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ON MULTIVALUED P-CONTRACTIVE MAPPINGS THAT BELONGS TO CLASS OF WEAKLY PICARD OPERATORS

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Abstract. In the present paper, by introducing the P-contractivity of a multivalued mapping, we give a new class of multivalued weakly Picard operators on complete metric spaces and show that the class of multivalued contractions is a proper subset of this new class. We also give a nontrivial example showing this fact.
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1. Introduction

The concept of multivalued weakly Picard operator plays an important role in both operator theory and fixed point theory. It has been introduced by Rus [22] as follows: Let (X, d) be a metric space. A mapping $T : X \to \mathcal{P}(X)$ is said to be multivalued weakly Picard (for short MWP) operator if there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0, which converges to a fixed point of T, where $\mathcal{P}(X)$ is the family of all nonempty subsets of X. We shall denote the class of all MWP operators on X by $\mathcal{MWP}(X)$. The aim of this paper is to present some new operators that belongs to $\mathcal{MWP}(X)$.

Throughout this paper (X, d) will be a metric space. $\mathcal{P}_C(X)$ will be the collection of all nonempty closed subsets of X, $\mathcal{P}_{CB}(X)$ will be the collection of all nonempty closed and bounded subsets of X and $\mathcal{P}_K(X)$ will be the collection of all nonempty compact subsets of X. For $A, B \in \mathcal{P}_C(X)$, let

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$. Then H is called generalized Pompeiu-Hausdorff distance on $\mathcal{P}_C(X)$. It is well known that H is a metric on $\mathcal{P}_{CB}(X)$, which is called Pompeiu-Hausdorff metric induced by d. We can find detailed information about the Pompeiu-Hausdorff metric in [4, 12].

Let $T : X \to \mathcal{P}(X)$ be a multivalued mapping and $x \in X$. Then x is said to be a fixed point of T, if $x \in Tx$. Nadler [16] initiated the study of fixed point theory for multivalued mappings on metric space by introducing the concept of multivalued contraction mapping. A mapping, which is $\mathcal{P}_{CB}(X)$ valued, is said to be multivalued contraction if there exists $L \in [0, 1)$ such that $H(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$ (see [16]). Therefore, Nadler [16] proved that every multivalued contraction on complete metric space has a fixed point. This result is an extension of Banach fixed point theorem to multivalued case. Then, the fixed point theory of multivalued mappings has been further developed in different directions by many authors, in particular, by Reich [21, 20], Mizoguchi-Takahashi [15], Klim-Wardowski [14], Berinde-Berinde [3], Ćirić [5] and many others [2, 6, 7, 11, 13, 17, 18, 23, 24]. Also, Feng and Liu [8] gave the following theorem without using generalized Pompeiu-Hausdorff distance. To state their result, we give the following notation for a multivalued mapping $T : X \to \mathcal{P}_C(X)$: let $b \in (0, 1)$ and $x \in X$ define

$$I_b^x = \{y \in Tx : bd(x, y) \leq d(x, Tx)\}.$$

Theorem 1.1. ([8]) *Let (X, d) be a complete metric space and $T : X \to \mathcal{P}_C(X)$ be a mapping. If there exists a constant $c \in (0, 1)$ such that there is $y \in I_b^x$ satisfying*

$$d(y, Ty) \leq cd(x, y), \tag{1.1}$$

for all $x \in X$, then T has a fixed point in X provided that $c < b$ and the function $x \to d(x, Tx)$ is lower semicontinuous.

As mentioned in Remark 1 of [8], we can see that Theorem 1.1 is a real generalization of Nadler's. If we examine in the proofs of both Nadler and Feng-Liu fixed point theorems we can see that there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0, converges to a fixed point of T. That is, the mentioned mappings in these theorems are MWP operators.

On the other hand, an interesting extension of Banach fixed point theorem for single valued mapping has been obtained by Popescu [19]. Popescu considered the following new contraction for a self mapping T of X: for all $x, y \in X$

$$d(Tx, Ty) \leq k \left[d(x, y) + |d(x, Tx) - d(y, Ty)| \right], \tag{1.2}$$

where $0 \leq k < 1$. We will call P-contraction to mapping T which satisfies (1.2). Briefly, Popescu proved that every P-contraction on complete metric space has a unique fixed point and further every Picard sequence converges to the fixed point of T. Therefore every P-contraction on complete metric space is a Picard operator. Then recently taking into account the P-contraction, some authors studied in this direction (see [1, 9, 10]).

In this paper, by inspiration the concept of P-contraction, we introduce some new contractive conditions for multivalued mappings. Therefore we present some fixed point theorems that includes Nadler's and Feng-Liu's results. Hence we will be provide some new operators that belongs to $\mathcal{MWP}(X)$.

