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SURJECTIVITY RESULTS FOR NONLINEAR WEAKLY SEQUENTIALLY CONTINUOUS OPERATORS AND APPLICATIONS

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Abstract. We introduce new surjectivity results for a couple (T, S) where T and S are two weakly sequentially continuous operators and satisfy some homogeneity conditions. Also we present new operator quantities using the measure of weak noncompactness. Our results extend in a broad sense some theorems of Fučik, Nečas, Souček and Souček in the weak topology setting. In addition we present an application for generalized Hammerstein type integral equations.
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1. Introduction

Let T and S be two maps acting from a Banach space X into a Banach space Y and $\lambda \neq 0$. Surjectivity results for nonlinear operator equations of the form $\lambda T - S = f$ play an important role in proving existence theorems for nonlinear differential and integral equations (see [10, 14, 15, 17, 18, 19]).

In 1973, S. Fučik, J. Nečas, J. Souček and V. Souček [11] presented various surjectivity results for nonlinear operator equations $\lambda T(x) - S(x) = f$ with T invertible and a (K, L, a)-homeomorphism (i.e. the operator $T : X \to Y$ (not necessarily linear) is said to be a (K, L, a)-homeomorphism if

(1) T is a homeomorphism of X onto Y,
(2) there exist real numbers $K > 0$, $a > 0$, $L > 0$ such that

$$L\|x\|^a \leq \|T(x)\| \leq K\|x\|^a \text{ for each } x \in X).$$

Also in [11] the authors assumed that T and S are two odd maps, S is completely continuous and different types of homogeneity conditions were considered. Some of their results have been extended by F. Pacella [20] to bounded weakly closed operators using topological degree when X and Y are two reflexive Banach spaces and Y is separable with strictly convex dual Y'. In 1999, W. Feng and J.R.L. Webb [10] established some surjectivity results for $\lambda T - S$ under weaker conditions (without oddness conditions) using the notion of the measure of noncompactness.

In this paper we discuss surjectivity results of the maps $\lambda T - S$ when T and S are two weakly sequentially continuous maps. We remove the oddness assumption on T and S (required in [11]). The proofs of surjectivity theorems in [11] are based on the Leray-Schauder degree (for completely continuous operators); as a result we cannot proceed as in [11]. We define new quantities $[f]_a^w$ and $[f]_A^w$ $(f : X \mapsto Y)$ and obtain some surjectivity results of Fučik, Nečas, Souček and Souček type.

This paper is organized as follows. In the first section we introduce new operator quantities $[f]_a^w$ and $[f]_A^w$, where f is a map acting between X and Y and we present properties that will be needed later. In Section 3, motivated partly by [10], we provide a surjectivity theorem for $\lambda T - S$ in Banach spaces (see Theorem 3.1) where we assume hypothesis on T and S, formulated in the weak topology setting, Theorem 3.1 generalizes Theorem 1.2 and Corollary 1.1 in Chapter II of [11]. Moreover, Theorem 3.1 is the analogue of Theorem 3.1 of [10] in the weak topology setting. As a consequence of our theorem (Theorem 3.1) we obtain surjectivity results similar to those in [11] in the weak topology setting (see Theorems 3.2 and 3.4). Also, using Theorem 3.1, we establish surjectivity theorems for weakly sequentially continuous maps, and these results are similar to those obtained in [20] for bounded weakly closed maps where the assumption that Y is a separable reflexive space with strictly convex dual Y' is removed. Note that in our results we do not assume that T is a (K, L, a)-homeomorphism. Finally using Theorem 3.1 we establish an existence principle for generalized Hammerstein type integral equations.

2. Preliminaries

Throughout this paper, X and Y are two Banach spaces (over a scalar field $\mathbb{K}$). As usual $\rightharpoonup$ will denote weak convergence while $\to$ will denote norm convergence. In [2] the authors used the upper and lower measure of noncompactness defined for a continuous map G between X and Y; recall these are

$$[G]_A = \sup \left\{ \frac{\alpha(G(\Omega))}{\alpha(\Omega)} : \Omega \subseteq X \text{ bounded, } \alpha(\Omega) > 0 \right\},$$

$$[G]_a = \inf \left\{ \frac{\alpha(G(\Omega))}{\alpha(\Omega)} : \Omega \subseteq X \text{ bounded, } \alpha(\Omega) > 0 \right\},$$

where $\alpha(\Omega)$ denotes the Kuratowski measure of noncompactness of a subset Ω of X (defined as the infimum of real numbers $\epsilon > 0$ such that Ω admits a finite covering by sets of diameter less than ϵ (see [8])). The quasi-norm (see [13], page 53) is defined by,

$$[G]_Q = \limsup_{\|x\|\to\infty} \frac{\|G(x)\|}{\|x\|} = \inf_{\rho>0} \sup_{\|x\|\geq\rho} \frac{\|G(x)\|}{\|x\|}.$$

For a comprehensive list of properties of $[G]_A$, $[G]_a$ and $[G]_Q$ we refer the reader to [2].

