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PSEUDOMONOTONE VARIATIONAL INEQUALITIES AND FIXED POINTS

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Abstract. We introduce two new iterative algorithms with line-search process for solving a variational inequality problem with pseudomonotone and Lipschitz continuous mapping and a common fixed-point problem of an asymptotically nonexpansive mapping and a strictly pseudocontractive mapping. The proposed algorithms are based on inertial subgradient extragradient method with line-search process, hybrid steepest-descent method, and viscosity approximation method. Under mild conditions, we prove strong convergence of the proposed algorithms in a real Hilbert space.
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1. Introduction-Preliminaries

Monotone variational inequalities act as an efficient mathematical modelling to solve a number of real problems in various engineering, medicine, economics etc. Their solutions have been studied by many authors via iterative methods; see, [7, 4, 3, 14, 12] and the references therein. From now on, we always assume that C is a convex, closed nonempty set in a real Hilbert space H. For each point $x \in H$, we know that there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \ \forall y \in C.$$

The mapping P_C is called the metric projection of H onto C. Let S be a mapping on C and denote by $\mathrm{Fix}(S)$ the set of fixed points of S. S is called an asymptotically nonexpansive mapping if $\exists \{\theta_n\} \subset [0, +\infty)$ with $\lim_{n\to\infty} \theta_n = 0$ such that

$$\|T^n x - T^n y\| \le (1 + \theta_n)\|x - y\|, \ \forall n \ge 1, \ x, y \in C.$$

In particular, if $\theta_n = 0$, then T is called a nonexpansive mapping. S is called a strictly pseudocontractive mapping if $\exists\zeta \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \zeta\|(I - T)x - (I - T)y\|^2, \ \forall x, y \in C.$$

Fixed points of (asymptotically) nonexpansive mappings and strictly pseudocontractive mappings were studied through iterative methods recently; see, [5, 6, 11, 13, 17] and the references therein.

Let $A : H \to H$ be a mapping. Recall that A is said to be

(i) L-Lipschitz continuous (or L-Lipschitzian) if $\exists L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \ \forall x, y \in C;$$

(ii) monotone if $\langle Tx - Ty, x - y\rangle \ge 0$, $\forall x, y \in C$;

(iii) pseudomonotone if $\langle Tx, y - x\rangle \ge 0 \Rightarrow \langle Ty, y - x\rangle \ge 0$, $\forall x, y \in C$;

(iv) α-strongly monotone if $\exists\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \alpha\|x - y\|^2, \ \forall x, y \in C;$$

(v) sequentially weakly continuous if $\forall\{x_n\} \subset C$, the relation holds:

$$x_n \rightharpoonup x \Rightarrow Tx_n \rightharpoonup Tx.$$

The classical variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \ge 0, \quad \forall x \in C. \tag{1.1}$$

The solution set of the VIP is denoted by $\mathrm{VI}(C, A)$. At present, one of the most popular methods for solving the VIP is the extragradient method introduced by Korpelevich [9] in 1976, that is, for any initial $x_0 \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(x_n - \tau Ax_n), \\ x_{n+1} = P_C(x_n - \tau Ay_n) \quad \forall n \ge 0, \end{cases} \tag{1.2}$$

with $\tau \in (0, \frac{1}{L})$. If $\mathrm{VI}(C, A) \ne \emptyset$, then the sequence $\{x_n\}$ generated by process (1.2) converges weakly to an element in $\mathrm{VI}(C, A)$. Recently, gradient-based methods have been considered by many authors in infinite dimensional spaces; see e.g., [1, 10, 16, 15] and references therein, to name but a few.

