 3.3577 |
| Case II | No. of Iter. | 4 | 5 | 11 | 53 |
| | CPU time (sec) | 0.6695 | 1.3355 | 0.6144 | 2.2959 |
| Case III | No. of Iter. | 4 | 8 | 11 | 60 |
| | CPU time (sec) | 1.0976 | 3.7500 | 0.6144 | 2.1449 |
| Case IV | No. of Iter. | 4 | 9 | 16 | 76 |
| | CPU time (sec) | 1.1848 | 8.7895 | 1.9476 | 3.7501 |



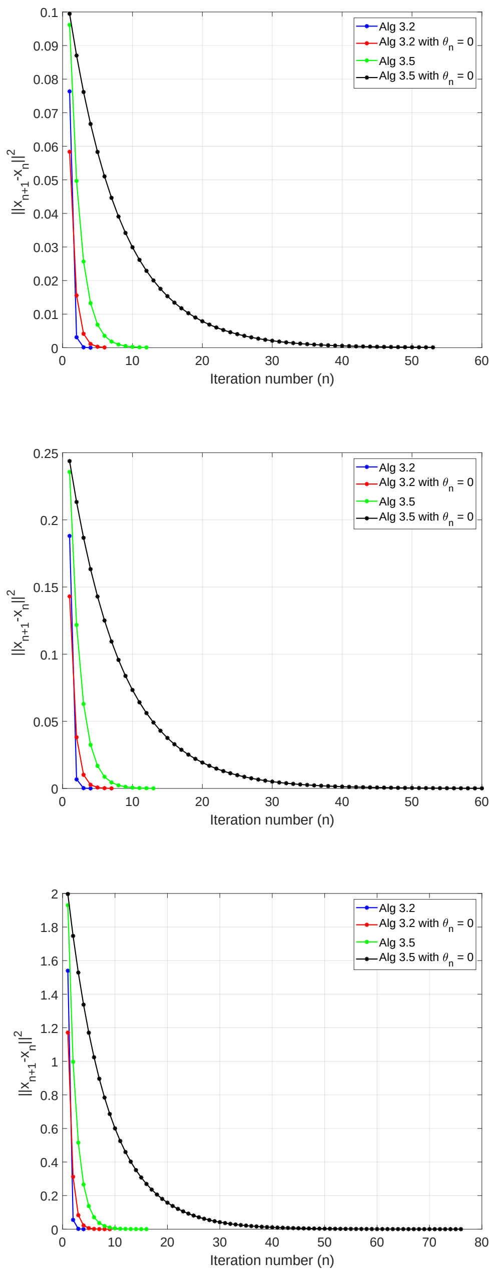



Figure 2. Example 4.4, Top Left: Case I; Top Right: Case II; Bottom Left Case III; Bottom Right: Case IV.

## 5. Conclusion

In this paper, we introduced two new inertial algorithms which consist of hybrid (or CQ) algorithm and viscosity approximation method with Meir-Keeler contraction mapping in real Hilbert space. We proved two strong convergence theorems for approximating solutions of monotone inclusion problems under some mild conditions in real Hilbert spaces. We also provide some numerical examples to show the efficiency and accuracy of our algorithms. Our contributions in this paper are highlighted as follow:

(i) The Meir-Keeler contraction mapping can be seen as a generalization of the contraction mapping. Hence, our results in this paper generalize the related results in [14, 15, 10].

(i) The strong convergence theorem achieved in this paper improved the corresponding weak convergence results of inertial algorithms in [5, 11] and some other related results.

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