In order to define the new quantities $[f]_A^w$ and $[f]_a^w$, we need a measure of weak noncompactness on X. Let $\mathfrak{B}$ the collection of all bounded sets of X. In addition, let $B_\varepsilon(X)$ denote the closed ball in X centered at 0_X with radius ε. The De Blasi measure of weak noncompactness (see [7]) is the map $\beta : \mathfrak{B} \to \mathbb{R}_+$ defined by

$$\beta(\Omega) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact set } D \text{ such that } \Omega \subseteq D + B_\varepsilon(X)\}.$$

Proposition 2.1. [7] *The De Blasi measure $\beta : \mathfrak{B} \to \mathbb{R}_+$ satisfies the following properties:*

1. $\beta(\Omega) = 0$ *if and only if* $\overline{\Omega^w}$ *is weakly compact (regularity),*
2. $\beta(\mu\Omega) = |\mu|\beta(\Omega)$ *for all* $\mu \in \mathbb{R}$ *and* $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$ *(seminorm),*
3. *If* $\Omega_1 \subseteq \Omega_2$*, then* $\beta(\Omega_1) \leq \beta(\Omega_1)$ *(monotonicity),*
4. $\beta(\Omega_1 \cup \Omega_2) = \max\{\beta(\Omega_1), \beta(\Omega_2)\}$ *(semi-additivity),*
5. $\beta(\Omega) = \beta(\overline{\Omega^w})$.

Note that the De Blasi measure is the first important example for the measure of weak noncompactness. For more properties on measures of weak noncompactness, we refer the reader to [1, 5, 4].

We are now ready to define $[f]_a^w$ and $[f]_A^w$. Let $f : X \to Y$ be a map (not necessary continuous). We define $[f]_A^w$ and $[f]_a^w$ as:

$$[f]_A^w = \sup\left\{\frac{\beta(f(\Omega))}{\beta(\Omega)} : \Omega \subseteq X \text{ bounded}, \beta(\Omega) > 0\right\},$$

and

$$[f]_a^w = \inf\left\{\frac{\beta(f(\Omega))}{\beta(\Omega)} : \Omega \subseteq X \text{ bounded}, \beta(\Omega) > 0\right\}.$$

The quasi-norm of f is

$$[f]_Q = \limsup_{\|x\|\to\infty} \frac{\|f(x)\|}{\|x\|}.$$

Definition 2.1. Let $f : X \to Y$ be a mapping. Then:

(a) f is called weakly sequentially continuous if $(x_n) \subset X$ with $x_n \rightharpoonup x$ implies $f(x_n) \rightharpoonup f(x)$.

(b) f is called weakly compact if for any nonempty and bounded subset Ω of X, the set $f(\Omega)$ is relatively weakly compact.

Definition 2.2. Let β be the De Blasi measure of weak noncompactness. An operator $f : X \to Y$ is said to be β-condensing if $\beta(f(\Omega)) < \beta(\Omega)$ for all bounded sets $\Omega \subseteq X$ with $\beta(\Omega) \neq 0$.

Remark 2.1. (See the proof of Theorem 1 in [3]) Consider a Banach space X and a weakly compact $D \subset X$. Then every sequentially weakly continuous map $f : D \to X$ is weakly continuous. This is an immediately consequence of the Eberlein-Šmulian's theorem.

We give some properties of $[f]_A^w$ that will be used later.

Proposition 2.2. *Let $f : X \to Y$ and $g : Y \to X$ be two maps. Then:*

(a) $[f]_a^w \leq [f]_A^w$.
(b) $[\lambda f]_A^w = |\lambda|[f]_A^w$, $\forall \lambda \in \mathbb{K}$.
(c) *If for every $\Omega \subseteq X$ bounded, $\beta(f(\Omega)) = 0 \Rightarrow \beta(\Omega) = 0$, then*

$$[g \circ f]_A^w \leq [g]_A^w [f]_A^w.$$

(d) $[f]_A^w = 0$ *if and only if f is weakly compact.*
(e) *If f is weakly sequentially continuous, f is bijective and f^{-1} is weakly sequentially continuous, then*

$$[f^{-1}]_A^w = ([f]_a^w)^{-1}.$$

Proof. (a) Is immediate.
(b) This follows from property (2) of Proposition 2.1.
(c)

$$\begin{aligned}
[g \circ f]_A^w &= \sup_{\infty > \beta(\Omega) > 0} \frac{\beta(g \circ f)(\Omega)}{\beta(\Omega)} = \sup_{\substack{\infty > \beta(\Omega) > 0 \\ \infty > \beta(f(\Omega)) > 0}} \frac{\beta(g \circ f)(\Omega)}{\beta(f(\Omega))} \frac{\beta(f(\Omega))}{\beta(\Omega)} \\
&\leq \sup_{\infty > \beta(N) > 0} \frac{\beta(g(N))}{\beta(N)} \sup_{\infty > \beta(\Omega) > 0} \frac{\beta(f(\Omega))}{\beta(\Omega)} = [g]_A^w [f]_A^w.
\end{aligned}$$

(d) If $[f]_A^w = 0$ then $\beta(f(\Omega)) = 0$ for any $\Omega \subseteq X$ bounded, and by (1) in Proposition 2.1 $f(\Omega)$ is weakly relatively compact and f is weakly compact.