2. Fixed point result

The following theorem is one of the main result of this section.

Theorem 2.1. *Let (X,d) complete metric space and $T : X \to \mathcal{P}_C(X)$ be a multivalued mapping. If for any $x \in X$ there exists $y \in I_b^x$ such that*

$$d(y,Ty) \leq c[d(x,y) + |d(x,Tx) - d(y,Ty)|], \tag{2.1}$$

where c is a positive real number satisfying $\frac{2c}{b(1+c)} < 1$, then T is a MWP operator provided that $f(x) = d(x,Tx)$ is lower semicontinuous.

Remark 2.2. Since c is positive and $\frac{2c}{b(1+c)} < 1$, we have $0 < c < b < 1$ and also we have

$$\left(\frac{c}{1-c}\right)\left(\frac{1-b}{b}\right) < 1.$$

Proof. (Proof of Theorem 2.1) Since $Tx \in \mathcal{P}_C(X)$ for any $x \in X$, then I_b^x is nonempty for any constant $b \in (0,1)$. By the assumption, for arbitrary point $x_0 \in X$, there exists $x_1 \in I_b^{x_0}$ such that

$$d(x_1,Tx_1) \leq c[d(x_0,x_1) + |d(x_0,Tx_0) - d(x_1,Tx_1)|],$$

and, for $x_1 \in X$, there exists $x_2 \in I_b^{x_1}$ such that

$$d(x_2,Tx_2) \leq c[d(x_1,x_2) + |d(x_1,Tx_1) - d(x_2,Tx_2)|].$$

Continuing this process, we can construct an iterative sequence $\{x_n\}$ such that $x_{n+1} \in I_b^{x_n}$ and

$$d(x_{n+1},Tx_{n+1}) \leq c[d(x_n,x_{n+1}) + |d(x_n,Tx_n) - d(x_{n+1},Tx_{n+1})|], \tag{2.2}$$

for $n = 0,1,2,\cdots$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0},Tx_{n_0}) = 0$, then x_{n_0} is a fixed point of T. Let assume $d(x_n,Tx_n) > 0$ for all $n \in \mathbb{N}$. Now, if there exists $m \in \mathbb{N}$ such that

$$d(x_{m+1},Tx_{m+1}) \geq d(x_m,Tx_m),$$

then from (2.2) we have (note that, $bd(x_m,x_{m+1}) \leq d(x_m,Tx_m)$ since $x_{m+1} \in I_b^{x_m}$)

$$\begin{aligned}
d(x_{m+1},Tx_{m+1}) &\leq c[d(x_m,x_{m+1}) + |d(x_m,Tx_m) - d(x_{m+1},Tx_{m+1})|] \\
&= c[d(x_m,x_{m+1}) + d(x_{m+1},Tx_{m+1}) - d(x_m,Tx_m)],
\end{aligned}$$

and so

$$\begin{aligned}
d(x_{m+1},Tx_{m+1}) &\leq \frac{c}{1-c}d(x_m,x_{m+1}) - \frac{c}{1-c}d(x_m,Tx_m) \\
&= \frac{c}{1-c}[d(x_m,x_{m+1}) - d(x_m,Tx_m)] \\
&\leq \frac{c}{1-c}[\frac{1}{b}d(x_m,Tx_m) - d(x_m,Tx_m)] \\
&\leq \left(\frac{c}{1-c}\right)\left(\frac{1-b}{b}\right)d(x_m,Tx_m) \\
&< d(x_m,Tx_m) \\
&\leq d(x_{m+1},Tx_{m+1}),
\end{aligned}$$

which is a contradiction. Therefore $d(x_{n+1}, Tx_{n+1}) < d(x_n, Tx_n)$ for all $n \in \mathbb{N}$. Thus, we have from (2.2)

$$d(x_{n+1}, Tx_{n+1}) \leq c[d(x_n, x_{n+1}) + d(x_n, Tx_n) - d(x_{n+1}, Tx_{n+1})]$$

and so

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) &\leq \frac{c}{1+c}\left[d(x_n, x_{n+1}) + d(x_n, Tx_n)\right] \\ &\leq \frac{2c}{1+c} d(x_n, x_{n+1}). \end{aligned}$$

Now since $x_{n+2} \in I_b^{x_{n+1}}$ we have

$$\begin{aligned} bd(x_{n+1}, x_{n+2}) &\leq d(x_{n+1}, Tx_{n+1}) \\ &\leq \frac{2c}{1+c} d(x_n, x_{n+1}), \end{aligned}$$

and so

$$d(x_{n+1}, x_{n+2}) \leq \frac{2c}{b(1+c)} d(x_n, x_{n+1}).$$

Therefore, we obtain

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1),$$

for all $n \in \mathbb{N}$, where $\lambda = \frac{2c}{b(1+c)} < 1$. Hence, for $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + \cdots + \lambda^{m-1} d(x_0, x_1) \\ &\leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1). \end{aligned}$$

