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# A NEW PARALLEL ALGORITHM TO SOLVING A SYSTEM OF QUASI-VARIATIONAL INCLUSION PROBLEMS AND COMMON FIXED POINT PROBLEMS IN BANACH SPACES

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**Abstract.** In this paper, a new parallel algorithm for finding a common solution of a system of quasi-variational inclusion problems and a common fixed point of a finite family of nonexpansive mappings in a q-uniformly Banach space is introduced and analyzed. A strong convergence theorem of the proposed algorithm is established under some control conditions. As a consequence, we apply our main results to solve convex minimization problems, multiple sets variational inequality problems and multiple sets equilibrium problems. Some numerical experiments of image restoration problems are also given for supporting the main results.
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## 1. Introduction

Let $C$ be a nonempty, closed and convex subset of a real Banach space $X$. Let $A : C \to X$ be a single-valued nonlinear mapping and let $B : X \to 2^X$ be a multi-valued mapping. The *quasi-variational inclusion problem* is the problem of finding a point $x \in X$ such that

$$0 \in Ax + Bx. \tag{1.1}$$

The set of all solutions of the problem (1.1) is denoted by $(A + B)^{-1}0$. This problem includes several important problems, as special cases, such as, optimization problems, variational inequality problems, split feasibility problems and equilibrium problems. Moreover, the problem has wide applications in the fields of economics, mechanics, structural analysis, signal processing and image restoration (see, e.g., [18, 10, 14, 7, 19], and the references therein).

A classical method to solving the problem (1.1) is the forward-backward splitting method in a Hilbert space $H$ [20, 25, 33, 12] which was first introduced by Combettes and Hirstoaga [11] in the following manner: $x_1 \in H$ and

$$x_{n+1} = (I + \lambda B)^{-1}(x_n - \lambda A x_n), \ \forall n \geq 1, \tag{1.2}$$

where $\lambda > 0$. They defined $J_\lambda^B = (I + \lambda B)^{-1}$ which is called the resolvent of $B$ for $\lambda$. We see that the iteration (1.2) involves with $I - \lambda A$ as the forward step and $J_\lambda^B$ as the backward step, but not the sum of $A$ and $B$. We also note that the forward-backward splitting method includes, as special cases, the proximal point algorithm [22, 27, 5, 15, 8] and the gradient method [13, 3]. In 2008, Zhang et al. [38] introduced an iterative scheme for finding elements in the set $F(S) \cap (A + B)^{-1}0$ as follows: $u, x_1 \in H$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S J_\lambda^B (x_n - \lambda A x_n), \ \forall n \geq 1, \tag{1.3}$$

where $S : H \to H$ is a nonexpansive mapping. They proved a strong convergence theorem of the sequence $\{x_n\}$ under some suitable conditions of parameter $\{\alpha_n\}$ and $\lambda$. On the other hand, Takahashi et al. [30] considered the problem of finding elements in the set $F(S) \cap (A+B)^{-1}0$ in a Hilbert space as well. They introduced the following iterative method: $x_1 = x \in C \subset H$ and let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S(\alpha_n x + (1 - \alpha_n) J_{\lambda_n}^B (x_n - \lambda_n A x_n)), \ \forall n \geq 1. \tag{1.4}$$

They also proved a strong convergence of $\{x_n\}$ under some mild conditions. In 2012, Lopez et al. [21] proved a strong convergence theorem for finding a solution of $(A + B)^{-1}0$ in a Banach space $X$ by using the following Halpern-type forward-backward method: $u, x_1 \in X$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J_{\lambda_n}^B (x_n - \lambda_n (A x_n + a_n)) + b_n), \ \forall n \geq 1. \tag{1.5}$$

Recently, Suantai et al. [28] proposed the explicit iteration to solving the fixed point problem of nonexpansive mappings and the quasi-variational inclusion problem in the framework of Banach spaces. They introduced the following iterative method: $x_1 \in C$ arbitrary,

$$\begin{cases} y_n = J_{\lambda_n}^B (\alpha_n u_n + (1 - \alpha_n) x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S y_n, \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{u_n\} \subset X$ is a perturbation for the $n$-step with $\lim_{n\to\infty} u_n = u' \in X$. They proved that the iterative sequence $\{x_n\}$ converges strongly to $x^* \in F(S) \cap (A+B)^{-1}0$ under some appropriate conditions.

Motivated and inspired by all of these researches going on in this direction, in this paper, we introduce a new algorithm for finding a solution of the quasi-variational inclusion problems for a common zero point of a sum of a finite family of $\alpha$-strongly monotone operators and maximal monotone operators and a common fixed point of a family of nonexpansive mappings in $q$-uniformly smooth Banach spaces. Our focus in this work is the following: let $C$ be a nonempty, closed and convex subset of a $q$-uniformly smooth Banach space $X$. Let $A_k : C \to X, \ k = 1, 2, ..., N$, be $\alpha_k$-inverse strongly accretive of order $q$ operators and let $B_k : D(B_k) \to 2^X, \ k = 1, 2, ..., N$, be

$m$-accretive operators. Let $S_i : C \to C,\ i = 1, 2, ..., M$, be nonexpansive mappings. Our proposed problem is to find a common element in the set

$$\Omega := \left(\bigcap_{i=1}^{M} F(S_i)\right) \cap \left(\bigcap_{k=1}^{N} (A_k + B_k)^{-1} 0\right)$$

under some suitable conditions. Then a strong convergence theorem will be established under some control and suitable conditions. Moreover, we apply the main results to solving convexly constrained minimization problems, variational inequality problems and equilibrium problems. We also perform some numerical experiments to solving the image restoration problems by utilizing our algorithms. The presented results in this work also extend and improve many well-known results in the literature.

## 2. Preliminaries

The intention of this section is to recall some useful definitions and results for proving our main results. We shall denote the notations $x_n \to x$ that the sequence $\{x_n\}$ converges strongly to $x$ and $x_n \rightharpoonup x$ that a sequence $\{x_n\}$ converges weakly to $x$. Let $X$ be a real Banach space with norm $\|\cdot\|$ and let $C$ be a nonempty, closed and convex subset of $X$. Let $S : C \to C$ be a mapping. A point $x \in C$ is called *fixed point* of $S$ if $x = Sx$. We use $F(S)$ to denote the set of all fixed points of $S$, i.e.,

$$F(S) := \{x \in C : x = Sx\}.$$

A mapping $S : C \to C$ is said to be *L-Lipschitzian*, if there exists a constant $L > 0$ such that

$$\|Sx - Sy\| \le L\|x - y\|,\ \forall x, y \in C.$$

If $L = 1$, then $S$ is said to be *nonexpansive*.

A Banach space $X$ is said to be *strictly convex* if for

$$x, y \in S(X) := \{x \in X : \|x\| = 1\}$$

and $x \neq y$, one has $\frac{\|x+y\|}{2} < 1$.

A Banach space $X$ is said to be *uniformly convex* if for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ (depending only on $\varepsilon$) such that $\frac{\|x+y\|}{2} \le 1 - \delta$, for all $x, y \in S(X)$ with $\|x - y\| \ge \varepsilon$. It is well-known that every uniformly convex Banach space is reflexive strictly convex and every Hilbert space $H$ is a uniformly convex space, (see [31] for more details). The modulus of smoothness of $X$ is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(\tau) := \sup\left\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 : x, y \in S(X)\right\}.$$

A Banach space $X$ is said to be *uniformly smooth* if $\frac{\rho_X(t)}{t} \to 0$ as $t \to 0$. Suppose that $q$ is a fixed real number with $q > 1$, a Banach space $X$ is said to be *q-uniformly smooth* (or to have a modulus of smoothness of power type $q$) if there exists a fixed constant $c > 0$ such that $\rho_X(t) \le ct^q$ for all $t > 0$. If $X$ is $q$-uniformly smooth, then $X$ is also uniformly smooth. Examples of $p$-uniformly smooth Banach spaces are $l_p, L_p, (p \ge 2)$, the Sobolev spaces $W_m^p, (p \ge 2)$ and all Hilbert spaces. In fact, every Hilbert space is 2-uniformly smooth.

Let $X^*$ be a dual space of a Banach space $X$. Let $q > 1$ be a real number. We denote by $J_q$ the *generalized duality mapping* $J_q : X \to 2^{X^*}$ defined by

$$J_q(x) := \{j_q(x) \in X^* : \langle x, j_q(x)\rangle = \|x\|^q, \|j_q(x)\| = \|x\|^{q-1}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between element of $X$ and $X^*$. In particular, $J_2$ is called the *normalized duality mapping* and it is known that $J_q(x) = \|x\|^{q-2}J_2(x)$ for $x \neq 0$ and if $X$ is a real Hilbert space, then $J_q = I$, where $I$ is the identity mapping and if a Banach space $X$ is uniformly smooth, then $J_q$ is a single-valued mapping and it will be denoted by $j_q$.

The generalized duality mapping $j_q$ is said to be *weakly sequentially continuous* if for each $\{x_n\}$ in $X$ with $x_n \rightharpoonup x$, we have $j_q(x_n) \overset{*}{\rightharpoonup} j_q(x)$.

**Lemma 2.1.** ([34]) *Let $q > 1$, $\lambda \in [0, 1]$ and $W_q(\lambda) := \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$. Let $X$ be a real $q$-uniformly smooth Banach space. Then there exists a constant $c_q > 0$ such that the following inequality holds:*

$$\|\lambda x + (1-\lambda)y\|^q \leq \lambda\|x\|^q + (1-\lambda)\|y\|^q - W_q(\lambda)c_q\|x-y\|^q, \ \forall x, y \in X.$$

**Lemma 2.2.** ([9]) *Let $q > 1$ and $X$ be a real normed space with the generalized duality mapping $J_q$. Then for any $x, y \in X$, we have*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle \text{ where } j_q(x+y) \in J_q(x+y).$$

**Lemma 2.3.** ([34]) *Let $q > 1$ be a fixed real number and $X$ a Banach space. Then $X$ is $q$-uniformly smooth if and only if there is a constant $\kappa_q > 0$ which is called the $q$-uniform smoothness coefficient of $X$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + \kappa_q\|y\|^q, \ \forall x, y \in X.$$

In particular, if $X = H$, then $\kappa_q = 1$.

**Lemma 2.4.** ([34]) *Let $q > 1$ and $r > 1$ be two fixed real numbers and $X$ a Banach space. Then $X$ is uniformly smooth if and only if there is a strictly increasing, continuous and convex function $g : (0, +\infty) \to (0, +\infty)$ such that $g(0) = 0$ and*

$$g(\|x-y\|) \leq \|x\|^q - q\langle x, j_q(y)\rangle + (q-1)\|y\|^q, \ \forall x, y \in B_r.$$

Let $A : X \to 2^X$ be a set-valued mapping. The domain and range of $A$ are denoted by $D(A) := \{x \in X : Ax \neq \emptyset\}$ and $R(A) := \bigcup\{Az : z \in D(A)\}$, respectively. The inverse of $A$ is denoted by $A^{-1}$ is defined as follows: $x \in A^{-1}y$ if and only if $y \in Ax$.

