Fixed Point Theory, 22(2021), No. 2, 645-662
DOI: 10.24193/fpt-ro.2021.2.42
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

APPROXIMATING COMMON FIXED POINT VIA ISHIKAWA'S ITERATION

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Abstract. In this work, we approximate a common fixed point of mappings $F, G : M \cup N \to M \cup N$, satisfying the conditions

(1) $G(M) \subseteq M, G(N) \subseteq N, F(M) \subseteq M$ and $F(N) \subseteq N$;
(2) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and
(3) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$,

where M and N are nonempty bounded closed convex subsets of a uniformly convex Banach space. We consider Ishikawa iteration associated with F and G and von Neumann sequence associated with Ishikawa iteration to approximate the common fixed point of F and G. We prove convergent results for common fixed point of F and G. Finally, we give corollaries on common best proximity point for cyclic mappings.

Key Words and Phrases: Nonexpansive mappings, best proximity points, fixed points, Banach space, Von Neumann sequences.
2020 Mathematics Subject Classification: 47H10, 46B20, 54H25.

1. Introduction

Let X be a nonempty set. For a given function $F : X \to X$, a point $u_0 \in X$ is said to be a fixed point of F if $F(u_0) = u_0$. Two fundamental theorems namely Banach contraction principle and Brouwer fixed point theorem which guarantee the existence of fixed point. Later, many authors contributed for the development of fixed point theory and its application in other branch of Mathematics. The study of common fixed point theorem plays crucial role in the theory of fixed point and it is attracted by many researchers. A point u is said to be common fixed point of two self mappings F and G, if it satisfies $F(u) = G(u) = u$. In recent years, many researchers are interested in convergent results of fixed point and common fixed point for a pair of mappings [6, 14, 21, 20]. In [17], the author proved the convergent result of common fixed point via Mann iteration. In [18], the authors studied the convergence of Ishikawa iteration to a common fixed point of a pair of mappings. In [9], the author defined the pair of mean nonexpansive mappings and proved the existence of the common fixed point results in Banach space settings. In [22], the authors gave approximation methods for common fixed points of mean nonexpansive mappings in Banach spaces. Recently, Eldred et al [3], approximated a fixed point using Mann's iterative process for the mapping of the form $G : M \cup N \to M \cup N$, which satisfies $(i)\, G(M) \subseteq M$ and $G(N) \subseteq N$ and (ii) $\|Gu - Gv\| \leq \|u - v\|$, $\forall u \in M, v \in N$, where M and N are nonempty bounded closed convex subsets of a uniformly convex Banach space.

Motivated by Eldred et al [3], in this paper, we define a relatively nonexpansive condition for the non-cyclic pair of mappings of the type $F, G : M \cup N \to M \cup N$, and we prove convergent results for the common fixed points through the Ishikawa's iteration process associated with the mappings F and G. Finally, we provide some corollaries on common best proximity point of a pair of cyclic mappings.

2. Preliminaries

Consider M and N are non-void subsets of a normed linear space. The following notations are used subsequently:

$$\begin{aligned}
&d(u, M) = inf\{\|u - v\| : v \in M\};\\
&P_M(u) = \{v \in M : \|u - v\| = d(u, M)\};\\
&dist(M, N) = inf\{\|u - v\| : u \in M, v \in N\};\\
&M_0 = \{u \in M : \|u - v'\| = dist(M, N) \text{ for some } v' \in N\};\\
&N_0 = \{v \in N : \|u' - v\| = dist(M, N) \text{ for some } u' \in M\}.
\end{aligned}$$

If M is convex, closed subset of a reflexive and strictly convex space, then $P_M(u)$ contains one element and if M and N are convex, closed subsets of a reflexive space, with either M or N is bounded, then $M_0 \neq \emptyset$.

The following definitions and theorems are very useful to prove our main results: First, we collect the following Ishikawa iteration sequence from [22]. Let X be a Banach space and let F, G be mappings from X to X. Then the Ishikawa iteration sequence $\{u_n\}$ of F and G is defined as

$$\begin{aligned}
v_n &= (1 - \eta_n)u_n + \eta_n F u_n,\\
u_{n+1} &= (1 - \delta_n)u_n + \delta_n G v_n, \qquad (2.1)
\end{aligned}$$

where $u_0 \in X$ and $\delta_n, \eta_n \in [0, 1]$.

Definition 2.1. [3] Let M and N be nonempty subsets of a Banach space X.
A mapping $G : M \cup N \to M \cup N$ is relatively non expansive if

(1) $G(M) \subseteq M, G(N) \subseteq N$,
(2) $\|Gu - Gv\| \leq \|u - v\|$, for all $u \in M, v \in N$.

Definition 2.2. [23] Let $(X, \|\cdot\|)$ be a Banach space. For every $\epsilon \in (0, 2]$, define the modulus of convexity of $\|\cdot\|$ by

$$\delta_X(\epsilon) = \inf\Big\{1 - \Big\|\frac{x + y}{2}\Big\| : x, y \in B_X, \|x - y\| \geq \epsilon\Big\},$$

where B_X is the unit ball of Banach space X.
The norm is called uniformly convex if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. The space $(X, \|\cdot\|)$ is then called uniformly convex space.

Definition 2.3. [22] Let X be a Banach space. The pair of mappings $F, G : X \to X$ is said to be mean nonexpansive if

$$\|Fu - Gv\| \leq a\|u - v\| + b\{\|u - Fu\| + \|v - Gv\|\} + c\{\|u - Gv\| + \|v - Fu\|\},$$

for all $u, v \in X, a, b, c \in [0, 1]$ and $a + 2b + 2c \leq 1$.

Remark 2.4. In Definition 2.3, for $a = 1, b = c = 0$, then the pair of mappings $F, G : X \to X$ is said to be nonexpansive.

Using the above definition for a pair of mappings, in [22], the authors proved the following convergence result for a common fixed point.

Theorem 2.5. [22] *Let K be a convex subset of a uniformly convex Banach space and suppose $G, F : K \to K$ is a pair of mean nonexpansive mappings with a nonempty common fixed points set; if $b > 0, 0 < \delta \leq \delta_n \leq 1/2, 0 \leq \eta_n \leq \eta < 1$, then the Ishikawa sequence $\{u_n\}$ converges to the common fixed points of F and G.*

Proposition 2.6. [8] *If X is a uniformly convex space and $\delta \in (0, 1)$ and $\epsilon > 0$, then for any $d > 0$, if $u, v \in X$ are such that $\|u\| \leq d, \|v\| \leq d, \|u - v\| \geq \epsilon$, then there exists $\delta = \delta(\frac{\epsilon}{d}) > 0$ such that $\|\delta u + (1 - \delta)v\| \leq \Big(1 - 2\delta(\frac{\epsilon}{d}) min(\delta, 1 - \delta)\Big)d$.*

Lemma 2.7. [4] *Suppose X be a uniformly convex Banach space. Suppose $0 < a < b < 1$, and $\{t_n\}$ is a sequence in $[a, b]$. Suppose $\{w_n\}, \{v_n\}$ are sequences in X such that $\|w_n\| \leq 1, \|v_n\| \leq 1$ for all n. Define $\{z_n\}$ in X by $z_n = (1 - t_n)w_n + t_n v_n$. If $\lim_{n\to\infty} \|z_n\| = 1$, then $\lim_{n\to\infty} \|w_n - v_n\| = 0$.*

We prove the following result which shows that, if F, G is a pair of nonexpansive mappings then the Ishikawa's iteration associated with F and G, converges to a common fixed point of F, G. Moreover, it is useful to prove our main results.