In the extragradient methods, one needs to compute two projections onto C for each iteration. It is known that the projection onto a closed convex set C is closely related to a minimum distance problem. If C is a general closed and convex set, this might require a prohibitive amount of computation time. In 2011, Censor et al. [1] modified Korpelevich's extragradient method and first introduced the subgradient extragradient method, in which the second projection onto C is replaced by a projection onto a half-space:

$$\begin{cases} y_n = P_C(x_n - \tau Ax_n), \\ C_n = \{x \in H : \langle x_n - \tau Ax_n - y_n, x - y_n\rangle \le 0\}, \\ x_{n+1} = P_{C_n}(x_n - \tau Ay_n) \quad \forall n \ge 0, \end{cases} \tag{1.3}$$

with $\tau \in (0, \frac{1}{L})$. In 2014, Kraikaew and Saejung [10] introduced the Halpern subgradient extragradient method for solving the VIP (1.1), and proved strong convergence of the proposed method to a solution of VIP (1.1). In 2018, by virtue of the inertial technique, Thong and Hieu [15] introduced the inertial subgradient extragradient method, and proved weak convergence of the proposed method to a solution of VIP (1.1). Very recently, Thong and Hieu [16] introduced two inertial subgradient extragradient algorithms with linear-search process for solving the VIP (1.1) with monotone and Lipschitz continuous mapping A and the fixed-point problem of a quasi-nonexpansive mapping T with a demiclosedness property in a real Hilbert space. Under mild conditions, Thong and Hieu [16] proved weak convergence of the proposed algorithms to an element of $\mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$. Inspired by the research work by Thong and Hieu [16], we introduce two asymptotic inertial subgradient extragradient algorithms with line-search process for solving the VIP (1.1) with pseudomonotone and Lipschitz continuous mapping and common fixed point problems of an asymptotically nonexpansive mapping and a strictly pseudocontractive mapping in H. Convergence theorems are established in Hilbert spaces.

The following tools are essential for our main results.

Lemma 1.1. [8] *Let $A : C \to H$ be pseudomonotone and continuous. Then $x^* \in C$ is a solution to the VIP $\langle Ax^*, x - x^* \rangle \geq 0$ $\forall x \in C$, if and only if*

$$\langle Ax, x - x^* \rangle \geq 0, \ \forall x \in C.$$

Lemma 1.2. [18] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the conditions: $a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \gamma_n$ $\forall n \geq 1$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real numbers such that*

(i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$, and

(ii) $\limsup_{n\to\infty} \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\lambda_n \gamma_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.3. [21] *Let $T : C \to C$ be a ζ-strict pseudocontraction. Then $I - T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$, where I is the identity mapping of H.*

Lemma 1.4. [19] *Let $\lambda \in (0, 1]$, $T : C \to H$ be a nonexpansive mapping, and the mapping $T^\lambda : C \to H$ be defined by $T^\lambda x := Tx - \lambda\mu F(Tx)$ $\forall x \in C$, where $F : H \to H$ is κ-Lipschitzian and η-strongly monotone. Then T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$, i.e.,*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \ \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Lemma 1.5. [2] *Let X be a Banach space which admits a weakly continuous duality mapping, C be a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$, where I is the identity mapping of X.*

Lemma 1.6. [20] *Let $T : C \to C$ be a ζ-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers. Assume $(\gamma + \delta)\zeta \leq \gamma$. Then*

$$\|\gamma(x - y) + \delta(Tx - Ty)\| \leq (\gamma + \delta)\|x - y\| \ \forall x, y \in C.$$

2. Main results

In this section, we assume the following.

$T : H \to H$ is an asymptotically nonexpansive mapping with $\{\theta_n\}$ and $S : H \to H$ is a ζ-strictly pseudocontractive mapping.

$A : H \to H$ is L-Lipschitz continuous, pseudomonotone on H, and sequentially weakly continuous on C, such that $\Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \neq \emptyset$.

$f : H \to H$ is a contraction with constant $\delta \in [0, 1)$, and $F : H \to H$ is η-strongly monotone and κ-Lipschitzian such that $\delta < \tau := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, \frac{2\eta}{\kappa^2})$.

$\{\sigma_n\} \subset [0, 1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$ such that

(i) $\sup_{n\geq 1} \frac{\sigma_n}{\alpha_n} < \infty$ and $\beta_n + \gamma_n + \delta_n = 1 \ \forall n \geq 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$ and $(\gamma_n + \delta_n)\zeta \leq \gamma_n \ \forall n \geq 1$;

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\liminf_{n\to\infty} \delta_n > 0$.