An operator $A : X \to 2^X$ is said to be *monotone* if

$$\langle u - v, x - y\rangle \geq 0, \ \forall u \in Ax, \forall v \in Ay.$$

A monotone operator $A$ on $X$ is said to be *maximal* if its graph

$$G(A) := \{(x, u) : u \in Ax\}$$

is not properly contained in the graph of any other monotone operator on $X$. That is, a monotone operator $A$ is maximal if and only if for $x \in D(A)$ and $u \in Ax$ such that $\langle u - v, x - y\rangle \geq 0$ implies $(y, v) \in G(A)$. A fundamental example of a maximally monotone operator is the subdifferential of a proper lower semicontinuous convex function.

For a proper lower semicontinuous convex function $f : X \to (-\infty, \infty]$, the *subdifferential mapping* $\partial f \in X \times X^*$ of $f$ defined by

$$\partial f(x) := \{x^* \in X^* : f(x) + \langle y - x, x^* \rangle \leq f(y), y \in X\},$$

for all $x \in X$, is a maximal monotone mapping (see [26] for more details).

Let $q > 1$. A set-valued mapping $A : X \to 2^X$ is said to be *accretive* of order $q$ if for each $x, y \in D(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y) \rangle \geq 0, \ \forall u \in Ax, \forall v \in Ay.$$

An accretive operator $A$ is said to be *m-accretive* if $R(I + \lambda A) = X$ for all $\lambda > 0$. In a real Hilbert space, an operator $A$ is $m$-acccretive if and only if $A$ is maximal monotone (see [31]).

Let $A$ be an $m$-accretive operator on $X$. We use $A^{-1}0$ to denote the set of all zeros of $A$, i.e., $A^{-1}0 := \{x \in D(A) : 0 \in Ax\}$. For an accretive operator $A$, we can define a single-valued operator $J_\lambda^A : R(I + \lambda A) \to D(A)$ by

$$J_\lambda^A = (I + \lambda A)^{-1}, \text{ for each } \lambda > 0,$$

which is call the *resolvent* of $A$ for $\lambda$. It is well-known that $J_\lambda^A$ is a nonexpansive mapping with $F(J_\lambda^A) = A^{-1}0$.

Let $f \in \Gamma_0(H)$, the set of proper lower semicontinuous convex functions from $H$ to $(-\infty, +\infty]$, and a parameter $\lambda > 0$. The operator $prox_{\lambda f} : H \to H$, say the proximity operator of parameter $\lambda$ of $f$ at $x \in H$, is defined by

$$prox_{\lambda f} x := \arg\min_{y \in H} \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}.$$

The well-known facts, if $f \in \Gamma_0(H)$, then $J_\lambda^{\partial f} = prox_{\lambda f}$ and if $f = \| \cdot \|_1$, then

$$prox_{\lambda \| \cdot \|_1} x := sgn(x) \max\{\|x\|_1 - \lambda, 0\},$$

where $\|x\|_1$ is $l_1$-norm, the sum of the absolute values of each components of $x$.

Let $\alpha > 0$ be a given constant and let $C$ be a subset of a real Hilbert space $H$.

A mapping $A : C \to H$ is said to be *$\alpha$-inverse strongly monotone* ($\alpha$-ism) if

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \ \forall x, y \in C.$$

Let $\alpha > 0$ and $q > 1$. A mapping $A : C \to X$ is said to be *$\alpha$-inverse strongly accretive* ($\alpha$-isa) of order $q$ if there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Ax - Ay, j_q(x - y) \rangle \geq \alpha \|Ax - Ay\|^q, \ \forall x, y \in C.$$

It is easy to see that if $A$ is $\alpha$-inverse strongly accretive of order $q$, then $A$ is $\frac{1}{\alpha}$-Lipschitz continuous. Moreover, we have from [21], if $A$ is an $\alpha$-inverse strongly accretive of order $q$ operator and $B$ is an $m$-accretive operator, then $F(J_\lambda^B(I - \lambda A)) = (A + B)^{-1}0$.

The following lemmas and propositions are useful for proving our main results.

**Lemma 2.5.** ([21]) *Let $C$ be a subset of a real $q$-uniformly smooth Banach space $X$ and $A : C \to X$ $\alpha$-inverse strongly accretive of order $q$. Then the following inequality holds:*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q - \lambda(\alpha q - \kappa_q \lambda^{q-1}) \|Ax - Ay\|^q, \ \forall x, y \in C.$$

*In particular, if* $0<\lambda\leq(\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$*, then* $I-\lambda A$ *is nonexpansive.*

**Lemma 2.6.** ([2]) *Let* $X$ *be a real Banach space. Let* $A$ *be an* $m$*-accretive operator. For any* $\lambda,\mu>0$*, then*

$$\|J_\lambda^A x-J_\mu^A x\|\leq\left|\frac{\lambda-\mu}{\lambda}\right|\|J_\lambda^A x-x\|,\ \forall x\in X.$$

**Proposition 2.7.** ([28]) *Let* $X$ *be a real* $q$*-uniformly smooth Banach space. Let* $A$ *be an* $m$*-accretive operator on* $X$ *and let* $J_\lambda^A$ *be the resolvent operator associated with* $A$ *and* $\lambda$*. Then* $J_\lambda^A$ *is firmly nonexpansive, i.e.,*

$$\|J_\lambda^A x-J_\lambda^A y\|^q\leq\langle x-y,j_q(J_\lambda^A x-J_\lambda^A y)\rangle,\ \forall x,y\in X.$$

**Lemma 2.8.** ([6]) *Let* $C$ *be a nonempty, closed and convex subset of a uniformly convex Banach space* $X$ *and let* $S:C\to C$ *be a nonexpansive mapping. Then* $I-S$ *is demiclosed at zero, i.e., for any sequence* $\{x_n\}$ *in* $C$ *such that* $x_n\rightharpoonup x\in C$ *and* $x_n-Sx_n\to 0$ *imply* $x=Sx$*.*

Let $C$ be a nonempty, closed convex subset of a Banach space $X$ and let $D$ be a nonempty subset of $C$. A mapping $Q:C\to D$ is said to be *sunny* if $Q(x+t(x-Qx))=Qx$ whenever $Qx+t(x-Qx)\in C$ for all $x\in C$ and $t\geq 0$. A *retraction* from $C$ to $D$ is a mapping $Q:C\to D$ such that $Qx=x$ for all $x\in R(Q)$.

A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. It is well-known that if $X=H$ is a Hilbert space, then a sunny nonexpansive retraction $Q_C$ is coincident with the metric projection from $H$ onto $C$, that is $Q_C=P_C$.

**Lemma 2.9.** *Let* $C$ *be a nonempty, closed convex subset of a smooth Banach space* $X$*. Let* $Q_C:X\to C$ *be a sunny nonexpansive retraction from* $X$ *onto* $C$*. Let* $x\in X$ *and let* $x_0\in C$*. Then* $x_0=Q_Cx$ *if and only if*

$$\langle x-x_0,j_q(y-x_0)\rangle\leq 0,\ \forall y\in C.$$

**Proposition 2.10.** ([23]) *Let* $q>1$*. Then the following inequality holds:*

$$a^q-b^q\leq qa^{q-1}(a-b),$$

*for arbitrary positive real numbers* $a,b$*.*

**Lemma 2.11.** ([29]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space and* $\{\beta_n\}$ *a sequence in [0,1] such that* $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$*. Suppose that*

$$x_{n+1}=(1-\beta_n)y_n+\beta_n x_n,$$

*for all* $n\in\mathbb{N}$ *and* $\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leq 0$*. Then*

$$\lim_{n\to\infty}\|x_n-y_n\|=0.$$

**Lemma 2.12.** ([35]) *Suppose that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1}\leq(1-\eta_n)a_n+\eta_n\delta_n,$$

*for all* $n\in\mathbb{N}$*, where* $\{\eta_n\}\subset(0,1)$ *such that* $\sum_{n=1}^\infty\eta_n=\infty$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that* $\limsup_{n\to\infty}\delta_n\leq 0$*. Then* $\lim_{n\to\infty}a_n=0$*.*

## 3. Main results

The main aim of this section is to introduce a parallel algorithm for finding a common element of solutions of quasi-variational inclusion problems and a common fixed point of a finite family of nonexpansive mappings in $q$-uniformly smooth Banach spaces. A strong convergence result of the proposed method is analyzed under some suitable conditions.

Let $C$ be a nonempty, closed and convex subset of a real uniformly convex and $q$-uniformly smooth Banach space $X$ which admits a weakly sequentially continuous generalized duality mapping $j_q$. Let $A_k : C \to X,\ k = 1, 2, ..., N$, be $\alpha_k$-inverse strongly accretive of order $q$ operators and let $B_k : D(B_k) \to 2^X,\ k = 1, 2, ..., N$, be $m$-accretive operators such that $D(B_k) \subset C$. Let $S_i : C \to C,\ i = 1, 2, ..., M$, be nonexpansive mappings. Let $\{\lambda_n\}$ be a positive real sequence and let $\{\sigma_n\}$,$\{\beta_n\}$,$\{\gamma_{i,n}\}$ for all $i = 1, 2, ..., M$ be sequences in (0,1). For $x_1 \in C$, we introduce the following parallel algorithm:

$$\begin{cases} y_{k,n} = J^{B_k}_{\lambda_n}(\sigma_n u_n + (1-\sigma_n)x_n - \lambda_n A_k x_n),\ k = 1, 2, ..., N, \\ \qquad \text{choose } k_n \text{ such that } \|y_{k_n,n} - x_n\| = \max_{k=1,2,...,N} \|y_{k,n} - x_n\|, \\ z_{i,n} = (1-\gamma_{i,n})S_i x_n + \gamma_{i,n} y_{k_n,n},\ i = 1, 2, ..., M, \\ \qquad \text{choose } i_n \text{ such that } \|z_{i_n,n} - x_n\| = \max_{i=1,2,...,M} \|z_{i,n} - x_n\|, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) z_{i_n,n}, \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{u_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} u_n = u' \in X$.

We next start by some lemmas.