Theorem 2.8. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space X and suppose $G, F : K \to K$ is a pair of nonexpansive mappings with a nonempty common fixed point set. Let $u_0 \in K$ and define*

$$u_{n+1} = (1 - \delta_n)u_n + \delta_n G((1 - \eta_n)u_n + \eta_n F u_n),$$

where $\delta_n, \eta_n \in (\epsilon, 1 - \epsilon)$, $n = 0, 1, 2, ...$ and $\epsilon \in (0, \frac{1}{2})$. Then

$$\lim_{n\to\infty} \|u_n - Gv_n\| = 0$$

and

$$\lim_{n\to\infty} \|u_n - Fu_n\| = 0.$$

Moreover, if $F(K)$ lies in a compact set, then $\{u_n\}$ and $\{v_n\}$ converge to a common fixed point of G and F.

Proof. By assumption, there exist $v \in K$ such that $Gv = Fv = v$. Now,

$$\begin{aligned}
\|u_{n+1} - v\| &= \|(1-\delta_n)u_n + \delta_n G\big((1-\eta_n)u_n + \eta_n Fu_n\big) - v\| \\
&= \|(1-\delta_n)u_n + \delta_n G\big((1-\eta_n)u_n + \eta_n Fu_n\big) - \big((1-\delta_n)v + \delta_n v\big)\| \\
&\leq (1-\delta_n)\|u_n - v\| + \delta_n\|G\big((1-\eta_n)u_n + \eta_n Fu_n\big) - Fv\| \\
&\leq (1-\delta_n)\|u_n - v\| + \delta_n\|(1-\eta_n)u_n + \eta_n Fu_n - v\| \\
&= (1-\delta_n)\|u_n - v\| \\
&\quad + \delta_n\|(1-\eta_n)u_n + \eta_n Fu_n - \big((1-\eta_n)v + \eta_n v\big)\| \\
&\leq (1-\delta_n)\|u_n - v\| + \delta_n\big(\|(1-\eta_n)(u_n - v)\| + \eta_n\|Fu_n - Gv\|\big) \\
&\leq \|u_n - v\|.
\end{aligned}$$

This implies that the sequence $\{\|u_n - v\|\}$ is nonincreasing and bounded below by 0. Hence there exists $d \geq 0$, such that $\|u_n - v\| \to d$.
Case (i) : If $\|u_n - v\| \to 0$.

$$\begin{aligned}
\|u_n - Fu_n\| &\leq \|u_n - v\| + \|v - Fu_n\| \\
&= \|u_n - v\| + \|Gv - Fu_n\| \\
&\leq \|u_n - v\| + \|v - u_n\|.
\end{aligned}$$

As $n \to \infty$, we get $\|u_n - Fu_n\| \to 0$.
And also $\|u_n - v_n\| \to 0$. Let $v_n = (1-\eta_n)u_n + \eta_n Fu_n$. Now

$$\begin{aligned}
\|v_n - v\| &= \|(1-\eta_n)u_n + \eta_n Fu_n - v\| \\
&= \|(1-\eta_n)u_n + \eta_n Fu_n - ((1-\eta_n)v + \eta_n v)\| \\
&\leq (1-\eta_n)\|u_n - v\| + \eta_n\|Fu_n - Gv\| \\
&\leq (1-\eta_n)\|u_n - v\| + \eta_n\|u_n - v\| \\
&= \|u_n - v\|.
\end{aligned}$$

And also

$$\begin{aligned}
\|u_n - Gv_n\| &\leq \|u_n - v\| + \|v - Gv_n\| \\
&= \|u_n - v\| + \|Fv - Gv_n\| \\
&\leq \|u_n - v\| + \|v - v_n\| \\
&\leq \|u_n - v\| + \|v - u_n\|.
\end{aligned}$$

As $n \to \infty$, we get $\|u_n - Gv_n\| \to 0$.
From the Ishikawa's iteration, we obtain

$$\|u_{n+1} - u_n\| = \delta_n\|Gv_n - u_n\|.$$

As $n \to \infty$, we get $\|u_{n+1} - u_n\| \to 0$.
Case (ii) : If $\|u_n - v\| \to d > 0$. We need to show that $\|u_n - Fu_n\| \to 0$. Suppose not. Then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and an $\epsilon > 0$ such that $\|u_{n_k} - Fu_{n_k}\| \geq \epsilon > 0$ for all k.
Since the modulus of convexity of δ of X is continuous and increasing function we

choose $\xi > 0$ as small that $\Big(1 - c\delta\big(\frac{\epsilon}{d+\xi}\big)\Big)(d+\xi) < d$, where $c > 0$.
Now we choose k, such that $\|u_{n_k} - v\| \le d + \xi$. By using Proposition 2.6,

$$\begin{aligned}
\|v - u_{n_k+1}\| &= \|v - \big((1-\delta_{n_k})u_{n_k} + \delta_{n_k} G\big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} F u_{n_k}\big)\big)\| \\
&= \|(1-\delta_{n_k})v + \delta_{n_k} v \\
&\quad - \big((1-\delta_{n_k})u_{n_k} + \delta_{n_k} G\big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} F u_{n_k}\big)\big)\| \\
&\le (1-\delta_{n_k})\|v - u_{n_k}\| + \delta_{n_k}\|Fv - G\big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} F u_{n_k}\big)\| \\
&\le (1-\delta_{n_k})(d+\xi) + \delta_{n_k}\|v - \big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} F u_{n_k}\big)\| \\
&= (1-\delta_{n_k})(d+\xi) + \delta_{n_k}\|(1-\eta_{n_k})(v - u_{n_k}) + \eta_{n_k}(v - F u_{n_k})\| \\
&\le (1-\delta_{n_k})(d+\xi) + \delta_{n_k}\Big(1 - 2\delta\Big(\frac{\epsilon}{d+\xi}\Big)\min\{\eta_{n_k}, 1-\eta_{n_k}\}\Big)(d+\xi) \\
&= \Big(1 - \delta_{n_k} + \delta_{n_k} - 2\delta_{n_k}\delta\Big(\frac{\epsilon}{d+\xi}\Big)\min\{\eta_{n_k}, 1-\eta_{n_k}\}\Big)(d+\xi) \\
&= \Big(1 - 2\delta\Big(\frac{\epsilon}{d+\xi}\Big)\min\{\delta_{n_k}\eta_{n_k}, \delta_{n_k}(1-\eta_{n_k})\}\Big)(d+\xi).
\end{aligned}$$

Since there exists $l > 0$ such that $2\min\{\delta_{n_k}\eta_{n_k}, \delta_{n_k}(1-\eta_{n_k})\} \ge l$,

$$\Big(1 - 2\delta\Big(\frac{\epsilon}{d+\xi}\Big)\min\{\delta_{n_k}\eta_{n_k}, \delta_{n_k}(1-\eta_{n_k})\}\Big)(d+\xi) \le \Big(1 - l\delta\Big(\frac{\epsilon}{d+\xi}\Big)\Big)(d+\xi).$$