If f is weakly compact then for every $\Omega \subseteq X$ bounded, $f(\Omega)$ is relatively weakly compact so $\beta(f(\Omega)) = 0$ and $[f]_A^w = 0$.
(e) Let $\Omega \subseteq X$ be bounded, f be bijective, f be weakly sequentially continuous and f^{-1} be weakly sequentially continuous. Then we claim $\beta(f(\Omega)) = 0$ if and only if $\beta(\Omega) = 0$.

In fact, if $\beta(f(\Omega)) = 0$ then $f(\Omega)$ is weakly relatively compact, and since f^{-1} is weakly sequentially continuous, then $f^{-1}(f(\Omega))$ is weakly relatively compact. Since f is bijective, $f^{-1}(f(\Omega)) = \Omega$ is weakly relatively compact and so $\beta(\Omega) = 0$. Now, if $\beta(\Omega) = 0$ then Ω is weakly relatively compact. Since f is weakly sequentially continuous from Remark 2.1 we have $f : \Omega \to Y$ is weakly continuous, so we have that $f(\Omega)$ is weakly relatively compact, that is, $\beta(f(\Omega)) = 0$. Thus our claim is true.

Also note

$$[f^{-1}]_A^w = \sup_{0 < \beta(\Omega) < \infty} \frac{\beta(f^{-1}(\Omega))}{\beta(\Omega)}.$$

Let $M = f^{-1}(\Omega)$ and then,

$$[f^{-1}]_A^w = \sup_{0<\beta(M)<\infty} \frac{\beta(M)}{\beta(f(M))} \qquad = \left(\inf_{0<\beta(M)<\infty} \frac{\beta(f(M))}{\beta(M))} \right)^{-1} = ([f]_a^w)^{-1}.$$

$\square$

Remark 2.2. If $f : X \to Y$ is weakly sequentially continuous, f is bijective and f^{-1} is weakly sequentially continuous, then f has the property : for every $\Omega \subseteq X$ bounded, $\beta(f(\Omega)) = 0 \Leftrightarrow \beta(\Omega) = 0$, (see the proof of statement (e)). In particular f satisfies the condition of statement (c) of Proposition 2.2.

Remark 2.3. If $[f]_A^w < 1$ then f is β-condensing.

Theorem 2.1. *(See [6] Theorem 3.2.) Let Ω be a non-empty, convex closed set of a Banach space X. Assume that $f : \Omega \to \Omega$ is a weakly sequentially continuous map and condensing with respect to β. In addition, suppose that $f(\Omega)$ is bounded. Then, f has a fixed point.*

Lemma 2.1. *Let $T : X \to Y$ bijective, $S : Y \mapsto X$ and $z_0 \in Y$. Define the map:*

$$F_{z_0,\lambda} : Y \to Y, \quad y \mapsto ST^{-1}\left(\frac{y}{\lambda}\right) + z_0.$$

Fix $\lambda \neq 0$. If $F_{z_0,\lambda}$ has a fixed point for all $z_0 \in Y$, then $\lambda T - S$ is onto

Proof. Let $z_0 \in Y$. If $F_{z_0}(y) = y$ then $ST^{-1}(\frac{y}{\lambda}) + z_0 = y$. We write $x = T^{-1}(\frac{y}{\lambda})$ and since T is bijective, we have $\lambda T(x) = y$, and then

$$\lambda T(x) - S(x) = z_0,$$

so $\lambda T - S$ is onto. $\square$

Definition 2.3. Let $F : X \mapsto Y$, $F_0 : X \mapsto Y$ and $a > 0$ a real number.

(a) F_0 is said to be a-homogeneous if $F_0(tx) = t^a F_0(x)$ for every $t \geq 0$ and $x \in X$.

(b) F is said to be a-quasihomogeneous relative to F_0 if F_0 is a-homogeneous and if

$$t_n \searrow 0, x_n \rightharpoonup x_0,\ t_n^a F\left(\frac{x_n}{t_n}\right) \to \phi \in Y \Rightarrow \phi = F_0(x_0).$$

(c) F is said to be a-strongly quasihomogeneous relative to F_0 if

$$t_n \searrow 0, u_n \rightharpoonup u_0 \Rightarrow t_n^a F\left(\frac{u_n}{t_n}\right) \to F_0(u_0) \in Y.$$

(d) F is said to be a-weakly quasihomogeneous relative to F_0 if

$$t_n \searrow 0, u_n \rightharpoonup u_0 \Rightarrow t_n^a F\left(\frac{u_n}{t_n}\right) \rightharpoonup F_0(u_0) \in Y.$$