**Lemma 3.1.** *Suppose that*

$$\Omega := \left(\bigcap_{i=1}^{M} F(S_i)\right) \cap \left(\bigcap_{k=1}^{N} (A_k + B_k)^{-1}0\right) \neq \emptyset.$$

*If* $0 < e \leq \lambda_n < \frac{\lambda_n}{1-\sigma_n} \leq f < (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$*, for some* $e, f \in \mathbb{R}^+$*, the set of all positive real numbers, and* $\alpha = \min\{\alpha_1, \alpha_2, ..., \alpha_N\}$*, then the sequence* $\{x_n\}$ *generated by (3.1) is bounded.*

*Proof.* Let $p \in \Omega$. Then we have $p = S_i p$, for all $i = 1, 2, ..., M$, and

$$p = J^{B_k}_{\lambda_n}(p - \lambda_n A_k p) = J^{B_k}_{\lambda_n}\left(\sigma_n p + (1-\sigma_n)\left(p - \frac{\lambda_n}{1-\sigma_n} A_k p\right)\right),$$

for all $k = 1, 2, ..., N$. Since $\{u_n\}$ is bounded, there exists a positive constant $M_1$ such that

$$M_1 = \sup_{n\geq 1} \|u_n - p\|.$$

Since $S_{i_n}, J_{\lambda_n}^{B_{k_n}}$ and $I-\frac{\lambda_n}{1-\sigma_n}A_{k_n}$ are nonexpansive (by Lemma 2.5), they follow that

$$\begin{aligned}
\|y_{k_n,n}-p\| &= \Big\|J_{\lambda_n}^{B_{k_n}}\left(\sigma_n u_n+(1-\sigma_n)\left(I-\frac{\lambda_n}{1-\sigma_n}A_{k_n}\right)x_n\right)\\
&\quad -J_{\lambda_n}^{B_{k_n}}\left(\sigma_n p+(1-\sigma_n)\left(I-\frac{\lambda_n}{1-\sigma_n}A_{k_n}\right)p\right)\Big\|\\
&\le \left\|\sigma_n(u_n-p)+(1-\sigma_n)\left[\left(I-\frac{\lambda_n}{1-\sigma_n}A_{k_n}\right)x_n-\left(I-\frac{\lambda_n}{1-\sigma_n}A_{k_n}\right)p\right]\right\|\\
&\le \sigma_n\|u_n-p\|+(1-\sigma_n)\left\|\left(I-\frac{\lambda_n}{1-\sigma_n}A_{k_n}\right)x_n-\left(I-\frac{\lambda_n}{1-\sigma_n}A_{k_n}\right)p\right\|\\
&\le \sigma_n\|u_n-p\|+(1-\sigma_n)\|x_n-p\|. \qquad (3.2)
\end{aligned}$$

This together with nonexpansitivity of $S_{i_n}$, we have

$$\begin{aligned}
\|z_{i_n,n}-p\| &= \|(1-\gamma_{i_n,n})S_{i_n}x_n+\gamma_{i_n,n}y_{k_n,n}-p\|\\
&\le (1-\gamma_{i_n,n})\|S_{i_n}x_n-p\|+\gamma_{i_n,n}\|y_{k_n,n}-p\|\\
&\le (1-\gamma_{i_n,n})\|S_{i_n}x_n-p\|+\gamma_{i_n,n}[\sigma_n\|u_n-p\|+(1-\sigma_n)\|x_n-p\|]\\
&\le (1-\gamma_{i_n,n})\|x_n-p\|+\gamma_{i_n,n}[\sigma_n\|u_n-p\|+(1-\sigma_n)\|x_n-p\|]\\
&= (1-\gamma_{i_n,n}\sigma_n)\|x_n-p\|+\gamma_{i_n,n}\sigma_n\|u_n-p\|. \qquad (3.3)
\end{aligned}$$

By (3.2) and (3.3), we get

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\beta_n x_n+(1-\beta_n)z_{i_n,n}-p\|\\
&\le \beta_n\|x_n-p\|+(1-\beta_n)\|z_{i_n,n}-p\|\\
&\le \beta_n\|x_n-p\|+(1-\beta_n)[(1-\gamma_{i_n,n}\sigma_n)\|x_n-p\|+\gamma_{i_n,n}\sigma_n\|u_n-p\|]\\
&= (1-(1-\beta_n)\gamma_{i_n,n}\sigma_n)\|x_n-p\|+(1-\beta_n)\gamma_{i_n,n}\sigma_n\|u_n-p\|\\
&\le \max\{\|x_n-p\|,M_1\}.
\end{aligned}$$

By mathematical induction, we obtain

$$\|x_{n+1}-p\|\le\max\{\|x_1-p\|,M_1\},$$

for all $n\ge 1$. Hence, $\{x_n\}$ is bounded. It follows that $\{y_{k_n,n}\}$, $\{z_{i_n,n}\}$, $\{A_{k_n}x_n\}$ and $\{S_{i_n}x_n\}$ are also bounded. □

**Lemma 3.2.** *Let $\{x_n\}$ be generated by (3.1) and $\alpha=\min\{\alpha_1,\alpha_2,...,\alpha_N\}$. If the control sequences satisfy the following conditions:*

*(i)* $\lim_{n\to\infty}\sigma_n=0$ *and* $\sum_{n=1}^{\infty}\sigma_n=\infty$*;*
*(ii)* $0<a_i\le\gamma_{i,n}\le b_i<1,\ i=1,2,...,M$ *and* $\lim_{n\to\infty}|\gamma_{i_n,n+1}-\gamma_{i_n,n}|=0$*;*
*(iii)* $0<c\le\beta_n\le d<1$*;*
*(iv)* $0<e\le\lambda_n<\frac{\lambda_n}{1-\sigma_n}\le f<(\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$ *and* $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n|=0$*,*

*for some $a_i,b_i,c,d,e,f\in\mathbb{R}^+$, then* $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

*Proof.* For $n\in\mathbb{N}$, we set

$$w_{k_n,n}:=\sigma_n u_n+(1-\sigma_n)x_n-\lambda_n A_{k_n}x_n.$$

By using Lemma 2.6, Lemma 3.1 and nonexpansitivity of the resolvent operator

$J^{B_{k_n}}_{\lambda_{n+1}}$, $I - \frac{\lambda_{n+1}}{1-\sigma_{n+1}} A_{k_n}$ and $I - \frac{\lambda_n}{1-\sigma_n} A_{k_n}$, we have

$$\begin{aligned}
&\|y_{k_n,n+1} - y_{k_n,n}\| \\
&= \|J^{B_{k_n}}_{\lambda_{n+1}} w_{k_n,n+1} - J^{B_{k_n}}_{\lambda_n} w_{k_n,n}\| \\
&\le \|J^{B_{k_n}}_{\lambda_{n+1}} w_{k_n,n+1} - J^{B_{k_n}}_{\lambda_{n+1}} w_{k_n,n}\| + \|J^{B_{k_n}}_{\lambda_{n+1}} w_{k_n,n} - J^{B_{k_n}}_{\lambda_n} w_{k_n,n}\| \\
&\le \|w_{k_n,n+1} - w_{k_n,n}\| + \|J^{B_{k_n}}_{\lambda_{n+1}} w_{k_n,n} - J^{B_{k_n}}_{\lambda_n} w_{k_n,n}\| \\
&= \|(\sigma_{n+1} u_{n+1} + (1-\sigma_{n+1}) x_{n+1} - \lambda_{n+1} A_{k_n} x_{n+1}) \\
&\quad - (\sigma_n u_n + (1-\sigma_n) x_n - \lambda_n A_{k_n} x_n)\| + \|J^{B_{k_n}}_{\lambda_{n+1}} w_{k_n,n} - J^{B_{k_n}}_{\lambda_n} w_{k_n,n}\| \\
&= \Bigg\| \sigma_{n+1}(u_{n+1} - u_n) + (\sigma_{n+1} - \sigma_n)(u_n - x_n) \\
&\quad + (1-\sigma_{n+1}) \left[ \left( I - \frac{\lambda_{n+1}}{1-\sigma_{n+1}} A_{k_n} \right) x_{n+1} - \left( I - \frac{\lambda_n}{1-\sigma_n} A_{k_n} \right) x_n \right] \\
&\quad + (\lambda_n - \lambda_{n+1}) A_{k_n} x_n \Bigg\| + \|J^{B_{k_n}}_{\lambda_{n+1}} w_{k_n,n} - J^{B_{k_n}}_{\lambda_n} w_{k_n,n}\| \\
&\le \sigma_{n+1}(\|u_{n+1}\| + \|u_n\|) + |\sigma_{n+1} - \sigma_n|(\|u_n\| + \|x_n\|) + (1-\sigma_{n+1})\|x_{n+1} - x_n\| \\
&\quad + |\lambda_{n+1} - \lambda_n| \|A_{k_n} x_n\| + \|J^{B_{k_n}}_{\lambda_{n+1}} w_{k_n,n} - J^{B_{k_n}}_{\lambda_n} w_{k_n,n}\| \\
&\le (1-\sigma_{n+1})\|x_{n+1} - x_n\| + \sigma_{n+1}(\|u_{n+1}\| + \|u_n\|) + |\sigma_{n+1} - \sigma_n|(\|u_n\| + \|x_n\|) \\
&\quad + |\lambda_{n+1} - \lambda_n| \|A_{k_n} x_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \|J^{B_{k_n}}_{\lambda_{n+1}} w_{k_n,n} - w_{k_n,n}\| \\
&\le (1-\sigma_{n+1})\|x_{n+1} - x_n\| + \left( \sigma_{n+1} + |\sigma_{n+1} - \sigma_n| + |\lambda_{n+1} - \lambda_n| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \right) M_2,
\end{aligned} \tag{3.4}$$

where $M_2 = \sup_{n \ge 1} \{\|u_{n+1}\| + \|u_n\|, \|u_n\| + \|x_n\|, \|A_{k_n} x_n\|, \|J^{B_{k_n}}_{\lambda_{n+1}} w_{k_n,n} - w_{k_n,n}\|\}$. By (3.4), we get

$$\begin{aligned}
\|z_{i_n,n+1} - z_{i_n,n}\| &= \|((1-\gamma_{i_n,n+1}) S_{i_n} x_{n+1} + \gamma_{i_n,n+1} y_{k_n,n+1}) \\
&\quad - ((1-\gamma_{i_n,n}) S_{i_n} x_n + \gamma_{i_n,n} y_{k_n,n})\| \\
&= \|\gamma_{i_n,n+1}(y_{k_n,n+1} - y_{k_n,n}) + (\gamma_{i_n,n+1} - \gamma_{i_n,n})(y_{k_n,n} - S_{i_n} x_n) \\
&\quad + (1-\gamma_{i_n,n+1})(S_{i_n} x_{n+1} - S_{i_n} x_n)\| \\
&\le \gamma_{i_n,n+1} \|y_{k_n,n+1} - y_{k_n,n}\| + |\gamma_{i_n,n+1} - \gamma_{i_n,n}| \|y_{k_n,n} - S_{i_n} x_n\| \\
&\quad + (1-\gamma_{i_n,n+1}) \|S_{i_n} x_{n+1} - S_{i_n} x_n\| \\
&\le (1-\gamma_{i_n,n+1}) \|x_{n+1} - x_n\| + |\gamma_{i_n,n+1} - \gamma_{i_n,n}| \|y_{k_n,n} - S_{i_n} x_n\| \\
&\quad + \gamma_{i_n,n+1} \Bigg[ (1-\sigma_{n+1}) \|x_{n+1} - x_n\| \\
&\quad + \left( \sigma_{n+1} + |\sigma_{n+1} - \sigma_n| + |\lambda_{n+1} - \lambda_n| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \right) M_2 \Bigg].
\end{aligned}$$

It follows from conditions of $\{\sigma_n\}$, $\{\gamma_{i_n,n}\}$ and $\{\lambda_n\}$ that

$$\limsup_{n\to\infty}(\|z_{i_n,n+1} - z_{i_n,n}\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.5}$$

This implies, by Lemma 2.11, that

$$\lim_{n\to\infty} \|x_n - z_{i_n,n}\| = 0. \tag{3.6}$$

Since $x_{n+1} - x_n = (1 - \beta_n)(z_{i_n,n} - x_n)$, we obtain $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. □

**Lemma 3.3.** *Let $\{x_n\}$ be generated by (3.1) and $\alpha = \min\{\alpha_1, \alpha_2, ..., \alpha_N\}$.*

*If the control sequences satisfy the same conditions as in Lemma 3.2, then $\lim_{n\to\infty} \|x_n - S_i x_n\| = 0$, for all $i = 1, 2, ..., M$.*

*Proof.* In order to show this, we will first show that

$$\lim_{n\to\infty} \|A_{k_n}x_n - A_{k_n}p\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - y_{k_n,n}\| = 0.$$