Suppose we choose very small $\xi > 0$, we have

$$\Big(1 - l\delta\Big(\frac{\epsilon}{d+\xi}\Big)\Big)(d+\xi) < d,$$

which is contradiction. This implies that

$$\lim_{n\to\infty}\|u_n - Fu_n\| = \lim_{n\to\infty}\|u_n - v_n\| = 0.$$

Now we prove that $\|u_{n+1} - u_n\| \to 0$. We know that

$$\|u_{n+1} - u_n\| = \delta_n\|Gv_n - u_n\|,$$

where $v_n = (1-\eta_n)u_n + \eta_n F u_n$. Now, we define

$$z_n = \frac{u_{n+1} - v}{\|u_n - v\|},\ y_n = \frac{Gv_n - v}{\|u_n - v\|} \text{ and } w_n = \frac{u_n - v}{\|u_n - v\|}.$$

One can note that $\|w_n\| = 1$. Now,

$$\begin{aligned}
\|Gv_n - v\| &= \|Gv_n - Fv\| \\
&\le \|v_n - v\| \\
&\le \|(1-\eta_n)u_n + \eta_n F u_n - v\| \\
&\le \|(1-\eta_n)u_n + \eta_n F u_n - ((1-\eta_n)v + \eta_n v)\| \\
&\le (1-\eta_n)\|u_n - v\| + \eta_n\|Fu_n - Gv\| \\
&\le (1-\eta_n)\|u_n - v\| + \eta_n\|u_n - v\| \\
&= \|u_n - v\|.
\end{aligned}$$

Therefore

$$\|y_n\| = \frac{\|Gv_n - v\|}{\|u_n - v\|} \leq \frac{\|u_n - v\|}{\|u_n - v\|} = 1.$$

From the Ishikawa's iteration, we obtain

$$u_{n+1} - v = (1 - \delta_n)(u_n - v) + \delta_n(Gv_n - v).$$

Dividing by $\|u_n - v\|$, we get

$$\frac{u_{n+1} - v}{\|u_n - v\|} = (1 - \delta_n)\frac{(u_n - v)}{\|u_n - v\|} + \delta_n \frac{(Gv_n - v)}{\|u_n - v\|}.$$

Then $z_n = (1 - \delta_n)w_n + \delta_n y_n$. Now we prove that $\|z_n\| \to 1$. Now,

$$\lim_{n\to\infty} \|z_n\| \quad = \quad \lim_{n\to\infty} \frac{\|u_{n+1} - v\|}{\|u_n - v\|} = \frac{d}{d} = 1.$$

By Lemma 2.7, $\|w_n - y_n\| \to 0$. This implies that $\|u_n - Gv_n\| \to 0$. Therefore

$$\|u_{n+1} - u_n\| \to 0.$$

Since $F(K)$ is contained in a compact set, $\{Fu_n\}$ has a subsequence $\{Fu_{n_k}\}$ that converges to a point $z \in K$. Also $\{u_{n_k}\}$ and $\{u_{n_k+1}\}$ converge to z. This implies that $\{u_n\}$ converges to z. And also $\{v_n\}$ converges to z. Then $Fu_n \to z, Gv_n \to z$. Since F and G are continuous, implies that $Fu_n \to Fz, Gv_n \to Gz$. Therefore $Fz = Gz = z$, which completes the proof. $\square$

Let M be a convex closed subset of a Hilbert Space X. Then for $u \in X$, we know that $P_M(u)$ is the unique nearest point of M to u. Also P_M is non expansive and distinguished by the Kolmogorov's criterion:
$\langle u - P_M u, P_M u - z\rangle \geq 0$, for all $u \in X$ and $z \in M$.

Let M and N be two convex closed subsets of X. Suppose define

$$P(u) = P_M(P_N(u)) \,\text{for each}\, u \in X,$$

then the sequences $\{P^n(u)\} \subset M$ and $\{P_N(P^n(u))\} \subset N$. The convergence of these sequences in norm were proved by von Neumann [15] when M and N are closed. The sequences $\{P^n(u)\}$ and $\{P_N(P^n(u))\}$ are called *von Neumann sequences* or *alternating projection algorithm* for two sets.

Definition 2.9. [5] Let M and N be nonempty closed convex subsets of a Hilbert space X. We say that (M, N) is boundedly regular if for each bounded subset F of X and for each $\epsilon > 0$ there exist $\delta > 0$ such that

$$\max\{d(u, M), d(u, N - v)\} \leq \delta \Rightarrow d(u, N) \leq \epsilon, \ \forall u \in F, \tag{2.2}$$

where $v = P_{\overline{N-M}}(0)$, the displacement vector from M to N. (v is the unique vector satisfying $\|v\| = dist(M, N)$).

Theorem 2.10. [5] *If (M, N) is boundedly regular, then the von Neumann sequences converges in norm.*

Theorem 2.11. [5] *If M or N is boundedly compact, then (M, N) is boundedly regular.*

Lemma 2.12. [4] *Let M be a nonempty closed and convex subset and N be nonempty closed subset of a uniformly convex Banach space. Let $\{u_n\}$ and $\{z_n\}$ be sequences in M and $\{v_n\}$ be a sequence in N satisfying:*

(1) $\|u_n - v_n\| \to dist(M, N)$, *and*
(2) $\|z_n - v_n\| \to dist(M, N)$. *Then* $\|u_n - z_n\|$ *converges to zero.*

Lemma 2.13. [4] *Let M be a nonempty closed convex subset and N be a nonempty closed subset of uniformly convex Banach space. Let $\{u_n\}$ be a sequence in M and $v_0 \in N$ such that $\|u_n - v_0\| \to dist(M, N)$. Then $\{u_n\}$ converges to $P_M(v_0)$.*

Proposition 2.14. [2] *Let M and N be two closed and convex subsets of a Hilbert space X. Then $P_N(M) \subseteq N, P_M(N) \subseteq M$, and $\|P_N u - P_M v\| \leq \|u - v\|$ for $u \in M$ and $v \in N$.*

Lemma 2.15. [3] *Let M and N be two closed and convex subsets of a Hilbert space X. For each $u \in X$,*

$$\|P^{n+1}(u) - z\| \leq \|P^n(u) - z\|, \text{for each } z \in M_0 \cup N_0.$$

3. Main results

In this section, we prove convergent results for common fixed point.

Theorem 3.1. *Let M and N be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $F, G : M \cup N \to M \cup N$ satisfy*

(1) $G(M) \subseteq M, G(N) \subseteq N, F(M) \subseteq M$ *and* $F(N) \subseteq N$;
(2) $\|Fu - Gv\| \leq \|u - v\|$ *for* $u \in M, v \in N$; *and*
(3) $\|Fu - Gv\| \leq \|u - v\|$ *for* $u \in N, v \in M$,

with a nonempty common fixed point set. Let $u_0 \in M$, and define

$$u_{n+1} = (1 - \delta_n)u_n + \delta_n G v_n,\ v_n = (1 - \eta_n)u_n + \eta_n F u_n,\ \delta_n, \eta_n \in (\epsilon, 1 - \epsilon),$$

where $\epsilon \in (0, 1/2)$ and $n = 0, 1, 2, \ldots$.
Suppose $d(u_n, M_0) \to 0$, then $\lim_{n\to\infty} \|u_n - Gv_n\| = 0$ and $\lim_{n\to\infty} \|u_n - Fu_n\| = 0$. Moreover, if $F(M)$ lies in a compact set, then $\{u_n\}$ and $\{v_n\}$ converges to a common fixed point of G and F.