Example 2.1. Let $X = Y$, $e \in X$ with $\|e\| = 1$ and $0 < \alpha < 1$. Let $T_\alpha : X \to X$ be defined by

$$T_\alpha(x) = \|x\|^\alpha e.$$

Then, T_α is compact and $[T_\alpha]_Q = \limsup_{\|x\|\to\infty} \frac{\|T_\alpha(x)\|}{\|x\|} = \limsup_{\|x\|\to\infty} \frac{\|x\|^\alpha e}{\|x\|} = 0$ since $0 < \alpha < 1$.
Assume that there exist $(u_n)_{n\in\mathbb{N}} \subseteq X$ with $u_n \rightharpoonup u_0$, $t_n \searrow 0$ such that

$$t_n T_\alpha(\frac{u_n}{t_n}) > \varepsilon > 0.$$

Then $\left(\left\|\frac{u_n}{t_n}\right\|\right)_{n\in\mathbb{N}}$ is unbounded. If $\left\|\frac{u_{n_k}}{t_{n_k}}\right\| \to \infty$, $(n_k \to \infty)$, then we have

$$\left\|t_{n_k} T_\alpha\left(\frac{u_{n_k}}{t_{n_k}}\right)\right\| = \frac{\left\|T_\alpha\left(\frac{u_{n_k}}{t_{n_k}}\right)\right\|}{\left\|\frac{u_{n_k}}{t_{n_k}}\right\|}\|u_{n_k}\| \to 0,$$

a contradiction.
Thus T_α is a 1-strongly quasihomogeneous operator relative to $T_{\alpha_0} = 0$.

Example 2.2. The identity I in any Banach space X is 1-weakly quasihomogeneous relative to I. In fact if $t_n \searrow 0$, $u_n \rightharpoonup u_0$ then $t_n I(\frac{u_n}{t_n}) = u_n \rightharpoonup u_0 = I(u_0)$.

Definition 2.4. Let Ω be a domain in $\mathbb{R}^n$ and X, Y be separable Banach spaces. A function $f : \Omega \times X \to Y$ is said to be a Carathéodory function if (i). for each fixed $x \in X$ the map $t \mapsto f(t,x)$ is Lebesgue measurable in Ω, and (ii). for a.e. $t \in \Omega$ the map $f(t,.) : X \to Y$ is continuous.

Theorem 2.2. *(See* [16] *Theorem* 4.*) Let Ω be a bounded domain in $\mathbb{R}^n$ and X, Y be separable Banach spaces. Let $f : \Omega\times X \to Y$ be a Carathéodory function and consider the Nemytskii operator N_f, generated by f, and defined by*

$$N_f(y)(t) := f(t, y(t)) \text{ for } y \in L^1(\Omega, X)$$

and suppose there exist $a \in L^1(\Omega)$ and $b \geq 0$ with $\|f(t,x)\| \leq a(t) + b\|x\|$ for $t \in \Omega$ and $x \in X$. If Y is reflexive then

$$N_f : L^1(\Omega, X) \to L^1(\Omega, Y)$$

is weakly sequentially continuous.

3. Surjectivity

Theorem 3.1. *Let $\lambda \in \mathbb{K}$ and $T : X \to Y$ be a weakly sequentially continuous map satisfying the following conditions:*

(1) *T is bijective and T^{-1} is weakly sequentially continuous.*
(2) *There exist real numbers $L > 0$, $a > 0$ and $b > 0$ such that*

$$\|T(x)\| \geq L\|x\|^a - b \text{ for every } x \in X.$$

(3) *$T$is bounded (i.e. maps bounded sets into bounded sets).*

Let $S : X \to Y$ be a bounded and weakly sequentially continuous map satisfying the following condition :

$$\alpha := \limsup_{x\in X, \|x\|\to\infty} \frac{\|S(x)\|}{\|x\|^a} < \infty.$$

Then, for $|\lambda| > \max\left\{\frac{\alpha}{L}, \frac{[S]^w_A}{[T]^w_a}\right\}$, $\lambda T - S : X \to Y$ is surjective.

Proof. To prove that $\lambda T - S$ is onto, it suffices to prove that $F_{z_0,\lambda}$ defined in Lemma 2.1 has a fixed point for all $z_0 \in Y$. Let $z_0 \in Y$ and $M \subseteq Y$ be a bounded set with $\beta(M) \neq 0$. Using Proposition 2.2(c) and Remark 2.2, one obtains

$$\begin{aligned}\beta(F_{z_0,\lambda}(M)) &= \beta\left(ST^{-1}(\frac{M}{\lambda}) + \{z_0\}\right)\\ &\le [ST^{-1}]_A^w \beta\left(\frac{M}{\lambda}\right)\\ &\le \frac{1}{|\lambda|}[S]_A^w[T^{-1}]_A^w\beta(M) = \frac{1}{|\lambda|}\frac{[S]_A^w}{[T]_a^w}\beta(M) < \beta(M).\end{aligned}$$

Therefore, F_{z_0} is β-condensing.