Using (3.2) together with Lemma 2.1 and Lemma 2.3, we get

$$\begin{aligned}
&\|y_{k_n,n} - p\|^q \\
&\leq \left\|\sigma_n(u_n - p) + (1 - \sigma_n)\left[\left(x_n - \frac{\lambda_n}{1-\sigma_n}A_{k_n}x_n\right) - \left(p - \frac{\lambda_n}{1-\sigma_n}A_{k_n}p\right)\right]\right\|^q \\
&\leq \sigma_n\|u_n - p\|^q + (1 - \sigma_n)\left\|\left(x_n - \frac{\lambda_n}{1-\sigma_n}A_{k_n}x_n\right) - \left(p - \frac{\lambda_n}{1-\sigma_n}A_{k_n}p\right)\right\|^q \\
&= \sigma_n\|u_n - p\|^q + (1 - \sigma_n)\left\|(x_n - p) - \frac{\lambda_n}{1-\sigma_n}(A_{k_n}x_n - A_{k_n}p)\right\|^q \\
&\leq \sigma_n\|u_n - p\|^q + (1 - \sigma_n)\left[\|x_n - p\|^q - \frac{q\lambda_n}{1-\sigma_n}\langle A_{k_n}x_n - A_{k_n}p, j_q(x_n - p)\rangle\right. \\
&\qquad \left. + \frac{\kappa_q\lambda_n^q}{(1-\sigma_n)^q}\|A_{k_n}x_n - A_{k_n}p\|^q\right] \\
&\leq \sigma_n\|u_n - p\|^q + (1 - \sigma_n)\left[\|x_n - p\|^q - \frac{\alpha q\lambda_n}{1-\sigma_n}\|A_{k_n}x_n - A_{k_n}p\|^q\right. \\
&\qquad \left. + \frac{\kappa_q\lambda_n^q}{(1-\sigma_n)^q}\|A_{k_n}x_n - A_{k_n}p\|^q\right] \\
&= \sigma_n\|u_n - p\|^q + (1 - \sigma_n)\|x_n - p\|^q - \lambda_n\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{(1-\sigma_n)^{q-1}}\right)\|A_{k_n}x_n - A_{k_n}p\|^q \\
&\leq \sigma_n\|u_n - p\|^q + \|x_n - p\|^q - \lambda_n\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{(1-\sigma_n)^{q-1}}\right)\|A_{k_n}x_n - A_{k_n}p\|^q.
\end{aligned} \tag{3.7}$$

Again, by Lemma 2.1 together with (3.7), we obtain

$$\begin{aligned}
&\|z_{i_n,n} - p\|^q \\
&= \|(1-\gamma_{i_n,n})(S_{i_n}x_n - p) + \gamma_{i_n,n}(y_{k_n,n} - p)\|^q \\
&\le (1-\gamma_{i_n,n})\|S_{i_n}x_n - p\|^q + \gamma_{i_n,n}\|y_{k_n,n} - p\|^q \\
&\le (1-\gamma_{i_n,n})\|x_n - p\|^q + \gamma_{i_n,n} \\
&\quad \times \left[\sigma_n\|u_n - p\|^q + \|x_n - p\|^q - \lambda_n\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{(1-\sigma_n)^{q-1}}\right)\|A_{k_n}x_n - A_{k_n}p\|^q\right] \\
&= \|x_n - p\|^q + \gamma_{i_n,n}\sigma_n\|u_n - p\|^q - \gamma_{i_n,n}\lambda_n\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{(1-\sigma_n)^{q-1}}\right)\|A_{k_n}x_n - A_{k_n}p\|^q.
\end{aligned}$$

By Lemma 2.1 and above inequality, we get

$$\begin{aligned}
\|x_{n+1} - p\|^q &= \|\beta_n(x_n - p) + (1-\beta_n)(z_{i_n,n} - p)\|^q \\
&\le \beta_n\|x_n - p\|^q + (1-\beta_n)\|z_{i_n,n} - p\|^q \\
&\le \beta_n\|x_n - p\|^q + (1-\beta_n)\bigg[\|x_n - p\|^q + \gamma_{i_n,n}\sigma_n\|u_n - p\|^q \\
&\qquad - \gamma_{i_n,n}\lambda_n\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{(1-\sigma_n)^{q-1}}\right)\|A_{k_n}x_n - A_{k_n}p\|^q\bigg] \\
&= \|x_n - p\|^q + (1-\beta_n)\gamma_{i_n,n}\sigma_n\|u_n - p\|^q \\
&\qquad - (1-\beta_n)\gamma_{i_n,n}\lambda_n\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{(1-\sigma_n)^{q-1}}\right)\|A_{k_n}x_n - A_{k_n}p\|^q,
\end{aligned}$$

which implies by Proposition 2.10 that

$$\begin{aligned}
&a_{i_n}e(1-d)(\alpha q - \kappa_q(f)^{q-1})\|A_{k_n}x_n - A_{k_n}p\|^q \\
&\qquad \le \|x_n - p\|^q - \|x_{n+1} - p\|^q + (1-\beta_n)\gamma_{i_n,n}\sigma_n\|u_n - p\|^q \\
&\qquad \le q\|x_n - q\|^{q-1}(\|x_n - p\| - \|x_{n+1} - p\|) + (1-\beta_n)\gamma_{i_n,n}\sigma_n\|u_n - p\|^q \\
&\qquad \le q\|x_n - q\|^{q-1}\|x_{n+1} - x_n\| + (1-\beta_n)\gamma_{i_n,n}\sigma_n\|u_n - p\|^q.
\end{aligned}$$

It follows from Lemma 3.1, Lemma 3.2 and $\sigma_n \to 0$ that

$$\lim_{n\to\infty}\|A_{k_n}x_n - A_{k_n}p\| = 0. \tag{3.8}$$

We next show that $\lim_{n\to\infty}\|x_n - y_{k_n,n}\| = 0$. By Proposition 2.7 and Lemma 2.4, we have

$$\begin{aligned}
&\|y_{k_n,n} - p\|^q \\
&= \|J_{\lambda_n}^{B_{k_n}}(\sigma_n u_n + (1-\sigma_n)x_n - \lambda_n A_{k_n}x_n) - J_{\lambda_n}^{B_{k_n}}(p - \lambda_n A_{k_n}p)\|^q \\
&\le \langle \sigma_n u_n + (1-\sigma_n)x_n - \lambda_n A_{k_n}x_n - (p - \lambda_n A_{k_n}p), j_q(y_{k_n,n} - p)\rangle \\
&\le \frac{1}{q}\bigg[\|\sigma_n u_n + (1-\sigma_n)x_n - \lambda_n A_{k_n}x_n - (p - \lambda_n A_{k_n}p)\|^q \\
&\quad + (q-1)\|y_{k_n,n} - p\|^q - g(\|\sigma_n u_n + (1-\sigma_n)x_n - \lambda_n(A_{k_n}x_n - A_{k_n}p) - y_{k_n,n}\|)\bigg],
\end{aligned}$$

which implies, by Lemma 2.1, that

$$\begin{aligned}
\|y_{k_n,n}-p\|^q &\le \|\sigma_n u_n+(1-\sigma_n)x_n-\lambda_n A_{k_n}x_n-(p-\lambda_n A_{k_n}p)\|^q \\
&\quad - g(\|\sigma_n u_n+(1-\sigma_n)x_n-\lambda_n(A_{k_n}x_n-A_{k_n}p)-y_{k_n,n}\|) \\
&= \left\|\sigma_n(u_n-p)+(1-\sigma_n)\left[\left(I-\frac{\lambda_n}{1-\sigma_n}A_{k_n}\right)x_n-\left(I-\frac{\lambda_n}{1-\sigma_n}A_{k_n}\right)p\right]\right\|^q \\
&\quad - g(\|\sigma_n u_n+(1-\sigma_n)x_n-\lambda_n(A_{k_n}x_n-A_{k_n}p)-y_{k_n,n}\|) \\
&\le \sigma_n\|u_n-p\|^q+(1-\sigma_n)\|x_n-p\|^q \\
&\quad - g(\|\sigma_n u_n+(1-\sigma_n)x_n-\lambda_n(A_{k_n}x_n-A_{k_n}p)-y_{k_n,n}\|) \\
&\le \sigma_n\|u_n-p\|^q+\|x_n-p\|^q \\
&\quad - g(\|\sigma_n u_n+(1-\sigma_n)x_n-\lambda_n(A_{k_n}x_n-A_{k_n}p)-y_{k_n,n}\|).
\end{aligned}$$

Using again Lemma 2.1 and above inequality, we get

$$\begin{aligned}
\|z_{i_n,n}-p\|^q &\le (1-\gamma_{i_n,n})\|S_{i_n}x_n-p\|^q+\gamma_{i_n,n}\|y_{k_n,n}-p\|^q \\
&\le (1-\gamma_{i_n,n})\|x_n-p\|^q+\gamma_{i_n,n}\Big[\sigma_n\|u_n-p\|^q+\|x_n-p\|^q \\
&\quad - g(\|\sigma_n u_n+(1-\sigma_n)x_n-\lambda_n(A_{k_n}x_n-A_{k_n}p)-y_{k_n,n}\|)\Big] \\
&= \|x_n-p\|^q+\gamma_{i_n,n}\sigma_n\|u_n-p\|^q \\
&\quad - \gamma_{i_n,n}g(\|\sigma_n u_n+(1-\sigma_n)x_n-\lambda_n(A_{k_n}x_n-A_{k_n}p)-y_{k_n,n}\|).
\end{aligned}$$

This together with Lemma 2.1 yield

$$\begin{aligned}
\|x_{n+1}-p\|^q &\le \beta_n\|x_n-p\|^q+(1-\beta_n)\|z_{i_n,n}-p\|^q \\
&\le \beta_n\|x_n-p\|^q+(1-\beta_n)\Big[\|x_n-p\|^q+\gamma_{i_n,n}\sigma_n\|u_n-p\|^q \\
&\quad - \gamma_{i_n,n}g(\|\sigma_n u_n+(1-\sigma_n)x_n-\lambda_n(A_{k_n}x_n-A_{k_n}p)-y_{k_n,n}\|)\Big] \\
&= \|x_n-p\|^q+(1-\beta_n)\gamma_{i_n,n}\sigma_n\|u_n-p\|^q \\
&\quad - (1-\beta_n)\gamma_{i_n,n}g(\|\sigma_n u_n+(1-\sigma_n)x_n-\lambda_n(A_{k_n}x_n-A_{k_n}p)-y_{k_n,n}\|),
\end{aligned}$$

which implies, by Proposition 2.10, that

$$\begin{aligned}
&a_{i_n}(1-d)g(\|\sigma_n u_n+(1-\sigma_n)x_n-\lambda_n(A_{k_n}x_n-A_{k_n}p)-y_{k_n,n}\|) \\
&\quad\le \|x_n-p\|^q-\|x_{n+1}-p\|^q+(1-\beta_n)\gamma_{i_n,n}\sigma_n\|u_n-p\|^q \\
&\quad\le q\|x_n-p\|^{q-1}(\|x_n-p\|-\|x_{n+1}-p\|)+(1-\beta_n)\gamma_{i_n,n}\sigma_n\|u_n-p\|^q \\
&\quad\le q\|x_n-p\|^{q-1}(\|x_{n+1}-x_n\|)+(1-\beta_n)\gamma_{i_n,n}\sigma_n\|u_n-p\|^q.
\end{aligned}$$

Then by Lemma 3.1, Lemma 3.2 and $\sigma_n \to 0$, we get

$$\lim_{n\to\infty} g(\|\sigma_n u_n+(1-\sigma_n)x_n-\lambda_n(A_{k_n}x_n-A_{k_n}p)-y_{k_n,n}\|)=0.$$

By continuity of $g$ and $g(0) = 0$, it follows from (3.8) that

$$\lim_{n\to\infty} \|x_n - y_{k_n,n}\| = 0. \tag{3.9}$$

From $z_{i_n,n} - x_n = (1-\gamma_{i_n,n})(S_{i_n}x_n - x_n) + \gamma_{i_n,n}(y_{k_n,n} - x_n)$, we obtain by (3.6) and (3.9) that

$$\|x_n - S_{i_n}x_n\| \le \frac{1}{1-\gamma_{i_n,n}}(\|x_n - z_{i_n,n}\| + \gamma_{i_n,n}\|x_n - y_{k_n,n}\|) \to 0 \text{ as } n \to \infty.$$

For each $i = 1, 2, ..., M$, we have

$$\begin{aligned}(1-\gamma_{i,n})\|x_n - S_ix_n\| &\le \|x_n - z_{i,n}\| + \gamma_{i,n}\|x_n - y_{k_n,n}\| \\ &\le \|x_n - z_{i_n,n}\| + \gamma_{i,n}\|x_n - y_{k_n,n}\|.\end{aligned}$$

Hence, we obtain that $\lim_{n\to\infty} \|x_n - S_ix_n\| = 0$, for all $i = 1, 2, ..., M$. □

Now, the strong convergence of algorithm (3.1) is given by the following theorem.