Algorithm 2.1.
Initialization: Given $\gamma > 0,\ l \in (0, 1),\ \mu \in (0, 1)$. Let $x_0, x_1 \in H$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Set $w_n = T^n x_n + \sigma_n(T^n x_n - T^n x_{n-1})$ and compute $y_n = P_C(w_n - \tau_n A w_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\tau\|Aw_n - Ay_n\| \leq \mu\|w_n - y_n\|. \tag{2.1}$$

Step 2. Compute $z_n = \alpha_n f(x_n) + (I - \alpha_n\rho F)T^n P_{C_n}(w_n - \tau_n A y_n)$ with

$$C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \leq 0\}.$$

Step 3. Compute

$$x_{n+1} = \beta_n x_n + \gamma_n z_n + \delta_n S z_n. \tag{2.2}$$

Again set $n := n + 1$ and go to Step 1.

Lemma 2.1. *The Armijo-like search rule (2.1) is well defined, and the inequality holds:* $\min\{\gamma, \frac{\mu l}{L}\} \leq \tau_n \leq \gamma$.

Proof. From the L-Lipschitz continuity of A, we get

$$\frac{\mu}{L}\|Aw_n - AP_C(w_n - \gamma l^m Aw_n)\| \leq \mu\|w_n - P_C(w_n - \gamma l^m Aw_n)\|.$$

Thus, (2.1) holds for all $\gamma l^m \leq \frac{\mu}{L}$. So τ_n is well defined. Obviously, $\tau_n \leq \gamma$. If $\tau_n = \gamma$, then the inequality is true. If $\tau_n < \gamma$, then we get from (2.1)

$$\|Aw_n - AP_C(w_n - \frac{\tau_n}{l}Aw_n)\| > \frac{\mu}{\frac{\tau_n}{l}}\|w_n - P_C(w_n - \frac{\tau_n}{l}Aw_n)\|.$$

From the L-Lipschitz continuity of A, we obtain $\tau_n > \frac{\mu l}{L}$. Hence the inequality is valid.

Lemma 2.2. *Let* $\{w_n\}, \{y_n\}$ *and* $\{z_n\}$ *be the sequences generated by Algorithm 2.1. Then*

$$\begin{aligned} \|z_n - p\|^2 \leq\ & \alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)(1 + \theta_n)\|w_n - p\|^2 \\ & - (1 - \alpha_n\tau)(1 + \theta_n)(1 - \mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \\ & + 2\alpha_n\langle(f - \rho F)p, z_n - p\rangle \ \forall p \in \Omega, n \geq n_0, \end{aligned} \tag{2.3}$$

for some $n_0 \geq 1$, *where* $u_n := P_{C_n}(w_n - \tau_n A y_n)$.

Proof. By fixing $p \in \Omega \subset C \subset C_n$, we have

$$\begin{aligned}\|u_n - p\|^2 &\leq \langle u_n - p, w_n - \tau_n A y_n - p\rangle \\ &= \tfrac{1}{2}\|u_n - p\|^2 + \tfrac{1}{2}\|w_n - p\|^2 - \tfrac{1}{2}\|u_n - w_n\|^2 - \langle u_n - p, \tau_n A y_n\rangle.\end{aligned}$$

So, it follows that $\|u_n - p\|^2 \leq \|w_n - p\|^2 - \|u_n - w_n\|^2 - 2\langle u_n - p, \tau_n A y_n\rangle$, which together with (2.1) and the pseudomonotonicity of A, we deduce that $\langle A y_n, p - y_n\rangle \leq 0$ and

$$\begin{aligned}\|u_n - p\|^2 \leq \|w_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - w_n\|^2 \\ + 2\langle w_n - \tau_n A y_n - y_n, u_n - y_n\rangle.\end{aligned} \tag{2.4}$$

Since $u_n = P_{C_n}(w_n - \tau_n A y_n)$ with $C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \leq 0\}$, we have $\langle w_n - \tau_n A w_n - y_n, u_n - y_n\rangle \leq 0$, which together with (2.1), implies that

$$\|u_n - p\|^2 \leq \|w_n - p\|^2 - (1-\mu)\|w_n - y_n\|^2 - (1-\mu)\|u_n - y_n\|^2 \quad \forall p \in \Omega. \tag{2.5}$$