Since $\lambda < 1$, the last inequality shows that $\{x_n\}$ is a Cauchy sequence. According to the completeness of X, there exists $z \in X$ such that $\{x_n\}$ converges to z. Since f is lower semicontinuous, we find that

$$\begin{aligned} 0 &\leq d(z, Tz) = f(z) \\ &\leq \liminf f(x_n) \\ &= \liminf d(x_n, Tx_n) \\ &\leq \liminf d(x_n, x_{n+1}) = 0, \end{aligned}$$

and so $d(z, Tz) = 0$. Since Tz is closed, it follows that $z \in Tz$. In the light of the proof technique, we can see that T is a MWP operator.

The following corollaries can be obtained from Theorem 2.1.

Corollary 2.3. *Let (X, d) complete metric space and $T : X \to \mathcal{P}_C(X)$ be a multivalued mapping. If there exists $c \in (0, 1)$ such that*

$$d(y, Ty) \leq c[d(x, y) + |d(x, Tx) - d(y, Ty)|],$$

for any $x \in X$ and $y \in Tx$, then T is a MWP operator provided that $f(x) = d(x, Tx)$ is lower semicontinuous.

Proof. Let $b \in (0,1)$ be such that $\frac{2c}{b(1+c)} < 1$. Since Tx is closed for all $x \in X$, then I_b^x is nonempty. Therefore for any $x \in X$ there exists $y \in I_b^x$ such that (2.1) holds. Thus, from Theorem 2.1, T has a fixed point.

The following result is a generalization of Nadler fixed point theorem for multivalued mappings.

Corollary 2.4. *Let (X,d) complete metric space and $T : X \to \mathcal{P}_{CB}(X)$ be a multivalued mapping. If there exists $c \in (0,1)$ such that*

$$H(Tx,Ty) \le c[d(x,y) + |d(x,Tx) - d(y,Ty)|], \tag{2.3}$$

for any $x, y \in X$. Then T is a MWP operator provided that $f(x) = d(x,Tx)$ is lower semicontinuous.

Proof. Since $d(y,Ty) \le H(Tx,Ty)$ for any $x \in X$ and $y \in Tx$, we have

$$\begin{aligned} d(y,Ty) &\le H(Tx,Ty) \\ &\le c[d(x,y) + |d(x,Tx) - d(y,Ty)|]. \end{aligned}$$

Therefore from Corollary 2.3, T has a fixed point.

Remark 2.5. If T is a multivalued contraction, then it is upper semicontinuous and so $f(x) = d(x,Tx)$ is lower semicontinuous. Therefore, Corollary 2.4 is a generalization of Nadler's result.

If we consider $\mathcal{P}_K(X)$ instead of $\mathcal{P}_C(X)$ in Theorem 2.1, we can relaxed the constant $b \in (0,1]$. Thus we have the following result.

Theorem 2.6. *Let (X,d) complete metric space and $T : X \to \mathcal{P}_K(X)$ be a multivalued mapping. If for any $x \in X$ there exists $y \in I_1^x$ such that*

$$d(y,Ty) \le c[d(x,y) + |d(x,Tx) - d(y,Ty)|],$$

where c is a positive real number satisfying $\frac{2c}{1+c} < 1$. Then T is a MWP operator provided that $f(x) = d(x,Tx)$ is lower semicontinuous.

Remark 2.7. By taking in to mind the proof of Theorem 1.1, we see that the mentioned mappings in Theorem 2.1, Theorem 2.6, Corollary 2.3 and Corollary 2.4 are MWP operators.

Now we present an illustrative example. Hence, due to the Theorem 2.1, the mapping T in this example belongs to $\mathcal{MWP}(X)$.

Example 2.8. Let $X = \{\frac{1}{2^n} : n = 1, 2, \cdots\} \cup \{0, 1\}$ with the usual metric defined by

$$d(x,y) = |x - y| \text{ for } x, y \in X.$$

Define a mapping $T : X \to \mathcal{P}_C(X)$ as

$$Tx = \begin{cases} \{0\} & , \quad x = 0 \\ \{0,1\} & , \quad x = 1 \\ \{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\} & , \quad x = \frac{1}{2^n}, n \ge 1 \end{cases}.$$

In this case since

$$f(x) = d(x, Tx) = \begin{cases} 0 & , \quad x \in \{0,1\} \\ \frac{1}{2^{n+1}} & , \quad x = \frac{1}{2^n}, n \geq 1 \end{cases},$$

then f is lower semicontinuous. Now we claim that the inequality (2.1) is satisfied for $b = \frac{9}{10}$ and $c = \frac{1}{3}$ (note that $\frac{2c}{b(1+c)} < 1$). To see this we have to consider the following cases:

Case 1. Let $x = 0$, then for $y = 0 \in I^0_{0,9}$ we have $d(y, Ty) = 0$ and so (2.1) holds.