Since T and S are weakly sequentially continous then $F_{z_0,\lambda}$ is weakly sequentially continuous. Now hypothesis (2) on T implies that there exist $a > 0$, $L > 0$ and $b > 0$ such that

$$\|T^{-1}(y)\| \le \left(\frac{\|y\| + b}{L}\right)^{\frac{1}{a}} \quad \forall\, y \in Y.$$

Then $T^{-1}(B_R(Y))$ is bounded. Also since S is bounded, then $ST^{-1}(\frac{B_R(Y)}{\lambda})$ is bounded and there exists a $c > 0$ such that

$$\|F_{z_0,\lambda}(y)\| \le c \qquad \forall y \in B_R(Y). \tag{3.1}$$

Also, we have, $[F_{z_0,\lambda}]_Q < 1$. To see this first note

$$\begin{aligned}\frac{\|F_{z_0,\lambda}(y)\|}{\|y\|} &= \frac{\|ST^{-1}(\frac{y}{\lambda}) + z_0\|}{\|y\|}\\ &\le \frac{\|ST^{-1}(\frac{y}{\lambda})\|}{\|y\|} + \frac{\|z_0\|}{\|y\|}.\end{aligned}$$

If we write $x = T^{-1}(\frac{y}{\lambda})$, then $T(x) = \frac{y}{\lambda}$, and we have,

$$\begin{aligned}[F_{z_0,\lambda}]_Q &= \limsup_{\|T(x)\|\to\infty} \frac{\|S(x)\|}{|\lambda|\|T(x)\|}\\ &= \limsup_{x\in X, \|x\|\to\infty} \frac{\|S(x)\|}{|\lambda|\|T(x)\|}\\ &\le \limsup_{x\in X, \|x\|\to\infty} \frac{\|S(x)\|}{|\lambda|(L\|x\|^a - b)}\\ &= \frac{\alpha}{|\lambda| L} < 1.\end{aligned}$$

Now since $[F_{z_0,\lambda}]_Q < 1$, there exist $q \in][F_{z_0,\lambda}]_Q, 1[$ and $R > 0$ such that

$$\|x\| \ge R \Rightarrow \|[F_{z_0,\lambda}(x)\| \le q\|x\|. \tag{3.2}$$

From (3.1) and (3.2), we have

$$\|F_{z_0,\lambda}(y)\| \le q\|y\| + c \quad \forall\, y \in Y. \tag{3.3}$$

Let $R' > 0$ and $y \in B_{R'}(Y)$. From (3.3), we have $\|F_{z_0,\lambda}(y)\| \leq qR' + c$. Choosing

$$R' > \frac{c}{1-q}$$

then $F_{z_0,\lambda} : B_{R'}(Y) \to B_{R'}(Y)$ (note $qR' + c < R'$) is weakly sequentially continuous and β- condensing. Now Theorem 2.1 guarantees that $F_{z_0,\lambda}$ has a fixed point $y \in Y$ and from Lemma 2.1, $\lambda T - S$ is a mapping from X onto Y. □

Remark 3.1. Note our theorem is the analogue of Theorem 3.1 of [10] in the weak topology setting.

Definition 3.1. A mapping $f : X \to Y$ is said to be regularly surjective from X onto Y if $f(X) = Y$ and for any $R > 0$, there exists a $r > 0$ such that $\|x\|_X \leq r$ for all $x \in X$ with $\|f(x)\|_Y \leq R$.

Example 3.1. Let $T : X \to X$ be defined by

$$T(x) = \|x\|x.$$

Then, T is regularly surjective. Note immediately that T is surjective. If we suppose that there exist a $M > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ with

$$\|x_n\| \to \infty \text{ and } \|T(x_n)\| \leq M \text{ for all } n \in \mathbb{N},$$

then this implies that $\|x_n\| \to \infty$ and $\|x_n\|^2 \leq M$ for all $n \in \mathbb{N}$. This is a contradiction. Thus T is regularly surjective.

Lemma 3.1. *Let X and Y be two Banach spaces and let T satisfy condition* (2) *of Theorem* 3.1. *Suppose $S : X \to Y$ is a mapping such that*

$$\alpha := \limsup_{x\in X, \|x\|\to\infty} \frac{\|S(x)\|}{\|x\|^a} < \infty.$$

Then for $|\lambda| > \frac{\alpha}{L}$ we have

$$\limsup_{x\in X, \|x\|\to\infty} \|\lambda T(x) - S(x)\| = \infty. \tag{3.4}$$

Proof. If (3.4) is false, then there exist a constant $M > 0$ and a sequence $(x_n) \subseteq X$, $\|x_n\| \to \infty$ such that

$$\|\lambda T(x_n) - S(x_n)\| \leq M,$$

for any positive integer n. Then

$$\frac{\lambda T(x_n)}{\|x_n\|^a} - \frac{S(x_n)}{\|x_n\|^a} \to 0,$$

and we have

$$\limsup_{n\to\infty} \frac{|\lambda|\|T(x)\|}{\|x_n\|^a} = \alpha.$$

Now since

$$\|T(x_n)\| \geq L\|x_n\|^a - b,$$

we have

$$|\lambda|L \leq \limsup_{n\to\infty}\left(\frac{|\lambda|\|T(x_n)\|}{\|x_n\|^a}+\frac{|\lambda|b}{\|x_n\|^a}\right)=\alpha.$$

This is a contradiction since $|\lambda| > \frac{\alpha}{L}$. □

Theorem 3.2. *Suppose that X is a reflexive Banach space, Y is a Banach space and T, S satisfy the conditions in Theorem*3.1. *Then for $|\lambda| > \frac{\alpha}{L}$, $\lambda T - S$ is regularly surjective.*