Taking into account $\lim_{n\to\infty} \frac{\theta_n(2+\theta_n)}{\alpha_n(1-\beta_n)} = 0$, we know that

$$\theta_n(2+\theta_n) \leq \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}, \quad \forall n \geq n_0$$

for some $n_0 \geq 1$. Hence we have that for all $n \geq n_0$,

$$\begin{aligned}\alpha_n\delta + (1-\alpha_n\tau)(1+\theta_n) &= 1 - \alpha_n(\tau-\delta) + (1-\alpha_n\tau)\theta_n \\ &\leq 1 - \alpha_n(\tau-\delta) + \theta_n \leq 1 - \tfrac{\alpha_n(\tau-\delta)}{2} \leq 1.\end{aligned}$$

Using Lemma 1.4, and the convexity of the function $h(t) = t^2$ $\forall t \in \mathbf{R}$, we obtain that, for all $n \geq n_0$,

$$\begin{aligned}&\|z_n - p\|^2 \\ &\leq [\alpha_n\delta\|x_n - p\| + (1-\alpha_n\tau)(1+\theta_n)\|u_n - p\|]^2 + 2\alpha_n\langle(f-\rho F)p, z_n - p\rangle \\ &\leq \alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)\|u_n - p\|^2 + 2\alpha_n\langle(f-\rho F)p, z_n - p\rangle \\ &= \alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)\|w_n - p\|^2 - (1-\alpha_n\tau)(1+\theta_n)(1-\mu) \\ &\times[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] + 2\alpha_n\langle(f-\rho F)p, z_n - p\rangle.\end{aligned}$$

This completes the proof.

Lemma 2.3. *Let* $\{w_n\}, \{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be bounded sequences generated by Algorithm 2.1. If* $T^n x_n - T^{n+1} x_n \to 0$, $x_n - x_{n+1} \to 0$, $w_n - x_n \to 0$, $w_n - z_n \to 0$ *and* $\exists\{w_{n_k}\} \subset \{w_n\}$ *such that* $w_{n_k} \rightharpoonup z \in H$, *then* $z \in \Omega$.

Proof. From Algorithm 2.1, we have $\|T^n x_n - x_n\| \leq \|w_n - x_n\| + (1+\theta_n)\|x_n - x_{n-1}\|$. Utilizing the assumptions $x_n - x_{n+1} \to 0$ and $w_n - x_n \to 0$, we have from $\theta_n \to 0$ that

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{2.6}$$

Combining the assumptions $w_n - x_n \to 0$ and $w_n - z_n \to 0$ implies that, as $n \to \infty$,

$$\|z_n - x_n\| \leq \|w_n - z_n\| + \|w_n - x_n\| \to 0.$$

Note that, for each $p \in \Omega$,

$$\begin{aligned} \|w_n - p\|^2 &\le (\|T^n x_n - p\| + \sigma_n \|T^n x_n - T^n x_{n-1}\|)^2 \\ &\le \|x_n - p\|^2 + \Gamma_n + \theta_n(2+\theta_n)(\|x_n - p\|^2 + \Gamma_n), \end{aligned}$$

where $\Gamma_n = \sigma_n \|x_n - x_{n-1}\|(2\|x_n - p\| + \sigma_n \|x_n - x_{n-1}\|)$. So it follows from (2.3) that for all $n \ge n_0$,

$$\begin{aligned} &(1-\alpha_n\tau)(1+\theta_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \\ &\le \alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)[\|x_n - p\|^2 + \Gamma_n \\ &\quad +\theta_n(2+\theta_n)(\|x_n - p\|^2 + \Gamma_n)] - \|z_n - p\|^2 + 2\alpha_n\|(f-\rho F)p\|\|z_n - p\| \\ &\le [1 - \tfrac{\alpha_n(\tau-\delta)}{2}]\|x_n - p\|^2 - \|z_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)[\Gamma_n \\ &\quad +\theta_n(2+\theta_n)(\|x_n - p\|^2 + \Gamma_n)] + 2\alpha_n\|(f-\rho F)p\|\|z_n - p\| \\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + (1+\theta_n)[\Gamma_n \\ &\quad +\theta_n(2+\theta_n)(\|x_n - p\|^2 + \Gamma_n)] + 2\alpha_n\|(f-\rho F)p\|\|z_n - p\|. \end{aligned}$$