Proof. If $dist(M, N) = 0$, then $M_0 = N_0 = M \cap N$ and by Theorem 2.8, we can prove the result from the truth that $F, G : M \cap N \to M \cap N$ is a pair of nonexpansive. Therefore let us take that $dist(M, N) > 0$. For a common fixed point $v \in N$ of F and G, we get

$$\begin{aligned}
\|u_{n+1} - v\| &= \|(1 - \delta_n)u_n + \delta_n G\big((1 - \eta_n)u_n + \eta_n F u_n\big) - v\| \\
&= \|(1 - \delta_n)u_n + \delta_n G\big((1 - \eta_n)u_n + \eta_n F u_n\big) - \big((1 - \delta_n)v + \delta_n v\big)\| \\
&\leq (1 - \delta_n)\|u_n - v\| + \delta_n \|G\big((1 - \eta_n)u_n + \eta_n F u_n\big) - Fv\| \\
&\leq (1 - \delta_n)\|u_n - v\| + \delta_n \|(1 - \eta_n)u_n + \eta_n F u_n - v\| \\
&= (1 - \delta_n)\|u_n - v\| + \delta_n \|(1 - \eta_n)u_n + \eta_n F u_n - \big((1 - \eta_n)v + \eta_n v\big)\| \\
&\leq (1 - \delta_n)\|u_n - v\| + \delta_n \big((1 - \eta_n)\|u_n - v\| + \eta_n \|F u_n - Gv\|\big) \\
&\leq \|u_n - v\|.
\end{aligned}$$

This implies that the sequence $\{\|u_n - v\|\}$ is nonincreasing. Then we can find $d > 0$ such that

$$\lim_{n\to\infty} \|u_n - v\| = d.$$

Suppose there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and an $\epsilon > 0$ such that

$$\|u_{n_k} - Fu_{n_k}\| \geq \epsilon > 0 \text{ for all } k.$$

Since the modulus of convexity of δ of X is continuous and increasing function we choose $\xi > 0$ as small that

$$\Big(1 - c\delta\big(\frac{\epsilon}{d+\xi}\big)\Big)(d+\xi) < d,$$

where $c > 0$.
Now we choose k, such that $\|u_{n_k} - v\| \leq d + \xi$. By using the Proposition 2.6,

$$\begin{aligned}
\|v - u_{n_k+1}\| &= \|v - \big((1-\delta_{n_k})u_{n_k} + \delta_{n_k} G\big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} Fu_{n_k}\big)\big)\| \\
&= \|(1-\delta_{n_k})v + \delta_{n_k} v \\
&\quad -\big((1-\delta_{n_k})u_{n_k} + \delta_{n_k} G\big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} Fu_{n_k}\big)\big)\| \\
&\leq (1-\delta_{n_k})\|v - u_{n_k}\| + \delta_{n_k}\|Fv - G\big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} Fu_{n_k}\big)\| \\
&\leq (1-\delta_{n_k})(d+\xi) + \delta_{n_k}\|v - \big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} Fu_{n_k}\big)\| \\
&= (1-\delta_{n_k})(d+\xi) + \delta_{n_k}\|(1-\eta_{n_k})(v - u_{n_k}) + \eta_{n_k}(v - Fu_{n_k})\| \\
&\leq (1-\delta_{n_k})(d+\xi) + \delta_{n_k}\Big(1 - 2\delta\Big(\frac{\epsilon}{d+\xi}\Big)\min\{\eta_{n_k}, 1-\eta_{n_k}\}\Big)(d+\xi) \\
&= \Big(1 - \delta_{n_k} + \delta_{n_k} - 2\delta_{n_k}\delta\Big(\frac{\epsilon}{d+\xi}\Big)\min\{\eta_{n_k}, 1-\eta_{n_k}\}\Big)(d+\xi) \\
&= \Big(1 - 2\delta\Big(\frac{\epsilon}{d+\xi}\Big)\min\{\delta_{n_k}\eta_{n_k}, \delta_{n_k}(1-\eta_{n_k})\}\Big)(d+\xi).
\end{aligned}$$

Since there exists $l > 0$ such that $2\min\{\delta_{n_k}\eta_{n_k}, \delta_{n_k}(1-\eta_{n_k})\} \geq l$,

$$\Big(1 - 2\delta\Big(\frac{\epsilon}{d+\xi}\Big)\min\{\delta_{n_k}\eta_{n_k}, \delta_{n_k}(1-\eta_{n_k})\}\Big)(d+\xi) \leq \Big(1 - l\delta\Big(\frac{\epsilon}{d+\xi}\Big)\Big)(d+\xi).$$

Suppose we choose very small $\xi > 0$, we have

$$\Big(1 - l\delta\Big(\frac{\epsilon}{d+\xi}\Big)\Big)(d+\xi) < d,$$

which is contradiction. This implies that

$$\lim_{n\to\infty} \|u_n - Fu_n\| = \lim_{n\to\infty} \|v_n - u_n\| = 0.$$

Next we prove that

$$\lim_{n\to\infty} \|u_n - Gv_n\| = 0.$$

Now we define

$$z_n = \frac{u_{n+1} - v}{\|u_n - v\|},\ y_n = \frac{Gv_n - v}{\|u_n - v\|}$$

and

$$w_n = \frac{u_n - v}{\|u_n - v\|}.$$

Here $\|w_n\| = 1$, and since

$$\begin{aligned}
\|Gv_n - v\| &= \|Gv_n - Fv\| \\
&\leq \|v_n - v\| \\
&= \|(1-\eta_n)u_n + \eta_n Fu_n - v\| \\
&= \|(1-\eta_n)u_n + \eta_n Fu_n - ((1-\eta_n)v + \eta_n v)\| \\
&\leq (1-\eta_n)\|u_n - v\| + \eta_n\|Fu_n - Gv\| \\
&\leq (1-\eta_n)\|u_n - v\| + \eta_n\|u_n - v\| \\
&= \|u_n - v\|.
\end{aligned}$$

Therefore

$$\|y_n\| = \frac{\|Gv_n - v\|}{\|u_n - v\|} \leq \frac{\|u_n - v\|}{\|u_n - v\|} = 1.$$

From the Ishikawa's iteration, we obtain

$$u_{n+1} - v = (1-\delta_n)(u_n - v) + \delta_n(Gv_n - v).$$

Dividing by $\|u_n - v\|$, we get

$$\frac{u_{n+1} - v}{\|u_n - v\|} = (1-\delta_n)\frac{(u_n - v)}{\|u_n - v\|} + \delta_n\frac{(Gv_n - v)}{\|u_n - v\|}.$$

Then $z_n = (1-\delta_n)w_n + \delta_n y_n$. Now we prove that $\|z_n\| \to 1$. Now

$$\lim_{n\to\infty}\|z_n\| \quad = \quad \lim_{n\to\infty}\frac{\|u_{n+1} - v\|}{\|u_n - v\|} = \frac{d}{d} = 1.$$

By Lemma 2.7, we get $\|w_n - y_n\| \to 0$. This implies that $\|u_n - Gv_n\| \to 0$. We know that $\|u_{n+1} - u_n\| = |\delta_n|\|Gv_n - u_n\|$. Therefore $\|u_{n+1} - u_n\| \to 0$.