**Theorem 3.4.** *Let $C$ be a nonempty, closed and convex subset of a real uniformly convex and $q$-uniformly smooth Banach space $X$ which admits a weakly sequentially continuous generalized duality mapping $j_q$. Let $A_k : C \to X$, $k = 1, 2, ..., N$, be $\alpha_k$-inverse strongly accretive of order $q$ operators and let $B_k : D(B_k) \to 2^X$, $k = 1, 2, ..., N$, be $m$-accretive operators such that $D(B_k) \subset C$. Let $S_i : C \to C$, $i = 1, 2, ..., M$, be nonexpansive mappings such that $\Omega := \left(\bigcap_{i=1}^M F(S_i)\right) \cap \left(\bigcap_{k=1}^N (A_k + B_k)^{-1}0\right) \neq \emptyset$. Let $\{\lambda_n\}$ be a positive real sequence and let $\{\sigma_n\}$,$\{\beta_n\}$,$\{\gamma_{i,n}\}$ for all $i = 1, 2, ..., M$ be sequences in (0,1). Suppose that the control sequences satisfy the same conditions as in Lemma 3.2. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to a point $x^* = Q_\Omega u'$ where $Q_\Omega$ is a sunny nonexpansive retraction from $C$ onto $\Omega$.*

*Proof.* First, we prove that

$$\limsup_{n\to\infty}\langle u' - x^*, j_q(y_{k_n,n} - x^*)\rangle \le 0,$$

where $x^* = Q_\Omega u'$. Choose a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that

$$\begin{aligned}\limsup_{n\to\infty}\langle u' - x^*, j_q(y_{k_n,n} - x^*)\rangle &= \limsup_{n\to\infty}\langle u' - x^*, j_q(x_n - x^*)\rangle \\ &= \lim_{m\to\infty}\langle u' - x^*, j_q(x_{n_m} - x^*)\rangle.\end{aligned}$$

By the reflexivity of $X$ and the boundedness of $\{x_{n_m}\}$, there exists a subsequence $\{x_{n_{m_i}}\}$ of $\{x_{n_m}\}$ such that $x_{n_{m_i}} \rightharpoonup w$ for some $w \in C$. Without loss of generality, we can assume that $x_{n_m} \rightharpoonup w$. Since $\|x_n - S_ix_n\| \to 0$, for all $i = 1, 2, ..., M$, by Lemma 2.8, we get that $w \in \bigcap_{i=1}^M F(S_i)$. From $\|x_n - y_{k_n,n}\| \to 0$ as $n \to \infty$, it is easy to see that $\|x_n - y_{k,n}\| \to 0$ as $n \to \infty$, for all $k = 1, 2, ..., N$. Let $v_k \in B_ku_k$, for each $k = 1, 2, .., N$. Since $y_{k,n} = J_{\lambda_n}^{B_k}(\sigma_nu_n + (1-\sigma_n)x_n - \lambda_nA_kx_n)$, we have

$$\sigma_nu_n + (1-\sigma_n)x_n - \lambda_nA_kx_n \in (I + \lambda_nB_k)y_{k,n},$$

so that,

$$\frac{1}{\lambda_n}(\sigma_nu_n + (1-\sigma_n)x_n - \lambda_nA_kx_n - y_{k,n}) \in B_ky_{k,n}.$$

Since $B_k$ is $m-$accretive, we have, for all $v_k \in B_k u_k$,

$$\left\langle \frac{1}{\lambda_n}(\sigma_n u_n + (1-\sigma_n)x_n - \lambda_n A_k x_n - y_{k,n}) - v_k, j_q(y_{k,n} - u_k) \right\rangle \geq 0.$$

Thus, $\langle \sigma_n u_n + (1-\sigma_n)x_n - \lambda_n A_k x_n - y_{k,n} - \lambda_n v_k, j_q(y_{k,n} - u_k)\rangle \geq 0$.
This follows that

$$\begin{aligned}
\langle A_k x_n + v_k, j_q(y_{k,n} - u_k)\rangle &\leq \frac{1}{\lambda_n}\langle x_n - y_{k,n}, j_q(y_{k,n} - u_k)\rangle \\
&\quad + \frac{\sigma_n}{\lambda_n}\langle u_n - x_n, j_q(y_{k,n} - u_k)\rangle \\
&\leq \frac{1}{\lambda_n}\|x_n - y_{k,n}\|\|y_{k,n} - u_k\|^{q-1} \\
&\quad + \frac{\sigma_n}{\lambda_n}\|u_n - x_n\|\|y_{k,n} - u_k\|^{q-1} \\
&\leq (\|x_n - y_{k,n})\| + \sigma_n)\Gamma_k, \qquad (3.10)
\end{aligned}$$

where

$$\Gamma_k = \sup_{n\geq 1}\left\{\frac{1}{\lambda_n}\|y_{k,n} - u_k\|^{q-1}, \frac{\sigma_n}{\lambda_n}\|u_n - x_n\|\|y_{k,n} - u_k\|^{q-1}\right\}.$$

Since $j_q$ is weakly sequentially continuous, $\|x_n - y_{k,n}\| \to 0$, $x_{n_j} \rightharpoonup w$ and $A_k$ is Lipschitz continuous, we get from (3.10) that $\langle A_k w + v_k, j_q(w - u_k)\rangle \leq 0$, that is, $\langle -A_k w - v_k, j_q(w - u_k)\rangle \geq 0$. Since $B_k$ is $m$-accretive, we obtain $-A_k w \in B_k w$. This implies that $w \in (A_k + B_k)^{-1}0$, for all $k = 1, 2, ..., N$. Thus,

$$w \in \left(\bigcap_{i=1}^{M} F(S_i)\right) \cap \left(\bigcap_{k=1}^{N}(A_k + B_k)^{-1}0\right).$$

Moreover, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle u' - x^*, j_q(y_{k_n,n} - x^*)\rangle &= \lim_{m\to\infty}\langle u' - x^*, j_q(x_{n_m} - x^*)\rangle \\
&= \langle u' - x^*, j_q(w - x^*)\rangle \leq 0.
\end{aligned}$$

We then prove that $\|x_n - x^*\| \to 0$ as $n \to \infty$. By (3.2) and Lemma 2.2, we have

$$\begin{aligned}
\|y_{k_n,n} - x^*\|^q &\leq \Bigg\|(1-\sigma_n)\left[\left(I - \frac{\lambda_n}{1-\sigma_n}A_{k_n}\right)x_n - \left(I - \frac{\lambda_n}{1-\sigma_n}A_{k_n}\right)x^*\right] \\
&\qquad + \sigma_n(u_n - x^*)\Bigg\|^q \\
&\leq (1-\sigma_n)^q\left\|\left(I - \frac{\lambda_n}{1-\sigma_n}A_{k_n}\right)x_n - \left(I - \frac{\lambda_n}{1-\sigma_n}A_{k_n}\right)x^*\right\|^q \\
&\qquad + q\sigma_n\langle u_n - x^*, j_q(y_{k_n,n} - x^*)\rangle \\
&\leq (1-\sigma_n)^q\|x_n - x^*\|^q + q\sigma_n\langle u_n - x^*, j_q(y_{k_n,n} - x^*)\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z_{i_n,n} - x^*\|^q &\le (1-\gamma_{i_n,n})\|S_{i_n}x_n - x^*\|^q + \gamma_{i_n,n}\|y_{k_n,n} - x^*\|^q \\
&\le (1-\gamma_{i_n,n})\|x_n - x^*\|^q + \gamma_{i_n,n}\|y_{k_n,n} - x^*\|^q \\
&\le (1-\gamma_{i_n,n})\|x_n - x^*\|^q \\
&\quad + \gamma_{i_n,n}\big[(1-\sigma_n)^q\|x_n - x^*\|^q + q\sigma_n\langle u_n - x^*, j_q(y_{k_n,n} - x^*)\rangle\big] \\
&\le (1-\gamma_{i_n,n})\|x_n - x^*\|^q \\
&\quad + \gamma_{i_n,n}\big[(1-\sigma_n)\|x_n - x^*\|^q + q\sigma_n\langle u_n - x^*, j_q(y_{k_n,n} - x^*)\rangle\big] \\
&= (1-\gamma_{i_n,n}\sigma_n)\|x_n - x^*\|^q + q\gamma_{i_n,n}\sigma_n\langle u_n - x^*, j_q(y_{k_n,n} - x^*)\rangle. \quad (3.11)
\end{aligned}$$

From (3.1) and (3.11), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &\le \beta_n\|x_n - x^*\|^q + (1-\beta_n)\|z_{i_n,n} - x^*\|^q \\
&\le \beta_n\|x_n - x^*\|^q + (1-\beta_n) \\
&\quad \times \left[(1-\gamma_{i_n,n}\sigma_n)\|x_n - x^*\|^q + q\gamma_{i_n,n}\sigma_n\langle u_n - x^*, j_q(y_{k_n,n} - x^*)\rangle\right] \\
&\le \big(1-\gamma_{i_n,n}\sigma_n(1-\beta_n)\big)\|x_n - x^*\|^q \\
&\quad + (1-\beta_n)q\gamma_{i_n,n}\sigma_n\langle u_n - u', j_q(y_{k_n,n} - x^*)\rangle \\
&\quad + (1-\beta_n)q\gamma_{i_n,n}\sigma_n\langle u' - x^*, j_q(y_{k_n,n} - x^*)\rangle \\
&\le \big(1-\gamma_{i_n,n}\sigma_n(1-\beta_n)\big)\|x_n - x^*\|^q \\
&\quad + (1-\beta_n)q\gamma_{i_n,n}\sigma_n\|u_n - u'\|\|y_{k_n,n} - x^*\|^{q-1} \\
&\quad + (1-\beta_n)q\gamma_{i_n,n}\sigma_n\langle u' - x^*, j_q(y_{k_n,n} - x^*)\rangle \\
&\le (1-\eta_n)\|x_n - x^*\|^q + \eta_n\delta_n,
\end{aligned}$$

where

$$\eta_n := \gamma_{i_n,n}\sigma_n(1-\beta_n)$$

and

$$\delta_n := q\|u_n - u'\|\|y_{k_n,n} - x^*\|^{q-1} + q\langle u' - x^*, j_q(y_{k_n,n} - x^*)\rangle.$$

It is easily checked that $\sum_{n=1}^{\infty}\eta_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Therefore, we apply Lemma 2.12, to conclude that $\|x_n - x^*\| \to 0$, as $n \to \infty$. The proof is now completed. □

## 4. Deduced results of our main theorem

The following result in a Hilbert space is directly obtained by Theorem 3.4.