Since $\alpha_n \to 0,\ \theta_n \to 0,\ \Gamma_n \to 0$ and $x_n - z_n \to 0$, we get

$$\lim_{n\to\infty} \|w_n - y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n - y_n\| = 0.$$

It follows that as $n \to \infty$,

$$\|w_n - u_n\| \le \|w_n - y_n\| + \|y_n - u_n\| \to 0 \quad \text{and} \quad \|x_n - u_n\| \le \|x_n - w_n\| + \|w_n - u_n\| \to 0.$$

By using Algorithm 2.1 we get

$$\delta_n\|Sz_n - z_n\| = \|x_{n+1} - x_n + (1-\beta_n)(x_n - z_n)\| \le \|x_{n+1} - x_n\| + \|x_n - z_n\|.$$

Since $x_n - x_{n+1} \to 0,\ z_n - x_n \to 0$ and $\liminf_{n\to\infty} \delta_n > 0$, we obtain

$$\lim_{n\to\infty} \|z_n - Sz_n\| = 0. \tag{2.7}$$

Note that

$$\frac{1}{\tau_n}\langle w_n - y_n, x - y_n\rangle + \langle Aw_n, y_n - w_n\rangle \le \langle Aw_n, x - w_n\rangle \quad \forall x \in C. \tag{2.8}$$

Since $\tau_n \ge \min\{\gamma, \frac{\mu l}{L}\}$, we get $\liminf_{k\to\infty}\langle Aw_{n_k}, x - w_{n_k}\rangle \ge 0\ \forall x \in C$.
Since $w_n - y_n \to 0$, we obtain from (2.8) that $\liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \ge 0\ \forall x \in C$.
Next we show that $x_n - Tx_n \to 0$. Indeed,

$$\begin{aligned} \|Tx_n - x_n\| &\le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^n x_n\| + \|T^n x_n - x_n\| \\ &\le (2+\theta_1)\|x_n - T^n x_n\| + \|T^{n+1}x_n - T^n x_n\|. \end{aligned}$$

From (2.6) and the assumption $T^n x_n - T^{n+1}x_n \to 0$ we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.9}$$

We now take a sequence $\{\varepsilon_k\} \subset (0,1)$ satisfying $\varepsilon_k \downarrow 0$ as $k \to \infty$. For all $k \ge 1$, we denote by m_k the smallest positive integer such that

$$\langle Ay_{n_j}, x - y_{n_j}\rangle + \varepsilon_k \ge 0 \quad \forall j \ge m_k. \tag{2.10}$$

Setting $\mu_{m_k} = \frac{Ay_{m_k}}{\|Ay_{m_k}\|^2}$, we get $\langle Ay_{m_k}, \mu_{m_k}\rangle = 1\ \forall k \ge 1$. From (2.10), we get

$$\langle Ay_{m_k}, x + \varepsilon_k\mu_{m_k} - y_{m_k}\rangle \ge 0,\ \forall k \ge 1.$$

From the pseudomonotonicity of A, we have

$$\langle Ax, x-y_{m_k}\rangle \geq \langle Ax-A(x+\varepsilon_k\mu_{m_k}), x+\varepsilon_k\mu_{m_k}-y_{m_k}\rangle-\varepsilon_k\langle Ax, \mu_{m_k}\rangle \quad \forall k\geq 1. \tag{2.11}$$

We claim that $\lim_{k\to\infty}\varepsilon_k\mu_{m_k}=0$. Indeed, from $w_{n_k}\rightharpoonup z$ and $w_n-y_n\to 0$, we obtain $y_{n_k}\rightharpoonup z$. So, $\{y_n\}\subset C$ guarantees $z\in C$. Again from the sequentially weak continuity of A, we know that $Ay_{n_k}\rightharpoonup Az$. Thus, $Az\neq 0$ (otherwise, z is a solution). Taking into account the sequentially weak lower semicontinuity of the norm $\|\cdot\|$, we get $0<\|Az\|\leq \liminf_{k\to\infty}\|Ay_{n_k}\|$. Note that $\{y_{m_k}\}\subset\{y_{n_k}\}$ and $\varepsilon_k\downarrow 0$ as $k\to\infty$. So it follows that

$$0\leq \limsup_{k\to\infty}\|\varepsilon_k\mu_{m_k}\| = \limsup_{k\to\infty}\frac{\varepsilon_k}{\|Ay_{m_k}\|}\leq \frac{\limsup_{k\to\infty}\varepsilon_k}{\liminf_{k\to\infty}\|Ay_{n_k}\|}=0.$$

Hence we get $\varepsilon_k\mu_{m_k}\to 0$.