Since $F(M)$ is contained in a compact set, then $\{Fu_n\}$ has a subsequence $\{Fu_{n_k}\}$, that converges to a point $z \in M$. Also $\{u_{n_k}\}$ and $\{u_{n_k+1}\}$ converge to z. Therefore, $u_n \to z$. Also $v_n \to z$.

Since $d(u_n, M_0) \to 0$, there exist $\{a_n\} \subseteq M_0$, such that $\|u_n - a_n\| \to 0$. Therefore, $a_{n_k} \to z$, which gives that $z \in M_0$.

Let $D = dist(M, N)$ and choose $w \in N_0$ such that $\|z - w\| = D$.

So we have $\|u_{n_k} - w\| \to \|z - w\| = D$, and $\|u_{n_k} - w\| \geq \|Fu_{n_k} - Gw\| \to \|z - Gw\|$. So $\|z - Gw\| = D$. By strict convexity of the norm, $Gw = w$.

And also $\|v_{n_k} - w\| \to \|z - w\| = D$, and $\|v_{n_k} - w\| \geq \|Gv_{n_k} - Fw\| \to \|z - Fw\|$. So $\|z - Fw\| = D$. Again by strict convexity of the norm, $Fw = w$.

And we have $\|Gz - Fw\| \leq \|z - w\| = D$, then $\|Gz - Fw\| = D$. By strict convexity of the norm, we obtain $Gz = z$.

And also $\|Fz - Gw\| \leq \|z - w\| = D$, then $\|Fz - Gw\| = D$. By strict convexity of the norm, we obtain $Fz = z$. Therefore, $Gz = Fz = z$, theorem follows. □

Corollary 3.1. *Let M and N be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $F, G : M \cup N \to M \cup N$ satisfy*

(1) $G(M) \subseteq M, G(N) \subseteq N, F(M) \subseteq M$ *and* $F(N) \subseteq N$; *and*
(2) $\|Fu - Gv\| \leq \|u - v\|$ *for* $u \in M, v \in N$.
(3) $\|Fu - Gv\| \leq \|u - v\|$ *for* $u \in N, v \in M$.

Let $u_0 \in M_0$, and define

$$u_{n+1} = (1-\delta_n)u_n + \delta_n G\big((1-\eta_n)u_n + \eta_n F u_n\big),\ \delta_n, \eta_n \in (\epsilon, 1-\epsilon),$$

where $\epsilon \in (0, 1/2)$ and $n = 0, 1, 2, ...,$ then

$$\lim_{n\to\infty} \|u_n - Gv_n\| = 0 \text{ and } \lim_{n\to\infty} \|u_n - Fu_n\| = 0.$$

Moreover, if $F(M)$ lies in a compact set, then $\{u_n\}$ and $\{v_n\}$ converges to a common fixed point of G and F.

Corollary 3.2. *Let M and N be nonempty bounded closed convex subsets of a Hilbert Space and Let G, F be as in Theorem 3.1. Let $u_0 \in M_0$, and define*

$$u_{n+1} = P^n\big((1-\delta_n)u_n + \delta_n G v_n\big),$$

where $v_n = (1-\eta_n)u_n + \eta_n F u_n$, $\delta_n, \eta_n \in (\epsilon, 1-\epsilon)$, where $\epsilon \in (0, 1/2)$ and $n = 0, 1, 2, ...,$ then

$$\lim_{n\to\infty} \|u_n - Gv_n\| = 0 \text{ and } \lim_{n\to\infty} \|u_n - Fu_n\| = 0.$$

Moreover, if $F(M)$ lies in a compact set, then $\{u_n\}$ and $\{v_n\}$ converges to a common fixed point of G and F.

Proof. One can note that $P^n\big((1-\delta_n)u_n + \delta_n G v_n\big) = (1-\delta_n)u_n + \delta_n G v_n$, by Corollary 3.1, the result follows. $\square$

We illustrate the above theorem through the following example.

Example 3.1. Let $(\mathbb{R}^2, \|.\|)$ with $\|(u_1, u_2) - (v_1, v_2)\| = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2}$. Let $M = \{(0, u) \in \mathbb{R}^2 : u \in [0, 1]\}$ and $N = \{(1, u) \in \mathbb{R}^2 : u \in [2, 3]\}$, then $dist(M, N) = \sqrt{2}$. And we define a pair of mappings $F, G : M \cup N \to M \cup N$ by $F(0, u) = (0, 1)$, $F(1, u) = (1, u)$ and $G(0, u) = (0, u)$, $G(1, u) = (1, 2)$.
For $(0, u) \in M, (1, v) \in N$, we have

$$\|G(0, u) - F(1, v)\| = \|(0, u) - (1, v)\|.$$

For $(0, u) \in M, (1, v) \in N$, we have

$$\|F(0, u) - G(1, v)\| = \|(0, 1) - (1, 2)\| = \sqrt{2} \leq \|(0, u) - (1, v)\|.$$

Clearly, the set $\{(0, 1), (1, 2)\}$ is common fixed points of F and G. Fix

$$\delta_n = \frac{3}{4},\ \eta_n = \frac{3}{4},\ \forall n.$$

Let $(0, u_0) \in M$, then the Ishikawa iteration becomes

$$\begin{aligned}
(0, v_n) &= \Big(1 - \frac{3}{4}\Big)(0, u_n) + \frac{3}{4}F(0, u_n) \\
&= \frac{1}{4}(0, u_n) + \frac{3}{4}(0, 1) \\
&= \Big(0, \frac{u_n}{4}\Big) + \Big(0, \frac{3}{4}\Big) \\
&= \Big(0, \frac{u_n + 3}{4}\Big)
\end{aligned}$$

and

$$\begin{aligned}(0,u_{n+1}) &= \Big(1-\frac{3}{4}\Big)(0,u_n)+\frac{3}{4}G(0,v_n)\\ &= \frac{1}{4}(0,u_n)+\frac{3}{4}G\Big(0,\frac{u_n+3}{4}\Big)\\ &= \Big(0,\frac{u_n}{4}\Big)+\frac{3}{4}\Big(0,\frac{u_n+3}{4}\Big)\\ &= \Big(0,\frac{u_n}{4}\Big)+\Big(0,\frac{3u_n+9}{16}\Big)\\ &= \Big(0,\frac{7u_n+9}{16}\Big).\end{aligned}$$

Using Matlab coding, we give the following table to show that the iteration $\{(0,u_{n+1})\}$ and $\{(0,v_n)\}$, converge to a common fixed point of F,G for a initial point

$$(0,u_0)=(0,0.2)\in M.$$

n	$(0,u_{n+1})$	$(0,v_n)$
22	(0,0.999999989893349)	(0,0.999999999999954)
23	(0,0.999999995578340)	(0,0.999999999999989)
24	(0,0.999999998065524)	(0,0.999999999999997)
25	(0,0.999999999153667)	(0,0.999999999999999)
26	(0,0.999999999629729)	**(0,1.000000000000000)**
⋮	⋮	⋮
41	(0,0.999999999999998)	
42	(0,0.999999999999999)	
43	**(0,1.000000000000000)**	

As a next main result, we want to approximate common best proximity pair using the Theorem 3.1.