Proof. We have T^{-1} is bounded. In fact, hypothesis (2) on T in Theorem 3.1 implies that there exist $a > 0$, $L > 0$ and $b > 0$ such that

$$\|T^{-1}(y)\| \leq \left(\frac{\|y\|+b}{L}\right)^{\frac{1}{a}} \quad \forall\, y \in Y.$$

Since X is a reflexive Banach space then T^{-1} is weakly compact. To see this note for any bounded set $M \subseteq Y$, we have $T^{-1}(M)$ is bounded and $T^{-1}(M) \subseteq X$, so $T^{-1}(M)$ is relatively weakly compact and we have $[T^{-1}]_A^w = 0$. Now the argument in Theorem 3.1 guarantees that $\lambda T - S$ is surjective for all $|\lambda| > \alpha/L$. In addition, this surjectivity is regular. If not there exists a $M > 0$ and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$, $\|x_n\| \to \infty$ such that

$$\|\lambda T(x_n) - S(x_n)\| \leq M.$$

However from Lemma 3.1, we have $\limsup\limits_{x\in X,\|x\|\to\infty} \|\lambda T(x) - S(x)\| = \infty$. This is a contradiction. □

Theorem 3.3. *Let X be reflexive and let T satisfy the conditions of Theorem* 3.1. *If S is a weakly sequentially continuous, bounded and a-weakly quasihomogeneous mapping relative to S_0 then*

$$\alpha := \limsup_{x\in X,\|x\|\to\infty} \frac{\|S(x)\|}{\|x\|^a} < \infty,$$

and for $|\lambda| > \frac{\alpha}{L}$, $\lambda T - S$ is regularly surjective.

Proof. The second part of the assertion follows from the previous theorem (if we prove the first part). It just remains for us to prove

$$\limsup_{x\in X,\|x\|\to\infty} \frac{\|S(x)\|}{\|x\|^a} < \infty.$$

Suppose the contrary. Then there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\|x_n\| \to \infty$ and $\frac{\|S(x_n)\|}{\|x_n\|^a} > n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $w_n = \frac{x_n}{\|x_n\|}$. Then $(w_n)_{n\in\mathbb{N}}$ is bounded and since X is reflexive, it follows that $(w_n)_{n\in\mathbb{N}}$ admits a subsequence weakly convergent to a $w_0 \in X$. Without loss of generality, we suppose that $w_n \rightharpoonup w_0$. Let $t_n = \frac{1}{\|x_n\|}$, $n \in \mathbb{N}$. Since S is a-weakly quasihomogeneous relative to S_0, it follows that

$$\frac{S(x_n)}{\|x_n\|^a} = \frac{S(\|x_n\|\frac{x_n}{\|x_n\|})}{\|x_n\|^a} = t_n^a S\left(\frac{w_n}{t_n}\right) \rightharpoonup S_0(w_0)$$

so, the sequence $\left(\frac{S(x_n)}{\|x_n\|^a}\right)_{n\in\mathbb{N}}$ is bounded, which is a contradiction. □

Theorem 3.4. *Let X be a reflexive Banach space and let T, a, b and L be as in Theorem 3.1. Let $S : X \to Y$ be a weakly sequentially continuous, bounded and b-strongly quasihomogeneous map relative to S_0. Suppose that $a > b$. Then for $\lambda \neq 0$, $\lambda T - S$ is regularly surjective.*

Proof. Since S is weakly sequentially continuous and $\lambda \neq 0$, then from Theorem 3.3, it suffices to prove that

$$\lim_{x\in X, \|x\|\to\infty} \frac{\|S(x)\|}{\|x\|^a} = 0.$$

If not, there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $\|x_n\| \to \infty$ and a $\varepsilon > 0$ such that $\frac{\|S(x_n)\|}{\|x_n\|^a} > \varepsilon$, for all sufficiently large n. For each $n \in \mathbb{N}$, let $w_n = \frac{x_n}{\|x_n\|}$ and $t_n = \frac{1}{\|x_n\|}$. Suppose as in the previous theorem that $w_n \rightharpoonup w_0 \in X$. Since S is b-strongly quasihomogeneous relative to S_0 then

$$\frac{S(x_n)}{\|x_n\|^b} = \frac{S(\|x_n\| w_n)}{\|x_n\|^b} \to S_0(w_0).$$

Now since $a > b$, we have $\frac{\|x_n\|^b}{\|x_n\|^a} \to 0$ and we obtain,

$$\frac{\|S(x_n)\|}{\|x_n\|^a} = \frac{\|x_n\|^b}{\|x_n\|^a} \frac{\|S(x_n)\|}{\|x_n\|^b} \to 0,$$

a contradiction. □

Remark 3.2. Theorem 4.1, page 63 in [11] guarantees that $\lambda T - S$ is surjective under the assumptions that T is an odd (K, L, a)-homeomorphism and S is an odd completely continuous b-strongly quasihomogeneous operator with respect to S_0 and $a > b$. In the previous theorem, we impose other conditions (in the weak topology) on T and S and we prove not only that $\lambda T - S$ is surjective but also $\lambda T - S$ is regularly surjective.

Next we recall Theorem 1.1, page 56 in [11].