Case 2. Let $x = 1$, then for $y = 1 \in I^1_{0,9}$ we have $d(y, Ty) = 0$ and so (2.1) holds.

Case 3. Let $x = \frac{1}{2^n}$ for $n \geq 1$, then for $y = \frac{1}{2^{n+1}} \in I^x_{0,9}$ we have

$$d(y, Ty) = \frac{1}{2^{n+2}}$$

and

$$d(x, y) + |d(x, Tx) - d(y, Ty)| = \frac{3}{2^{n+2}}.$$

Thus we have

$$d(y, Ty) = \frac{1}{2^{n+2}} = \frac{1}{3}\frac{3}{2^{n+2}} = \frac{1}{3}\left[d(x, y) + |d(x, Tx) - d(y, Ty)|\right].$$

Therefore all conditions of Theorem 2.1 (and also Theorem 2.6) are satisfied and so T is a MWP operator.

In the following theorem we will consider another approach for P-contractivity of multivalued mappings.

Theorem 2.9. *Let (X, d) be a complete metric space and $T : X \to \mathcal{P}_C(X)$ be a multivalued mapping and $k \in (0, 1)$. If for all $x, y \in X$ and $u \in Tx$ there exists $v \in Ty$ such that*

$$d(u, v) \leq k[d(x, y) + |d(x, u) - d(y, v)|]. \tag{2.4}$$

Then T is a MWP operator provided that $f(x) = d(x, Tx)$ is lower semicontinuous.

Proof. Let $x_0 \in X$ be arbitrary and $x_1 \in Tx_0$. From (2.4) there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq k\left[d(x_0, x_1) + |d(x_0, x_1) - d(x_1, x_2)|\right].$$

Similarly, since $x_2 \in Tx_1$ there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq k\left[d(x_1, x_2) + |d(x_1, x_2) - d(x_2, x_3)|\right].$$

Continuing this process, we can obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) \leq k\left[d(x_{n-1}, x_n) + |d(x_{n-1}, x_n) - d(x_n, x_{n+1})|\right],$$

for all $n \geq 1$. Here we can assume $x_n \notin Tx_n$ for all $n \in \mathbb{N}$ (Otherwise T must have a fixed point). Now, denote by $d_n = d(x_n, x_{n+1})$, then we have

$$d_n \leq k\left[d_{n-1} + |d_{n-1} - d_n|\right]. \tag{2.5}$$

If there exists $n_0 \in \mathbb{N}$ such that $d_{n_0-1} \leq d_{n_0}$, then from (2.5) we have

$$d_{n_0} \leq k d_{n_0},$$

which is a contradiction since $k < 1$ and $d_{n_0} > 0$. Therefore $d_{n-1} > d_n$ for all $n \in \mathbb{N}$. Thus, we have from (2.5)

$$\begin{aligned} d_n &\leq k(d_{n-1} + d_{n-1} - d_n) \\ &\leq 2kd_{n-1} - kd_n, \end{aligned}$$

and so

$$\begin{aligned} d_n &\leq \frac{2k}{k+1} d_{n-1} \\ &= \lambda d_{n-1} \\ &\vdots \\ &\leq \lambda^n d_0, \end{aligned} \tag{2.6}$$

where $\lambda = \frac{2k}{k+1} < 1$. Taking the limit as $n \to \infty$ in (2.6) we get

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ with $m > n$, then we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \lambda^n d_0 + \lambda^{n+1} d_0 + \cdots + \lambda^{m-1} d_0 \\ &\leq \lambda^n d_0 (1 + \lambda + \lambda^2 + \cdots + \lambda^{m-n-1} + \cdots) \\ &= \frac{\lambda^n}{1-\lambda} d_0. \end{aligned} \tag{2.8}$$

Taking the limit as $n \to \infty$ in (2.8) we get $\{x_n\}$ is a Cauchy sequence. By completeness of (X, d), $\{x_n\}$ converges to some point $z \in X$. Since f is lower semicontinuous, we get

$$\begin{aligned} 0 &\leq f(z) \\ &\leq \liminf f(x_n) \\ &= \liminf d(x_n, Tx_n) \\ &\leq \liminf d(x_n, x_{n+1}) = 0, \end{aligned}$$

and thus $f(z) = d(z, Tz) = 0$. Since Tz is closed, it follows that $z \in Tz$.

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