Definition 3.2. [12] Let (M,N) be a nonempty pair of subsets of a metric space (X,d) and $F:M\cup N\to M\cup N$ be a noncyclic mapping. A point $(p,q)\in M\times N$ is said to be a best proximity pair for the noncyclic mapping F if

$$p=Fp,\quad q=Fq,\quad d(p,q)=dist(A,B).$$

Definition 3.3. [12] Let F and G be two noncyclic mappings defined on $M\cup N$, where (M,N) is a nonempty pair in a normed linear space X. A point $(p,q)\in M\times N$ is called a common best proximity pair for the noncyclic pair (F,G) if (p,q) is a best proximity pair for both F and G, that is, p and q are two common fixed points of the mappings F and G in M and N respectively, such that $\|p-q\|=dist(A,B)$.

Lemma 3.4. [11] *Let (M,N) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space X. Define $P:M_0\cup N_0\to M_0\cup N_0$ as*

$$P(x)=\begin{cases}P_{M_0}(x) & \text{if}\quad x\in N_0,\\ P_{N_0}(x) & \text{if}\quad x\in M_0.\end{cases}\tag{3.1}$$

Then the following statements hold.

(1) *$\|x-Px\| = dist(M,N)$ for any $x \in M_0 \cup N_0$ and $P(M_0) \subseteq N_0$, $P(N_0) \subseteq M_0$.*
(2) *P is an isometry, that is, $\|Px - Py\| = \|x - y\|$ for all $(x, y) \in M_0 \times N_0$.*
(3) *P is affine.*

Definition 3.5. [16] If $M_0 \neq \emptyset$ then the pair (M, N) is said to have P-property if for any $u_1, u_2 \in M_0$ and $v_1, v_2 \in N_0$

$$\begin{cases} d(u_1, v_1) = dist(M, N) \\ d(u_2, v_2) = dist(M, N) \end{cases} \Rightarrow d(u_1, u_2) = d(v_1, v_2).$$

Lemma 3.6. [1] *Every, nonempty, bounded, closed and convex pair in a uniformly convex Banach space X has the P-property.*

Lemma 3.7. [12] *Let (M, N) be a nonempty, closed and convex pair in a uniformly convex Banach space X. Then for the projection mapping $P : M_0 \cup N_0 \to M_0 \cup N_0$ defined in (3.1) we have both $P|_{M_0}$ and $P|_{N_0}$ are continuous.*

Here we prove the convergence result:

Theorem 3.8. *Let M and N be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $F, G : M \cup N \to M \cup N$ satisfy*
(1) *$G(M) \subseteq M, G(N) \subseteq N, F(M) \subseteq M$ and $F(N) \subseteq N$;*
(2) *$\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and*
(3) *$\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$,*

with a nonempty common fixed point set. Let $u_0 \in M$, and define

$$u_{n+1} = (1 - \delta_n)u_n + \delta_n G v_n,\ v_n = (1 - \eta_n)u_n + \eta_n F u_n,\ \delta_n, \eta_n \in (\epsilon, 1 - \epsilon),$$

where $\epsilon \in (0, 1/2)$ and $n = 0, 1, 2, ...$ and $w_{n+1} = Pu_{n+1}$, where P is the projection mapping defined in (3.1). Assume $d(u_n, M_0) \to 0$. Suppose $F(M)$ lies in a compact set, then the sequence $\{(u_n, w_n)\} \in M \times N$ converges to a best proximity pair of the mappings G and F.

Proof. Let $x \in M_0$. Then we have

$$\|Fx - GPx\| \leq \|x - Px\| = dist(M, N).$$

Therefore, $\|Fx - GPx\| = dist(M, N) = \|Fx - PFx\|$. By Lemma 3.6, we get $GPx = PFx$. In the same way, we can prove $FPx = PGx$. Also by Theorem 3.1, the Ishikawa's iteration $\{u_n\}$ converges to a common fixed point $z \in M_0$ of G and F. From Lemma 3.7, we know that $P|_{M_0}$ is continuous. Then $w_n = Pu_n \to Pz := z'$. Clearly $\|z - z'\| = dist(M, N)$. So we can obtain

$$Fz' = FPz = PGz = Pz = z'.$$

Also,

$$Gz' = GPz = PFz = Pz = z'.$$

So the result follows. □

4. Approximating common fixed point using von Neumann sequences

In the next result, we provide a stronger version to approximate the common fixed point via von Neumann sequences.

Theorem 4.1. *Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G : M \cup N \to M \cup N$ satisfy*

(1) $G(M) \subseteq M, G(N) \subseteq N, F(M) \subseteq M$ *and* $F(N) \subseteq N$;
(2) $\|Fu - Gv\| \le \|u - v\|$ *for* $u \in M, v \in N$; *and*
(3) $\|Fu - Gv\| \le \|u - v\|$ *for* $u \in N, v \in M$.

with nonempty common fixed point set. Let $u_0 \in M$, and define

$$u_{n+1} = P^n\big((1-\delta_n)u_n + \delta_n G v_n\big),\ v_n = (1-\eta_n)u_n + \eta_n F u_n,\ \delta_n, \eta_n \in (\epsilon, 1-\epsilon),$$

where $\epsilon \in (0, 1/2)$ and $n = 1, 2, \ldots$, then

$$\lim_{n\to\infty} \|u_n - Fu_n\| = 0.$$

Moreover, if $F(M)$ lies in a compact set and $\|u_n - Gv_n\| \to 0$, then $\{u_n\}$ and $\{v_n\}$ converges to a common fixed point of G and F.

Proof. If $dist(M, N) = 0$, then $M_0 = N_0 = M \cap N$ and $F, G : M \cap N \to M \cap N$ is a pair of nonexpansive with

$$\begin{aligned} u_{n+1} &= P^n\big((1-\delta_n)u_n + \delta_n G\big((1-\eta_n)u_n + \eta_n F u_n\big)\big) \\ &= (1-\delta_n)u_n + \delta_n G\big((1-\eta_n)u_n + \eta_n F u_n\big), \end{aligned}$$

the usual Ishikawa's iteration. So let us take that $dist(M, N) > 0$. For a common fixed point $v \in N$ of F and G. Now,

$$\begin{aligned} \|u_{n+1} - v\| &= \|P^n\big((1-\delta_n)u_n + \delta_n G\big((1-\eta_n)u_n + \eta_n F u_n\big)\big) - v\| \\ &\le \|(1-\delta_n)u_n + \delta_n G\big((1-\eta_n)u_n + \eta_n F u_n\big) - v\| \\ &= \|(1-\delta_n)u_n + \delta_n G\big((1-\eta_n)u_n + \eta_n F u_n\big) - \big((1-\delta_n)v + \delta_n v\big)\| \\ &\le (1-\delta_n)\|u_n - v\| + \delta_n\|G\big((1-\eta_n)u_n + \eta_n F u_n\big) - Fv\| \\ &\le (1-\delta_n)\|u_n - v\| + \delta_n\|(1-\eta_n)u_n + \eta_n F u_n - v\| \\ &= (1-\delta_n)\|u_n - v\| \\ &\quad + \delta_n\|(1-\eta_n)u_n + \eta_n F u_n - \big((1-\eta_n)v + \eta_n v\big)\| \\ &\le (1-\delta_n)\|u_n - v\| + \delta_n\big((1-\eta_n)\|u_n - v\| + \eta_n\|Fu_n - Gv\|\big) \\ &\le \|u_n - v\|. \end{aligned}$$