**Corollary 4.1.** *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $A_k : C \to H$, $k = 1,2,...,N$, be $\alpha_k$-ism and let $B_k : D(B_k) \to 2^H$, $k = 1,2,...,N$, be maximal monotone operators such that $D(B_k) \subset C$. Let $S_i : C \to C$, $i = 1,2,...,M$, be nonexpansive mappings such that*

$$\Omega := \left(\bigcap_{i=1}^{M} F(S_i)\right) \cap \left(\bigcap_{k=1}^{N} (A_k + B_k)^{-1}0\right) \ne \emptyset.$$

*Let $\{\lambda_n\}$ be a positive real sequence and let $\{\sigma_n\}$,$\{\beta_n\}$,$\{\gamma_{i,n}\}$ for all $i = 1, 2, ..., M$ be sequences in (0,1). Let $x_1 \in C$, and let $\{x_n\} \subset C$ be a sequence generated by the following*

$$\begin{cases} y_{k,n} = J^{B_k}_{\lambda_n}(\sigma_n u_n + (1-\sigma_n)x_n - \lambda_n A_k x_n),\ k = 1, 2, ..., N, \\ \qquad \text{choose } k_n \text{ such that } \|y_{k_n,n} - x_n\| = \max_{k=1,2,...,N} \|y_{k,n} - x_n\|, \\ z_{i,n} = (1-\gamma_{i,n})S_i x_n + \gamma_{i,n} y_{k_n,n},\ i = 1, 2, ..., M, \\ \qquad \text{choose } i_n \text{ such that } \|z_{i_n,n} - x_n\| = \max_{i=1,2,...,M} \|z_{i,n} - x_n\|, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) z_{i_n,n}, \forall n \geq 1, \end{cases} \tag{4.1}$$

*where $J^B_\lambda = (I + \lambda B)^{-1}$ and $\{u_n\}$ is a sequence in $H$ such that $\lim_{n\to\infty} u_n = u' \in H$. Suppose that the control sequences satisfy the following conditions:*

(i) $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_{n=1}^{\infty} \sigma_n = \infty$*;*
(ii) $0 < a_i \leq \gamma_{i,n} \leq b_i < 1,\ i = 1, 2, ..., M$ *and* $\lim_{n\to\infty} |\gamma_{i_n,n+1} - \gamma_{i_n,n}| = 0$*;*
(iii) $0 < c \leq \beta_n \leq d < 1$*;*
(iv) $0 < e \leq \lambda_n < \frac{\lambda_n}{1-\sigma_n} \leq f < 2\alpha$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

*for some $a_i, b_i, c, d, e, f \in \mathbb{R}^+$ and $\alpha = \min\{\alpha_1, \alpha_2, ..., \alpha_N\}$. Then $\{x_n\}$ generated by (4.1) converges strongly to a point $x^* = Q_\Omega u'$.*

If we put $M = N = 1$ in Theorem 3.4, then we obtain the following result.

**Corollary 4.2.** *Let $C$ be a nonempty, closed and convex subset of a real uniformly convex and q-uniformly smooth Banach space $X$ which admits a weakly sequentially continuous generalized duality mapping $j_q$. Let $A : C \to X$ be an $\alpha$-isa of order q operator and let $B : D(B) \to 2^X$ be an m-accretive operator such that $D(B) \subset C$. Let $S : C \to C$ be a nonexpansive mapping such that $\Omega := F(S) \cap (A+B)^{-1}0 \neq \emptyset$. Let $\{\lambda_n\}$ be a positive real sequence and let $\{\sigma_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in (0,1). Let $x_1 \in C$, and let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} y_n = J^B_{\lambda_n}(\sigma_n u_n + (1-\sigma_n)x_n - \lambda_n A x_n), \\ z_n = (1-\gamma_n)S x_n + \gamma_n y_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) z_n, \forall n \geq 1, \end{cases} \tag{4.2}$$

*where $\{u_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} u_n = u' \in X$. Suppose that the control sequences satisfy the following conditions:*

(i) $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_{n=1}^{\infty} \sigma_n = \infty$*;*
(ii) $0 < a \leq \gamma_n \leq b < 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_{,n}| = 0$*;*
(iii) $0 < c \leq \beta_n \leq d < 1$*;*
(iv) $0 < e \leq \lambda_n < \frac{\lambda_n}{1-\sigma_n} \leq f < (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

*for some $a, b, c, d, e, f \in \mathbb{R}^+$. Then $\{x_n\}$ generated by (4.2) converges strongly to $x^* = Q_\Omega u'$.*

Moreover, Theorem 3.4 can be reduced to find zeros of maximal monotone operators. Let $\widetilde{S}_i : C \to 2^C,\ i = 1, 2, ..., M$, be $m$-accretive operators on $C$. Put $S_i = J^{\widetilde{S}_i}_\lambda$, for $\lambda > 0$ and $i = 1, 2, ..., M$. Then $S_i$ is nonexpansive and $F(S_i) = \widetilde{S}_i^{-1}0$ for all $i = 1, 2, ..., M$. The following result is directly obtained by Theorem 3.4.

**Corollary 4.3.** *Let $C$ be a nonempty, closed and convex subset of a real uniformly convex and $q$-uniformly smooth Banach space $X$ which admits a weakly sequentially continuous generalized duality mapping $j_q$. Let $A_k : C \to X,\ k = 1, 2, ..., N$, be $\alpha_k$-isa of order $q$ operators and let $B_k : D(B_k) \to 2^X,\ k = 1, 2, ..., N$, be $m$-accretive such that $D(B_k) \subset C$. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be a resolvent of $B$ for $\lambda > 0$ and let $\widetilde{S}_i : C \to 2^C,\ i = 1, 2, ..., M$, be $m$-accretive operators on $C$ such that*

$$\Omega := \left(\bigcap_{i=1}^{M} \widetilde{S}_i^{-1}0\right) \cap \left(\bigcap_{k=1}^{N} (A_k + B_k)^{-1}0\right) \neq \emptyset.$$

*Let $\{\lambda_n\}$ be a positive real sequence and let $\{\sigma_n\}$,$\{\beta_n\}$,$\{\gamma_{i,n}\}$ for all $i = 1, 2, ..., M$ be sequences in (0,1). Let $x_1 \in C$, and let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} y_{k,n} = J_{\lambda_n}^{B_k}(\sigma_n u_n + (1 - \sigma_n)x_n - \lambda_n A_k x_n),\ k = 1, 2, ..., N, \\ \qquad \textit{choose } k_n \textit{ such that } \|y_{k_n,n} - x_n\| = \max\limits_{k=1,2,...,N} \|y_{k,n} - x_n\|, \\ z_{i,n} = (1 - \gamma_{i,n})J_\lambda^{\widetilde{S}_i} x_n + \gamma_{i,n} y_{k_n,n},\ i = 1, 2, ..., M, \\ \qquad \textit{choose } i_n \textit{ such that } \|z_{i_n,n} - x_n\| = \max\limits_{i=1,2,...,M} \|z_{i,n} - x_n\|, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) z_{i_n,n}, \forall n \geq 1, \end{cases} \tag{4.3}$$

*where $\{u_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} u_n = u' \in X$. Suppose that the control sequences satisfy the same conditions as in Lemma 3.2. Then $\{x_n\}$ generated by (4.3) converges strongly to a point $x^* = Q_\Omega u'$.*

We also can apply Theorem 3.4 for finding a common zero of the sum of $\alpha$-inverse strongly monotone operators and maximal monotone operators in a $q$-uniformly smooth Banach space.

**Corollary 4.4.** *Let $C$ be a nonempty, closed and convex subset of a real uniformly convex and $q$-uniformly smooth Banach space $X$ which admits a weakly sequentially continuous generalized duality mapping $j_q$. Let $A_k : C \to X,\ k = 1, 2, ..., N$, be $\alpha_k$-isa of order $q$ and let $B_k : D(B_k) \subset C \to 2^X,\ k = 1, 2, ..., N$, be $m$-accretive operators. Assume that $\bigcap_{k=1}^{N}(A_k + B_k)^{-1}0 \neq \emptyset$. Let $\{\lambda_n\}$ be a positive real sequence and let $\{\sigma_n\}$,$\{\beta_n\}$ are sequences in (0,1). Let $x_1 \in C$, and let $\{x_n\} \subset C$ be a sequence generated by the following*

$$\begin{cases} y_{k,n} = J_{\lambda_n}^{B_k}(\sigma_n u_n + (1 - \sigma_n)x_n - \lambda_n A_k x_n),\ k = 1, 2, ..., N, \\ \qquad \textit{choose } k_n \textit{ such that } \|y_{k_n,n} - x_n\| = \max\limits_{k=1,2,...,N} \|y_{k,n} - x_n\|, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) y_{k_n,n}, \forall n \geq 1, \end{cases} \tag{4.4}$$

*where $\{u_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} u_n = u' \in X$. Suppose that the control sequences satisfy the following conditions:*

*(i)* $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_{n=1}^{\infty} \sigma_n = \infty$;

*(ii)* $0 < c \leq \beta_n \leq d < 1$;

*(iii)* $0 < e \le \lambda_n < \frac{\lambda_n}{1-\sigma_n} \le f < (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

*for some* $c, d, e, f \in \mathbb{R}^+$ *and* $\alpha = \min\{\alpha_1, \alpha_2, ..., \alpha_N\}$. *Then* $\{x_n\}$ *generated by (4.4) converges strongly to a point* $x^* \in \bigcap_{k=1}^N (A_k + B_k)^{-1}0$.

In the following corollary, we apply Theorem 3.4 for finding a common fixed point of a finite family of nonexpansive operators.

**Corollary 4.5.** *Let* $C$ *be a nonempty, closed and convex subset of a real uniformly convex and* $q$*-uniformly smooth Banach space* $X$. *Let* $S_i : C \to C$, $i = 1, ..., M$, *be nonexpansive mappings such that* $\bigcap_{i=1}^M F(S_i) \ne \emptyset$. *Let* $\{\gamma_{i,n}\} \subset (0,1), i = 1, ..., M$. *Let* $x_1 \in C$, *and let* $\{x_n\} \subset C$ *be a sequence generated by*

$$\begin{cases} z_{i,n} = (1-\gamma_{i,n})S_i x_n + \gamma_{i,n} x_n, \ i = 1, 2, ..., M, \\ \quad \text{choose } i_n \text{ such that } \|z_{i_n,n} - x_n\| = \max_{i=1,2,...,M} \|z_{i,n} - x_n\|, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) z_{i_n,n}, \forall n \ge 1, \end{cases} \tag{4.5}$$

*Suppose that the control sequences satisfy the following conditions:*

*(i)* $0 < a_i \le \gamma_{i,n} \le b_i < 1$, $i = 1, 2, ..., M$ *and* $\lim_{n\to\infty} |\gamma_{i_n,n+1} - \gamma_{i_n,n}| = 0$;

*(ii)* $0 < c \le \beta_n \le d < 1$, *for some* $a_i, b_i, c, d \in \mathbb{R}^+$.