Next we show that $z\in\Omega$. Indeed, from $w_n-x_n\to 0$ and $w_{n_k}\rightharpoonup z$, we get $x_{n_k}\rightharpoonup z$. From (2.9) we have $x_{n_k}-Tx_{n_k}\to 0$. Note that Lemma 1.5 guarantees the demiclosedness of $I-T$ at zero. Thus $z\in \mathrm{Fix}(T)$. Meantime, from $w_n-z_n\to 0$ and $w_{n_k}\rightharpoonup z$, we get $z_{n_k}\rightharpoonup z$. From (2.7) we have $z_{n_k}-Sz_{n_k}\to 0$. From Lemma 1.3, it follows that $I-S$ is demiclosed at zero. Hence we get $(I-S)z=0$, i.e., $z\in\mathrm{Fix}(S)$. On the other hand, letting $k\to\infty$, we deduce that the right hand side of (2.11) tends to zero by the uniform continuity of A, the boundedness of $\{y_{m_k}\}$, $\{\mu_{m_k}\}$ and the limit $\lim_{k\to\infty}\varepsilon_k\mu_{m_k}=0$. Thus, we get $\langle Ax, x-z\rangle=\liminf_{k\to\infty}\langle Ax, x-y_{m_k}\rangle\geq 0$ $\forall x\in C$. By Lemma 1.1, we have $z\in \mathrm{VI}(C,A)$. Therefore,

$$z\in \mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)=\Omega.$$

This completes the proof.

Theorem 2.1. *Let the sequence $\{x_n\}$ be generated by Algorithm 1.1. Assume that $T^nx_n-T^{n+1}x_n\to 0$. Then*

$$x_n\to x^*\in\Omega \;\Leftrightarrow\; \begin{cases} x_n-x_{n+1}\to 0,\\ x_n-y_n\to 0\end{cases}$$

where $x^*\in\Omega$ is a unique solution to the VIP: $\langle(\rho F-f)x^*, p-x^*\rangle\geq 0$ $\forall p\in\Omega$.

Proof. From $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$, we may assume, without loss of generality, that $\{\beta_n\}\subset[a,b]\subset(0,1)$. We claim that $P_\Omega(f+I-\rho F)$ is a contraction. Indeed, by Lemma 1.4, we have that $P_\Omega(f+I-\rho F)$ is a contraction. Banach's Contraction Mapping Principle guarantees that $P_\Omega(f+I-\rho F)$ has a unique fixed point. Say $x^*\in H$, that is, $x^*=P_\Omega(f+I-\rho F)x^*$. Thus, there exists a unique solution $x^*\in\Omega=\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$ to the VIP

$$\langle(\rho F-f)x^*, p-x^*\rangle\geq 0\quad\forall p\in\Omega. \tag{2.12}$$

It is now easy to see that the necessity of the theorem is valid.
Indeed, if $x_n\to x^*\in\Omega=\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$, then $x^*=Tx^*$, $x^*=Sx^*$ and $x^*=P_C(x^*-\tau_nAx^*)$, which together with Algorithm 2.1, implies that

$$\|w_n-x^*\|\leq(1+\theta_n)(\|x_n-x^*\|+\sigma_n\|x_n-x_{n-1}\|)\to 0\ (n\to\infty),$$

and hence

$$\begin{aligned}\|y_n - x_n\| &\le \|P_C(w_n - \tau_n A w_n) - P_C(x^* - \tau_n A x^*)\| + \|x_n - x^*\| \\ &\le (1+\gamma L)\|w_n - x^*\| + \|x_n - x^*\| \to 0 \ (n \to \infty).\end{aligned}$$

In addition, it is clear that

$$\|x_n - x_{n+1}\| \le \|x_n - x^*\| + \|x_{n+1} - x^*\| \to 0 \ (n \to \infty).$$