This implies that the sequence $\{\|u_n - v\|\}$ is nonincreasing. Then we can find $d > 0$ such that

$$\lim_{n\to\infty} \|u_n - v\| = d.$$

Suppose there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and an $\epsilon > 0$ such that

$$\|u_{n_k} - Fu_{n_k}\| \ge \epsilon > 0 \text{ for all } k.$$

Since the modulus of convexity of δ of X is continuous and increasing function we choose $\xi > 0$ as small that

$$\Big(1 - c\delta\big(\frac{\epsilon}{d+\xi}\big)\Big)(d+\xi) < d,$$

where $c > 0$.
Now we choose k, such that $\|u_{n_k} - v\| \leq d + \xi$. By using the proposition 2.6,

$$\begin{aligned}
\|v - u_{n_k+1}\| &= \|v - P^n\big((1-\delta_{n_k})u_{n_k} + \delta_{n_k} G\big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} F u_{n_k}\big)\big)\| \\
&\leq \|v - \big((1-\delta_{n_k})u_{n_k} + \delta_{n_k} G\big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} F u_{n_k}\big)\big)\| \\
&= \|(1-\delta_{n_k})v + \delta_{n_k} v \\
&\quad -\big((1-\delta_{n_k})u_{n_k} + \delta_{n_k} G\big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} F u_{n_k}\big)\big)\| \\
&\leq (1-\delta_{n_k})\|v - u_{n_k}\| + \delta_{n_k}\|Fv - G\big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} F u_{n_k}\big)\| \\
&\leq (1-\delta_{n_k})(d+\xi) + \delta_{n_k}\|v - \big((1-\eta_{n_k})u_{n_k} + \eta_{n_k} F u_{n_k}\big)\| \\
&= (1-\delta_{n_k})(d+\xi) + \delta_{n_k}\|(1-\eta_{n_k})(v - u_{n_k}) + \eta_{n_k}(v - F u_{n_k})\| \\
&\leq (1-\delta_{n_k})(d+\xi) + \delta_{n_k}\Big(1 - 2\delta\Big(\frac{\epsilon}{d+\xi}\Big)\min\{\eta_{n_k}, 1-\eta_{n_k}\}\Big)(d+\xi) \\
&= \Big(1 - \delta_{n_k} + \delta_{n_k} - 2\delta_{n_k}\delta\Big(\frac{\epsilon}{d+\xi}\Big)\min\{\eta_{n_k}, 1-\eta_{n_k}\}\Big)(d+\xi) \\
&= \Big(1 - 2\delta\Big(\frac{\epsilon}{d+\xi}\Big)\min\{\delta_{n_k}\eta_{n_k}, \delta_{n_k}(1-\eta_{n_k})\}\Big)(d+\xi).
\end{aligned}$$

Since there exists $l > 0$ such that $2\min\{\delta_{n_k}\eta_{n_k}, \delta_{n_k}(1-\eta_{n_k})\} \geq l$,

$$\Big(1 - 2\delta\Big(\frac{\epsilon}{d+\xi}\Big)\min\{\delta_{n_k}\eta_{n_k}, \delta_{n_k}(1-\eta_{n_k})\}\Big)(d+\xi) \leq \Big(1 - l\delta\Big(\frac{\epsilon}{d+\xi}\Big)\Big)(d+\xi).$$

Suppose we choose very small $\xi > 0$, we have

$$\Big(1 - l\delta\Big(\frac{\epsilon}{d+\xi}\Big)\Big)(d+\xi) < d,$$

which is contradiction. This implies that

$$\lim_{n\to\infty}\|u_n - Fu_n\| = \lim_{n\to\infty}\|v_n - u_n\| = 0.$$

Since $\|u_n - Gv_n\| \to 0$, and we know that

$$\|u_{n+1} - u_n\| = \|P^n\big((1-\delta_n)u_n + \delta_n G v_n\big) - u_n\| \leq \delta_n\|Gv_n - u_n\|,$$

we obtain $\|u_{n+1} - u_n\| \to 0$.
Since $F(M)$ is contained in a compact set, then $\{Fu_n\}$ has a subsequence $\{Fu_{n_k}\}$ that converges to a point $v_0 \in M$. Also $\{u_{n_k}\}, \{v_{n_k}\}, \{Gv_{n_k}\}$ and $\{u_{n_k+1}\}$ converge to v_0, which implies that $u_n \to v_0$. Also we have $v_n \to v_0$ as $n \to \infty$.
Now, $\|Fu_{n_k} - G(P_N(v_0))\| \leq \|u_{n_k} - P_N(v_0)\|$ which implies that

$$\|v_0 - G(P_N(v_0))\| \leq \|v_0 - P_N(v_0)\|.$$

Hence $G(P_N(v_0)) = P_N(v_0)$.
Similarly, $\|Gv_{n_k} - F(P_N(v_0))\| \leq \|v_{n_k} - P_N(v_0)\|$ which implies that

$$\|v_0 - F(P_N(v_0))\| \leq \|v_0 - P_N(v_0)\|.$$

Hence $F(P_N(v_0)) = P_N(v_0)$. Also,

$$\|G(P(v_0)) - P_N(v_0)\| = \|G(P(v_0)) - F(P_N(v_0))\| \leq \|P(v_0) - P_N(v_0)\|.$$

So $G(P(v_0)) = P(v_0)$. And also,

$$\|F(P(v_0)) - P_N(v_0)\| = \|F(P(v_0)) - G(P_N(v_0))\| \leq \|P(v_0) - P_N(v_0)\|.$$

So $F(P(v_0)) = P(v_0)$. Now,

$$\|GP_N(P(v_0)) - P(v_0)\| = \|GP_N(P(v_0)) - F(P(v_0))\| \leq \|P_N(P(v_0)) - P(v_0)\|.$$

Thus $GP_N(P(v_0)) = P_N(P(v_0))$.
For any n, we have $F(P^n(v_0)) = P^n(v_0)$ and $GP_N(P^n(v_0)) = P_N(P^n(v_0))$.
Similarly,

$$\|FP_N(P(v_0)) - P(v_0)\| = \|FP_N(P(v_0)) - G(P(v_0))\| \leq \|P_N(P(v_0)) - P(v_0)\|.$$