*Then* $\{x_n\}$ *generated by (4.5) converges strongly to a point* $x^* \in \bigcap_{i=1}^M F(S_i)$.

## 5. Applications

Now, we discuss some applications of our main results for convex minimization problems, multiple sets variational inequality problems and multiple sets equilibrium problems in the framework of Hilbert spaces. Throughout in this section, let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$.

5.1. **Application for Convex Minimization Problems.** The convex minimization problem is formulated as follows:

$$\text{find a point } \widehat{x} \in C \text{ such that } f(\widehat{x}) + g(\widehat{x}) \le \min_{x\in C}\{f(x) + g(x)\}, \tag{5.1}$$

where $f : H \to \mathbb{R}$ is a convex smooth function and $g : H \to \mathbb{R}$ is a convex, lower semicontinuous and nonsmooth function. We denote the set of solutions of the problem (5.1) by $CMP(f, g)$. It is well-known, by Fremat's rule, that the convex minimization problem is equivalent to the following problem:

$$\text{find a point } \widehat{x} \in C \text{ such that } 0 \in \nabla f(\widehat{x}) + \partial g(\widehat{x}), \tag{5.2}$$

where $\nabla f$ is a gradient of $f$ and $\partial g$ is a subdifferential mapping of $g$. From [1], we have that if $\nabla f$ is $L$-Lipschitzian continuous, then it is also $\frac{1}{L}-$inverse strongly monotone operator. Moreover, it is known that $\partial g$ is maximal monotone. Therefore, if we take $A_k = \nabla f_k$ and $B_k = \partial g_k$ for $k = 1, 2, ..., N$, in Theorem 3.4, then we obtain the following result.

**Theorem 5.1.** *Let* $f_k : H \to \mathbb{R}$, $k = 1, ..., N$, *be convex and differentiable function with* $L_k$*-Lipschitzian continuous with gradient* $\nabla f_k$ *and let* $g_k : H \to \mathbb{R}$, $k = 1, ..., N$, *be convex lower semicontinuous functions such that* $D(\partial g_k) \subset C$.

*Let $S_i : C \to C,\ i = 1, 2, ..., M,$ be nonexpansive mappings such that*

$$\left(\bigcap_{i=1}^{M} F(S_i)\right) \cap \left(\bigcap_{k=1}^{N} CMP(f_k, g_k)\right) \neq \emptyset.$$

*Let $\{\lambda_n\}$ be a positive real sequence and let $\{\sigma_n\}$,$\{\beta_n\}$,$\{\gamma_{i,n}\}$ for all $i = 1, ..., M$ be sequences in (0,1). Let $x_1 \in C$, and let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} y_{k,n} = J_{\lambda_n}^{\partial g_k}(\sigma_n u_n + (1-\sigma_n)x_n - \lambda_n \nabla f_k(x_n)),\ k = 1, 2, ..., N, \\ \qquad \text{choose } k_n \text{ such that } \|y_{k_n,n} - x_n\| = \max_{k=1,2,...,N} \|y_{k,n} - x_n\|, \\ z_{i,n} = (1-\gamma_{i,n})S_i x_n + \gamma_{i,n} y_{k_n,n},\ i = 1, 2, ..., M, \\ \qquad \text{choose } i_n \text{ such that } \|z_{i_n,n} - x_n\| = \max_{i=1,2,...,M} \|z_{i,n} - x_n\|, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) z_{i_n,n}, \forall n \geq 1, \end{cases} \tag{5.3}$$

*where $\{u_n\}$ is a sequence in $H$ such that $\lim_{n\to\infty} u_n = u' \in H$. Suppose that the control sequences satisfy the following conditions:*

(*i*) $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_{n=1}^{\infty} \sigma_n = \infty$;
(*ii*) $0 < a_i \leq \gamma_{i,n} \leq b_i < 1,\ i = 1, 2, ..., M$ *and* $\lim_{n\to\infty} |\gamma_{i_n,n+1} - \gamma_{i_n,n}| = 0$;
(*iii*) $0 < c \leq \beta_n \leq d < 1$;
(*iv*) $0 < e \leq \lambda_n < \frac{\lambda_n}{1-\sigma_n} \leq f < \frac{2}{L}$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

*for some $a_i, b_i, c, d, e, f \in \mathbb{R}^+$ and $L = \max\{L_1, L_2, ..., L_N\}$. Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \left(\bigcap_{i=1}^{M} F(S_i)\right) \cap \left(\bigcap_{k=1}^{N} CMP(f_k, g_k)\right)$.*

5.2. **Application for Multiple Sets Variational Inequality Problems.** Now, we apply our main theorem for finding a common solution of multiple sets variational inequality problems. Let $A : C \to H$ be a nonlinear monotone operators. The variational inequality problem is to find a point $\widehat{x} \in C$ satisfies

$$\langle A\widehat{x}, y - \widehat{x}\rangle \geq 0,\ \forall y \in C. \tag{5.4}$$

The set of solutions of the problem (5.4) is denoted by $VI(C, A)$. It is well-known that

$$\widehat{x} \in VI(C, A) \text{ if and only if } \widehat{x} = P_C(\widehat{x} - \lambda A\widehat{x}),$$

for all $\lambda > 0$ where $P_C$ is the metric projection from $H$ onto $C$. Let us consider the indicator function of $C$, denoted by $\iota_C$, which is defined as follows:

$$\iota_C(x) = \begin{cases} 0 & \text{if } x \in C; \\ +\infty & \text{if } x \notin C. \end{cases}$$

We know that $\iota_C$ is a proper, convex and lower semicontinuous function on $H$ and $\partial\iota_C$ is a maximal monotone operator. Moreover, we have $J_\lambda^{\partial\iota_C} = (I + \lambda\partial\iota_C)^{-1}$ for each $\lambda > 0$ and for $x \in H$, $y = J_\lambda^{\partial\iota_C} x$ if and only if $y = P_C x$. Further, we have $(A + \partial\iota_C)^{-1}0 = VI(C, A)$. By applying our main result, Theorem 3.4, we obtain the following result for a common fixed point and multiple sets variational inequality problems in Hilbert spaces.

**Theorem 5.2.** *Let $C_k$ be nonempty, closed and convex subsets of a real Hilbert space $H$. Let $A_k : C_k \to H$, $k = 1, ..., N$, be $\alpha_k$-ism and let $S_i : C \to C$, $i = 1, ..., M$, be a finite family of nonexpansive mappings such that*

$$\left(\bigcap_{i=1}^{M} F(S_i)\right) \cap \left(\bigcap_{k=1}^{N} VI(C_k, A_k)\right) \neq \emptyset.$$

*Let $\{\lambda_n\}$ be a positive real sequence and let $\{\sigma_n\}$,$\{\beta_n\}$,$\{\gamma_{i,n}\}$ for all $i = 1, ..., M$ be sequences in (0,1). Let $x_1 \in H$, and let $\{x_n\} \subset H$ be a sequence generated by the following*

$$\begin{cases} y_{k,n} = P_{C_k}(\sigma_n u_n + (1-\sigma_n)x_n - \lambda_n A_k x_n),\ k = 1, 2, ..., N, \\ \qquad \text{choose } k_n \text{ such that } \|y_{k_n,n} - x_n\| = \max_{k=1,2,\dots,N} \|y_{k,n} - x_n\|, \\ z_{i,n} = (1-\gamma_{i,n})S_i x_n + \gamma_{i,n} y_{k_n,n},\ i = 1, 2, ..., M, \\ \qquad \text{choose } i_n \text{ such that } \|z_{i_n,n} - x_n\| = \max_{i=1,2,\dots,M} \|z_{i,n} - x_n\|, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) z_{i_n,n}, \forall n \geq 1, \end{cases} \tag{5.5}$$

*where $\{u_n\}$ is a sequence in $H$ such that $\lim_{n\to\infty} u_n = u' \in H$. Suppose that the control sequences satisfy the same conditions as in Corollary 4.1. Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \left(\bigcap_{i=1}^{M} F(S_i)\right) \cap \left(\bigcap_{k=1}^{N} VI(C_k, A_k)\right)$.*

5.3. **Application for Multiple Sets Equilibrium Problems.** Let $G : C \times C \to \mathbb{R}$ be a bi-function. The equilibrium problem for $G$ is the problem of finding a point $\widehat{x} \in C$ such that

$$G(\widehat{x}, y) \geq 0, \tag{5.6}$$

for all $y \in C$. We denote $EP(G)$ by the set of solutions of equilibrium problem. For solving the equilibrium problem, we assume that $G : C \times C \to \mathbb{R}$ satisfies the following properties:

$(a)$ $G(x, x) = 0$ for all $x \in C$;
$(b)$ $G$ is monotone, i.e., $G(x, y) + G(y, x) \leq 0$ for all $x, y \in C$;
$(c)$ for each $x, y, z \in C$, $\limsup_{t\downarrow 0} G(tz + (1-t)x, y) \leq G(x, y)$;
$(d)$ for each $x \in C$, $y \mapsto G(x, y)$ is convex and lower semicontinuous.

Next, we have the following lemma which can be found in [4, 11].

**Lemma 5.3.** *Let $G : C \times C \to \mathbb{R}$ be a bi-function satisfying properties $(a) - (d)$. For $\lambda > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$G(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0,$$

*for all $y \in C$. Further, define a mapping $T_\lambda : H \to C$, say the resolvent of $G$, for $\lambda > 0$ as follows:*

$$T_\lambda(x) := \left\{ z \in C : G(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\},$$

*for all $\lambda > 0$ and $x \in H$. Then the following hold:*

*(aa)* $T_\lambda$ *is single-valued;*
*(bb)* $T_\lambda$ *is firmly nonexpansive, that is, for* $x, y \in H$,

$$\|T_\lambda x - T_\lambda y\|^2 \leq \langle T_\lambda x - T_\lambda y, x - y\rangle;$$

*(cc)* $F(T_\lambda) = EP(G)$;
*(dd)* $EP(G)$ *is closed and convex.*

**Lemma 5.4.** ([30]) *Let* $G : C \times C \to \mathbb{R}$ *be a bi-function satisfying properties* $(a)-(d)$. *Let* $A_G$ *be a multi-valued mapping of* $H$ *into itself defined by*

$$A_G x = \begin{cases} \{z \in H : G(x, y) \geq \langle y - x, z\rangle, \forall y \in C\} & \text{if } x \in C; \\ \emptyset & \text{if } x \notin C. \end{cases}$$

*Then* $EP(G) = A_G^{-1}0$ *and* $A_G$ *is maximal monotone with* $D(A_G) \subset C$. *Further, for any* $x \in H$ *and* $\lambda > 0$, *the resolvent* $T_\lambda$ *of* $G$ *coincides with the resolvent of* $A_G$, *i.e.,* $T_\lambda x = (I + \lambda A_G)^{-1}x$.