Next we show the sufficiency of the theorem. To the aim, we assume

$$\lim_{n\to\infty} (\|x_n - x_{n+1}\| + \|x_n - y_n\|) = 0$$

and divide the proof of the sufficiency into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Fixing $p \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$, we have that $Tp = p,\ Sp = p$, and (2.5) holds, i.e.,

$$\|u_n - p\|^2 \le \|w_n - p\|^2 - (1-\mu)\|w_n - y_n\|^2 - (1-\mu)\|u_n - y_n\|^2. \tag{2.13}$$

This immediately implies that

$$\|u_n - p\| \le \|w_n - p\| \quad \forall n \ge 1. \tag{2.14}$$

From the definition of w_n, we get

$$\begin{aligned}\|w_n - p\| &\le \|T^n x_n - p\| + \sigma_n \|T^n x_n - T^n x_{n-1}\| \\ &\le (1+\theta_n)(\|x_n - p\| + \alpha_n \cdot \tfrac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|).\end{aligned} \tag{2.15}$$

Since $\sup_{n\ge1} \frac{\sigma_n}{\alpha_n} < \infty$ and $\sup_{n\ge1} \|x_n - x_{n-1}\| < \infty$, we know that

$$\sup_{n\ge1} \frac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\| < \infty,$$

which hence implies that there exists a constant $M_1 > 0$ such that

$$\frac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\| \le M_1 \quad \forall n \ge 1. \tag{2.16}$$

Combining (2.14), (2.15) and (2.16), we obtain

$$\|u_n - p\| \le \|w_n - p\| \le (1+\theta_n)(\|x_n - p\| + \alpha_n M_1) \quad \forall n \ge 1. \tag{2.17}$$

From Algorithm 2.1, Lemma 1.4 and (2.17), it follows that for all $n \ge n_0$,

$$\begin{aligned}\|z_n - p\| &\le \alpha_n \delta \|x_n - p\| + (1-\alpha_n\tau)(1+\theta_n)\|u_n - p\| + \alpha_n \|(f - \rho F)p\| \\ &\le [\alpha_n\delta + 1 - \alpha_n\tau + \theta_n(2+\theta_n)](\|x_n - p\| + \alpha_n M_1) + \alpha_n\|(f-\rho F)p\| \\ &\le (1 - \tfrac{\alpha_n(\tau-\delta)}{2})\|x_n - p\| + \alpha_n(M_1 + \|(f-\rho F)p\|),\end{aligned}$$

which together with Lemma 1.6 and $(\gamma_n + \delta_n)\zeta \le \gamma_n$, implies that, for all $n \ge n_0$,

$$\begin{aligned}\|x_{n+1} - p\| &\le \beta_n \|x_n - p\| + (1-\beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n - p) + \delta_n(Sz_n - p)]\| \\ &\le [1 - \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - p\| + \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2} \cdot \tfrac{2(M_1 + \|(f-\rho F)p\|)}{\tau-\delta} \\ &\le \max\left\{\|x_n - p\|, \tfrac{2(M_1+\|(f-\rho F)p\|)}{\tau-\delta}\right\}.\end{aligned}$$

By induction, we obtain

$$\|x_n - p\| \le \max\left\{\|x_{n_0} - p\|, \frac{2(M_1 + \|(\rho F - f)p\|)}{\tau - \delta}\right\},\ \forall n \ge n_0.$$

Thus, $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{Sz_n\}$, $\{T^n u_n\}$ and $\{T^n x_n\}$.

Step 2. We show that for all $n \geq n_0$,

$$\begin{aligned}&(1-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]\\ &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_4,\end{aligned}$$

with constant $M_4 > 0$. Indeed, utilizing Lemma 2.2 and the convexity of $\|\cdot\|^2$, from $(\gamma_n+\delta_n)\zeta \leq \gamma_n$ we obtain that for all $n \geq n_0$,