Theorem 3.5. *Let X, Y be Banach spaces and let $T : X \to Y$ be an odd (K, L, a)-homeomorphism and $S : X \to Y$ be an odd completely continuous map. Then for each $\lambda \neq 0$ if*

$$\lim_{\|x\|\to\infty} \|\lambda T(x) - S(x)\| = \infty, \tag{3.5}$$

then $\lambda T - S$ maps X onto Y.

If we replace the conditions on T by the conditions in Theorem 3.1, and S is a bounded weakly sequentially continuous map such that the map $\lambda T - S$ satisfies condition (3.5), then the map $\lambda T - S$ might not be onto.

Example 3.2. Let X be a separable Hilbert space. Denote by $\{e_n\}_{n\in\mathbb{N}}$ an orthonormal basis in X, and define $L : X \to X$ by

$$L(x) = \sum_i \alpha_i e_{i+1} \text{ for all } x \in X \text{ where } x = \sum_i \alpha_i e_i \text{ and } \alpha_i = (x, e_i).$$

It is easy to see that L is a linear bounded weakly continuous map such that

$$\|x\| = \|L(x)\| \text{ for all } x \in X,$$

but L is not surjective. In fact, for $y \in X$ with $(y, e_1) \neq 0$ we have $y \notin L(X)$ because $L(X) = \{z \in X \quad \text{such that} \quad (z, e_1) = 0\}$. Take $T = I_X$ the identity map. Then T satisfies the conditions in Theorem 3.1. In fact, I_X is a bijective bounded weakly sequentially continuous and we can choose $a = 1$. Let $S = I_X - L$ and $\lambda = 1$. Now S is a linear bounded weakly continuous (so weakly sequentially continuous) operator and

$$\begin{aligned} \lim_{\|x\|\to\infty} \|\lambda I_X(x) - S(x)\| &= \lim_{\|x\|\to\infty} \|x - (x - L(x))\| \\ &= \lim_{\|x\|\to\infty} \|L(x)\| \\ &= \lim_{\|x\|\to\infty} \|x\| = \infty. \end{aligned}$$

Thus $T = I_X$ satisfies the conditions in Theorem 3.1 and S is bounded weakly sequentially continuous map but $\lambda I_X - S$ is not surjective.

In [20] the author showed if $\lambda \neq 0$ and $\lambda T - S$ is regularly surjective then λ is not eigenvalue for the couple (T, S) when T, S are a-homogeneous weakly closed mapping and X,Y are two reflexive Banach spaces with Y separable and Y' is strictly convex. In our next theorem we obtain a similar result to Theorem 2.4 in [20] when T and S are weakly sequentially continuous and the condition that Y is a separable reflexive space with Y' strictly convex is removed.

Theorem 3.6. *Let X be a reflexive Banach space and Y a Banach space. Suppose $T : X \to Y$ satisfies the conditions in Theorem* 3.1 *and T is a-homogeneous. Assume $S : X \to Y$ is a a-homogeneous bounded weakly sequentially continuous map. Then we have*

(1) $\alpha := \limsup\limits_{\|x\|\to\infty} \frac{\|S(x)\|}{\|x\|^a} < \infty$ *and $\lambda T - S$ is regularly surjective if $\lambda > \frac{\alpha}{L}$.*

(2) *If $\lambda \neq 0$ and $\lambda T - S$ is regularly surjective then λ is not an eigenvalue for the couple (T, S).*

Recall that λ is said to be an eigenvalue of the couple T, S if there is a $x_0 \neq 0$ such that $\lambda T(x_0) - S(x_0) = 0$.

Proof. Suppose the $\limsup$ in (1) is not finite. Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\|x_n\| \to \infty$ and $\frac{\|S(x_n)\|}{\|x_n\|^a} > n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $w_n = \frac{x_n}{\|x_n\|}$. Then $(w_n)_{n\in\mathbb{N}}$ is bounded and since X is reflexive then $(w_n)_{n\in\mathbb{N}}$ admits a subsequence weakly convergent to w_0. Without loss of generality, we suppose that $w_n \rightharpoonup w_0 \in X$. Since S is weakly sequentially continuous, we have $S(w_n) \rightharpoonup S(w_0)$ and then $(S(w_n))_{n\in\mathbb{N}}$ is bounded. On the other hand, since S is a-homogeneous, then

$$\|S(w_n)\| = \left\| S\left(\frac{x_n}{\|x_n\|}\right)\right\| = \frac{\|S(x_n)\|}{\|x_n\|^a} > n,$$

a contradiction since $(S(w_n))_{n\in\mathbb{N}}$ is bounded. Now Theorem 3.2 guarantees that $\lambda T - S$ is regularly surjective if $\lambda > \frac{\alpha}{L}$.