Thus $FP_N(P(v_0)) = P_N(P(v_0))$.
For any n, we have

$$G(P^n(v_0)) = P^n(v_0) \text{ and } FP_N(P^n(v_0)) = P_N(P^n(v_0)).$$

By Theorem 2.10, for each $u \in M$ the sequence $\{P^n(u)\}$ converges to some $r(u) \in M_0$.
Now,

$$\begin{aligned}
\|G(r(v_0)) - P_N(r(v_0))\| &\leq \lim_{n\to\infty} \|G(r(v_0)) - P_N(P^n(v_0))\| \\
&= \lim_{n\to\infty} \|G(r(v_0)) - F(P_N(P^n(v_0)))\| \\
&\leq \lim_{n\to\infty} \|r(v_0) - P_N(P^n(v_0))\| \\
&= \|r(v_0) - P_N(r(v_0))\|.
\end{aligned}$$

So

$$\|G(r(v_0)) - P_N(r(v_0))\| \leq \|r(v_0) - P_N(r(v_0))\|.$$

Therefore $G(r(v_0)) = r(v_0)$ and similarly, we get $GP_N(r(v_0)) = P_N(r(v_0))$.
In the same way, we prove that $F(r(v_0)) = r(v_0)$ and $FP_N(r(v_0)) = P_N(r(v_0))$.
Now we define $g_n : M \to \mathbb{R}$ by $g_n(u) = \|P^n(u) - r(u)\|$.
Since $\|r(u) - r(v)\| = \lim_{n\to\infty} \|P^n(u) - P^n(v)\| \leq \|u - v\|$, then we conclude that u is continuous. Therefore $g_n(u)$ is continuous and converges pointwise to zero. Since $r(u) \in M_0$, by Lemma 2.15, we obtain $g_{n+1} \leq g_n$. Therefore g_n converges uniformly on the compact set

$$\mathcal{F} = \{(1 - \delta_{n_k})u_{n_k} + \delta_{n_k} Gv_{n_k}\} \cup \{v_0\}.$$

Therefore

$$\lim_{k\to\infty} \|P^{n_k}((1 - \delta_{n_k})u_{n_k} + \delta_{n_k} Gv_{n_k}) - r((1 - \delta_{n_k})u_{n_k} + \delta_{n_k} Gv_{n_k})\| = 0.$$

Since $r((1 - \delta_{n_k})u_{n_k} + \delta_{n_k} Gv_{n_k}) \to r(v_0)$, we get $u_{n_k+1} \to r(v_0)$, which gives that $r(v_0) = v_0$. Therefore $Gv_0 = G(r(v_0)) = r(v_0) = v_0$ and $Fv_0 = F(r(v_0)) = r(v_0) = v_0$, which completes the proof. $\square$

Suppose X is a Hilbert space and let M and N be nonempty bounded closed convex subsets of X and suppose $F, G : M \cup N \to M \cup N$ satisfy

(1) $G(M) \subseteq N, G(N) \subseteq M, F(M) \subseteq N$ and $F(N) \subseteq M$;
(2) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and
(3) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$.

Consider $P_M G : M \to M, P_N F : N \to N, P_N G : N \to N$ and $P_M F : M \to M$. From the Proposition 2.14, $\|P_M F(u) - P_N G(v)\| \leq \|u - v\|$ for $u \in M$ and $v \in N$ and $\|P_N F(u) - P_M G(v)\| \leq \|u - v\|$ for $u \in N$ and $v \in M$, by Theorem 4.1, we give the following results on convergence of best proximity points.

Corollary 4.1. *Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G : M \cup N \to M \cup N$ satisfy*

(1) $G(M) \subseteq N, G(N) \subseteq M, F(M) \subseteq N$ *and* $F(N) \subseteq M$;
(2) $\|Fu - Gv\| \leq \|u - v\|$ *for* $u \in M, v \in N$; *and*
(3) $\|Fu - Gv\| \leq \|u - v\|$ *for* $u \in N, v \in M$.

If $F(M)$ is mapped into a compact subset of N, then for any $u_0 \in M_0$ the sequence defined by

$$u_{n+1} = (1 - \delta_n)u_n + \delta_n P_M\big(G((1 - \eta_n)u_n + \eta_n P_M F u_n)\big)$$

converges to u in M_0 such that $\|u - Gu\| = \|u - Fu\| = dist(M, N)$.

Corollary 4.2. *Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G : M \cup N \to M \cup N$ satisfy*

(1) $G(M) \subseteq N, G(N) \subseteq M, F(M) \subseteq N$ *and* $F(N) \subseteq M$;
(2) $\|Fu - Gv\| \leq \|u - v\|$ *for* $u \in M, v \in N$; *and*
(3) $\|Fu - Gv\| \leq \|u - v\|$ *for* $u \in N, v \in M$.

If $F(M)$ is mapped into a compact subset of N, then for any $u_0 \in M$ the sequence defined by

$$u_{n+1} = (1 - \delta_n)u_n + \delta_n P_M\big(G((1 - \eta_n)u_n + \eta_n P_M F u_n)\big)$$

converges to u in M_0 such that $\|u - Gu\| = \|u - Fu\| = dist(M, N)$, provided that $d(u_n, M_0) \to 0$.

Corollary 4.3. *Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G : M \cup N \to M \cup N$ satisfy*

(1) $G(M) \subseteq N, G(N) \subseteq M, F(M) \subseteq N$ *and* $F(N) \subseteq M$;
(2) $\|Fu - Gv\| \leq \|u - v\|$ *for* $u \in M, v \in N$; *and*
(3) $\|Fu - Gv\| \leq \|u - v\|$ *for* $u \in N, v \in M$.

If $F(M)$ is mapped into a compact subset of N, then for any $u_0 \in M_0$ the sequence defined by

$$u_{n+1} = P^n\big((1 - \delta_n)u_n + \delta_n P_M\big(G((1 - \eta_n)u_n + \eta_n P_M F u_n)\big)\big)$$

converges to u in M_0 such that $\|u - Gu\| = \|u - Fu\| = dist(M, N)$.

Proof. The result follows by Corollary 4.1. □

Corollary 4.4. *Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G : M \cup N \to M \cup N$ satisfy*

(1) $G(M) \subseteq N, G(N) \subseteq M, F(M) \subseteq N$ *and* $F(N) \subseteq M$;

(2) $\|Fu - Gv\| \leq \|u - v\|$ *for* $u \in M, v \in N$; *and*
(3) $\|Fu - Gv\| \leq \|u - v\|$ *for* $u \in N, v \in M$.

If $F(M)$ *is mapped into a compact subset of* N *and* $\|u_n - P_M G v_n\| \to 0$, *then for any* $u_0 \in M$ *the sequence defined by*

$$u_{n+1} = P^n\big((1-\delta_n)u_n + \delta_n P_M\big(G((1-\eta_n)u_n + \eta_n P_M F u_n)\big)\big)$$

converges to u *in* M_0 *such that* $\|u - Gu\| = \|u - Fu\| = dist(M, N)$.

Proof. The result follows by Theorem 4.1. □

Acknowledgements. The authors would like to thank the National Board for Higher Mathematics (NBHM), DAE, Govt. of India for providing a financial support under the grant no. 02011/22/2017/R&D II/14080.

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Received: October 3, 2019; Accepted: May 16, 2020.