**Theorem 5.5.** *Let* $C_k$ *be nonempty, closed and convex subsets of a real Hilbert space* $H$. *Let* $A_k : C_k \to H,\ k = 1, 2, ..., N$, *be* $\alpha_k$*-inverse strongly monotone operators and let* $G_k : C_k \times C_k \to \mathbb{R},\ k = 1, 2, ..., N$, *be bi-functions which satisfy the properties* $(a) - (d)$. *Let* $S_i : C \to C,\ i = 1, 2, ..., M$, *be nonexpansive mappings such that*

$$\left(\bigcap_{i=1}^{M} F(S_i)\right) \cap \left(\bigcap_{k=1}^{N} EP(G_k)\right) \neq \emptyset.$$

*Let* $\{\lambda_n\}$ *be a positive real sequence and let* $\{\sigma_n\}$,$\{\beta_n\}$,$\{\gamma_{i,n}\}$ *for all* $i = 1, 2, ..., M$ *be sequences in (0,1). Let* $x_1 \in H$, *and let* $\{x_n\} \subset H$ *be a sequence generated by the following*

$$\begin{cases} y_{k,n} = T_{\lambda_n}(\sigma_n u_n + (1-\sigma_n)x_n - \lambda_n A_k x_n),\ k = 1, 2, ..., N, \\ \qquad \text{choose } k_n \text{ such that } \|y_{k_n,n} - x_n\| = \max_{k=1,2,\dots,N} \|y_{k,n} - x_n\|, \\ z_{i,n} = (1-\gamma_{i,n})S_i x_n + \gamma_{i,n} y_{k_n,n},\ i = 1, 2, ..., M, \\ \qquad \text{choose } i_n \text{ such that } \|z_{i_n,n} - x_n\| = \max_{i=1,2,\dots,M} \|z_{i,n} - x_n\|, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) z_{i_n,n}, \forall n \geq 1, \end{cases} \tag{5.7}$$

*where* $\{u_n\}$ *is a sequence in* $H$ *such that* $\lim_{n\to\infty} u_n = u' \in H$. *Suppose that the control sequences satisfy the same conditions as in Corollary 4.1. Then the sequence* $\{x_n\}$ *converges strongly to a point* $x^* \in \left(\bigcap_{i=1}^{M} F(S_i)\right) \cap \left(\bigcap_{k=1}^{N} EP(G_k)\right)$.

## 6. Numerical examples

In this section, we present some numerical examples to solving the image restoration problems by using our proposed algorithms. Throughout this section, we let $\|\cdot\|_1$ is $l_1$-norm and $\|\cdot\|_2$ is $l_2$-norm. From the problem (5.1), we now consider the convex minimization problem of sum of two functions as follows:

$$\hat{x} := \arg\min_x \{f(x) + g(x)\}, \tag{6.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex loss function and differentiable with $L$-Lipschitz continuous gradient (for constant $L$), i.e.,

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|,$$

for all $x, y \in \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is a proper convex and lower semicontinuous function.

The problem (6.1) can be solved by using the proximal gradient technique which was presented by Parikh and Boyd [24], i.e., if $\hat{x}$ is a solution of this problem (6.1), then it satisfies the following fixed point problem:

$$\hat{x} = T\hat{x}, \tag{6.2}$$

where $T$ is a class of forward-backward operator defined by $T := prox_{cg}(I - c\nabla f)$ for $c > 0$. It is also known that $T$ is a nonexpansive mapping when $c \in \left(0, \frac{2}{L}\right)$.

We next discuss some preliminary numerical results for the image restoration problem as the following form:

$$Ax = b + w, \tag{6.3}$$

where $x \in \mathbb{R}^{n\times 1}$ is an original image, $b \in \mathbb{R}^{m\times 1}$ is an observed image, $A \in \mathbb{R}^{m\times n}$ represents a blurring operator and $w \in \mathbb{R}^{m\times 1}$ is an additive noise. For image restoration problem, the size of the unknown true image $x$ is assumed to be the same as that of $w$, i.e., $m = n$. In order to solve this problem can be related to the problem (6.1) and it is of the following form:

$$\hat{x} := \arg\min_x\{\|Ax - b\|_2^2 + \lambda\|x\|_1\},$$

for some regularization parameter $\lambda > 0$, that is, we propose to estimate the original image $x$ by minimizing the additive noise.

In this situation, we choose the regularization parameter $\lambda = 5e^{-5}$. For these examples, we look at the $256 \times 256$ camera man (original image). We use a Gausssian blur of size $9 \times 9$ and standard deviation $\sigma = 4$ to create the blurred and noisy image (observed image).

In 2009, Thung and Raveendran [32] introduced Peak Signal-to-Noise Ratio (PSNR) to measure a quality of restored images for each $x_n$ as the following:

$$PSNR(x_n) = 10\log\left(\frac{255^2}{MSE}\right),$$

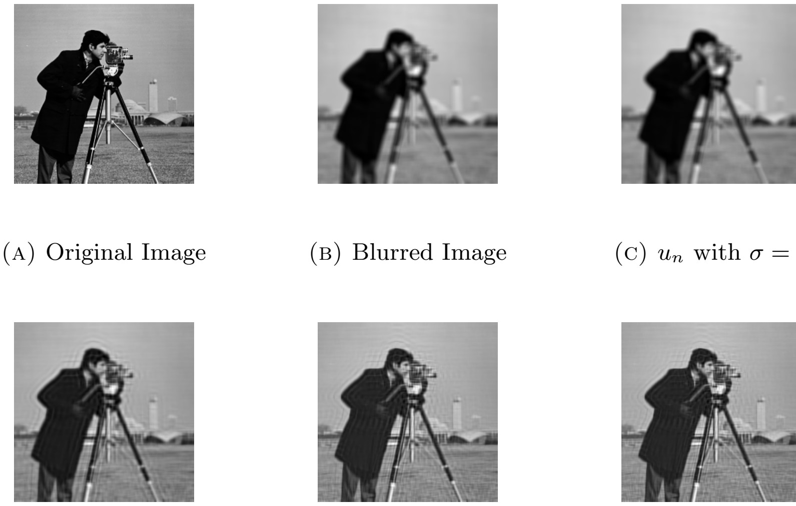
where the well-known Mean Square Error formula is $MSE = \frac{1}{256^2}\|x_n - x\|^2$ for $x$ is the original image. A higher PSNR implies that the restored image is of higher quality.

In Theorem 3.4, we take the same as Theorem 5.1 and set the nonexpansive mappings $S_i$ as follows:

$$S_i := prox_{c_i g}(I - c_i\nabla f),$$

for $i = 1, 2, 3$ when $f(x) = \|Ax - b\|_2^2$ and $g(x) = \lambda\|x\|_1$. We also choose the Lipschitz constant $L$ of the gradient $\nabla f$ is the maximal value of eigenvalues of the matrix $A^T A$ and the following parameters

$$\sigma_n = \frac{1}{n},\ \lambda_n = \frac{31n}{20L(n+1)},\ \gamma_{i,n} = \frac{n}{2n+1}, i = 1, 2, 3,$$



(A) Original Image (B) Blurred Image (C) $u_n$ with $\sigma = 4$

(D) $100^{th}$ iter. (E) $500^{th}$ iter. (F) $1000^{th}$ iter.

FIGURE 1. Results of the restoration camera man for case $u_n$ with $\sigma = 4$

$$\beta_n = \frac{n}{300n+1},\ c_1 = \frac{1}{2L},\ c_2 = \frac{1}{L},\ c_3 = \frac{31}{20L}.$$

So, these control parameters satisfy all conditions of Theorem 3.4. Moreover, for here, we will consider two difference cases of the chosen constant sequence $\{u_n\}$ in Theorem 3.4:

Case 1. $\{u_n\}$ is the blurred image by a Gausssian blur of size $9 \times 9$ and standard deviation $\sigma = 4$ (PSNR = 21.367303).

Case 2. $\{u_n\}$ is the blurred image by a Gausssian blur of size $9 \times 9$ and standard deviation $\sigma = 2$ (PSNR = 22.712128).

Then we obtain the restoration images of $100^{th}$, $500^{th}$ and $1000^{th}$ iterations of Camera man in Figure 1 (Case 1) and Figure 2 (Case 2). Their PSNR values of the studied algorithms are also presented in Figure 3 and Table 1.

Finally, we observe that the PSNR values obtained by Case 1 and Case 2 at iteration 1000 are equal to 27.132789 and 27.447153, respectively, that increased from the blurred image and $u_n$. Therefore, our experiments show that the proposed algorithm can be applied to solve the images restoration problems. However, the performances of our methods depend on choosing any parameters and the sequence $u_n$.

(A) Original Image

(B) Blurred Image

(C) $u_n$ with $\sigma = 2$

(D) $100^{th}$ iter.

(E) $500^{th}$ iter.

(F) $1000^{th}$ iter.

FIGURE 2. Results of the restoration camera man for case $u_n$ with $\sigma = 2$

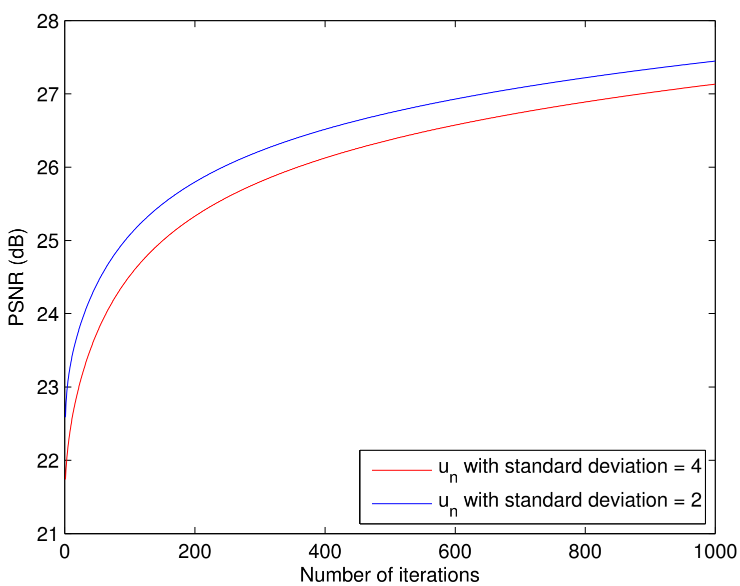



FIGURE 3. Plotting of the values of PSNR of camera man

TABLE 1. The values of PSNR at $x_1$, $x_5$, $x_{10}$, $x_{50}$, $x_{100}$, $x_{500}$ and $x_{1000}$

| No. iterations | PSNR | |
|---|---|---|
| | $u_n$ with $\sigma = 4$ | $u_n$ with $\sigma = 2$ |
| 1 | 21.738898 | 22.587446 |
| 5 | 22.194396 | 23.092401 |
| 10 | 22.510790 | 23.368955 |
| 50 | 23.751543 | 24.415867 |
| 100 | 24.518550 | 25.076645 |
| 500 | 26.372678 | 26.743317 |
| 1000 | 27.132789 | 27.447153 |

## 7. Conclusion

In this work, we establish a new iterative scheme for approximation a solution of a common element of the set of all solutions of a finite family of quasi-variational inclusion problems on monotone operators and the set of all common fixed points of a finite family of nonexpansive mappings in Banach spaces. Under some mild conditions, we obtaine the strong convergence theorem of this iteration. It also extend and reduce to the corresponding results in the literature. Further, we also give some applications including convex minimization problems, variational inequality problems and equilibrium problems. Some numerical experiments of image restoration problems supporting our main results are presented.

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