To prove (2), we suppose that λ is an eigenvalue for (T,S). Then there exists $x_0 \in X$, $x_0 \neq 0$ with $\lambda T(x_0) - S(x_0) = 0$. For each $n \in \mathbb{N}$, let $y_n = \frac{1}{n}$ and $w_n = \frac{x_0}{y_n} = nx_0$. Then $\lim\limits_{n\to\infty} \|w_n\| = \infty$ and since T and S are both a-homogeneous then

$$\lambda T(w_n) - S(w_n) = \frac{\lambda T(x_0) - S(x_0)}{y_n^a} = 0 \text{ for all } n \in \mathbb{N},$$

so $\|\lambda T(w_n) - S(w_n)\| < R$ for each $R > 0$ and all $n \in \mathbb{N}$, and this contradicts the assumption that $\lambda T - S$ is regularly surjective. □

4. Application

In this section we study a generalized Hammerstein type integral equation. Let X be a separable Banach space and Y be a separable reflexive Banach space, D be a bounded subset of $\mathbb{R}^n$, $\eta \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $E = L^1(D,X)$.
Let $G : E \to E$ and $H : E \to E$ be defined by $H(y) = \int_D k(t,s)f(s,y(s))ds$.

Consider the nonlinear operator $F : E \to E$ given by

$$F(y) \equiv G(y) + H(y) \equiv G(y) + \eta \int_D k(t,s)f(s,y(s))ds.$$

We are concerned with the solvability of the following generalized Hammerstein type integral equation:

$$\lambda y = F(y) \equiv G(y) + H(y) \equiv G(y) + \eta \int_D k(t,s)f(s,y(s))ds, \quad (|\lambda| \geq 1) \tag{4.1}$$

in $E = L^1(D,X)$. Suppose that G, f, and k satisfy the following conditions:

(1) $G : E \to E$ is a weakly sequentially continuous, weakly compact map,
(2) $f : D \times X \to Y$ is a Carathéodory's function,
(3) There are $a \in L^1(D)$ and $b \geq 0$ such that
$$\|f(t,x)\| \leq a(t) + b\|x\|, t \in D, x \in X,$$
(4) $k : D \times D \to L(Y,X)$ (the space of bounded linear operators from Y into X) is strongly measurable and the linear operator K, defined by
$$K(z)(t) = \int_D k(t,s)z(s)ds,$$
maps $L^1(D,Y)$ into $L^1(D,X)$ continuously,
(5) The function $s \to k(t,s)$ is in $L^\infty(D, L(Y,X))$ for almost all $t \in D$;
(6) Set $\gamma = \limsup\limits_{\|x\|\to\infty} \frac{\|G(x)\|}{\|x\|}$ and suppose $|\eta|b\|K\| < 1$ (here $\|K\|$ denotes the operator norm of K) and $\gamma < 1$.

Theorem 4.1. *Assume that conditions* (1) – (6) *are satisfied. Then (4.1) has a solution in* $E = L^1(D,X)$.

Proof. First, we prove that H is a weakly sequentially continuous, β-condensing operator. From assumptions (2) and (3) and Theorem 2.2 we see that the Nemytskii operator, generated by f and defined by $N_f(y)(t) := f(t,y(t)), y \in L^1(D,X)$ is weakly sequentially continuous from $L^1(D,X)$ into $L^1(D,Y)$ and takes bounded sets into bounded sets. From assumption (4), K is a linear operator and it is weakly continuous so weakly sequentially continuous. Thus the operator $H = \eta K N_f$ is weakly sequentially continuous. Using assumptions (2), (3), and (6) and arguing as in [9], we have immediately that the operator H is β-condensing since $|\eta|b\|K\| < 1$. Let $y \in E$. Then we have,

$$\begin{aligned}
\|H(y)\| &= \|\eta \int_D k(t,s)f(s,y(s))ds\| \\
&\leq |\eta| \int_D \| \int_D k(t,s)f(s,y(s))ds\|dt \\
&\leq |\eta| \int_D \int_D \|k(t,s)f(s,y(s))\|dsdt \\
&\leq |\eta|\|K\| \int_D \|f(s,y(s))\|dsdt \\
&\leq |\eta|\|K\| \int_D \|a(s)\| + b\|y(s)\|ds \\
&\leq |\eta|\|K\|(\|a\| + b),
\end{aligned}$$

so

$$\begin{aligned}
\limsup_{\|y\|\to\infty} \frac{\|F(y)\|}{\|y\|} &\leq \limsup_{\|y\|\to\infty} \frac{\|G(y)\|}{\|y\|} + \limsup_{\|y\|\to\infty} \frac{\|L(y)\|}{\|y\|} \\
&\leq \gamma + \limsup_{\|y\|\to\infty} \frac{\||\eta|\|K\|(\|a\| + b)\|}{\|y\|} = \gamma < 1.
\end{aligned}$$

Take $T = I$, $S = F$, $L = 1$, $a = 1$ and $\alpha = \gamma$ in Theorem 3.1. Note G is a weakly compact, weakly sequentially continuous operator and hence F is a weakly sequentially continuous and β-condensing operator, that is, $[F]_A^w < 1$, and therefore

$$\max\{\alpha, [F]_A^w\} < 1.$$

Now Theorem 3.1 guarantees that $\lambda I - F$ is surjective for $|\lambda| > \max\{\alpha, [F]_A^w\}$, so (4.1) has at least a solution in E (note above we choose $|\lambda| \geq 1$ so $|\lambda| > \max\{\alpha, [F]_A^w\}$). □

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