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n-p)+\delta_n(Tz_n-p)]\|^2\\ &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)\|w_n-p\|^2\\ &\quad-(1-\alpha_n\tau)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]+\alpha_n M_2\},\end{aligned} \tag{3.18}$$

where $\sup_{n\geq 1} 2\|(f-\rho F)p\|\|z_n-p\| \leq M_2$ for some $M_2 > 0$. Also, from (2.17) we have

$$\begin{aligned}\|w_n-p\|^2 &\leq [1+\theta_n(2+\theta_n)][\|x_n-p\|^2+\alpha_n(2M_1\|x_n-p\|+\alpha_n M_1^2)]\\ &\leq \|x_n-p\|^2+\alpha_n M_3,\end{aligned} \tag{2.19}$$

where

$$\sup_{n\geq 1}\{2M_1\|x_n-p\|+\alpha_n M_1^2+\frac{\theta_n}{\alpha_n}(2+\theta_n)[\|x_n-p\|^2+\alpha_n(2M_1\|x_n-p\|+\alpha_n M_1^2)]\} \leq M_3$$

for some $M_3 > 0$. Note that $\alpha_n\delta+(1-\alpha_n\tau)(1+\theta_n) \leq 1-\frac{\alpha_n(\tau-\delta)}{2}$ for all $n \geq n_0$. Substituting (2.19) for (2.18), we deduce that for all $n \geq n_0$,

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\{(1-\tfrac{\alpha_n(\tau-\delta)}{2})\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)\alpha_n M_3\\ &\quad-(1-\alpha_n\tau)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]+\alpha_n M_2\}\\ &\leq \|x_n-p\|^2-(1-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]\\ &\quad+\alpha_n M_4,\end{aligned} \tag{2.20}$$

where $\sup_{n\geq 1}(M_2+(1+\theta_n)M_3) \leq M_4$ for some $M_4 > 0$. This immediately implies that for all $n \geq n_0$,

$$\begin{aligned}&(1-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]\\ &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_4.\end{aligned} \tag{2.21}$$

Step 3. We show that for all $n \geq n_0$,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-p\|^2\\ &\quad+\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}[\tfrac{4}{\tau-\delta}\langle(f-\rho F)p, z_n-p\rangle\\ &\quad+\tfrac{4M}{\tau-\delta}\cdot\tfrac{\sigma_n}{\alpha_n}\cdot\|x_n-x_{n-1}\|+\tfrac{4M^2}{\tau-\delta}\cdot\tfrac{\theta_n}{\alpha_n}],\end{aligned}$$

with constant $M > 0$. Indeed, we have

$$\begin{aligned}\|w_n-p\|^2 &\leq \|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|(2\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|)\\ &\quad+\theta_n(2+\theta_n)(\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|)^2\\ &\leq \|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|M+\theta_n M^2,\end{aligned} \tag{2.22}$$

where $\sup_{n\geq 1}(2+\theta_n)(\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|)\leq M$ for some $M>0$. Combining (2.18) and (2.22), we have that for all $n\geq n_0$,

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)[\|x_n-p\|^2\\
&+\sigma_n\|x_n-x_{n-1}\|M+\theta_nM^2]+2\alpha_n\langle(f-\rho F)p,z_n-p\rangle\}\\
&\leq [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-p\|^2+(1-\beta_n)[\sigma_n\|x_n-x_{n-1}\|2M+\theta_n2M^2]\\
&+2\alpha_n(1-\beta_n)\langle(f-\rho F)p,z_n-p\rangle\\
&=[1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-p\|^2+\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}[\tfrac{4}{\tau-\delta}\langle(f-\rho F)p,z_n-p\rangle\\
&+\tfrac{4M}{\tau-\delta}\cdot\tfrac{\sigma_n}{\alpha_n}\cdot\|x_n-x_{n-1}\|+\tfrac{4M^2}{\tau-\delta}\cdot\tfrac{\theta_n}{\alpha_n}].
\end{aligned} \tag{2.23}$$

Step 4. We show that $\{x_n\}$ converges strongly to a unique solution $x^*\in\Omega$ to the VIP (2.12). Indeed, putting $p=x^*$, we deduce from (2.23) that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-x^*\|^2+\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}\\
&\times[\tfrac{4}{\tau-\delta}\langle(f-\rho F)x^*,z_n-x^*\rangle+\tfrac{4M}{\tau-\delta}\cdot\tfrac{\sigma_n}{\alpha_n}\cdot\|x_n-x_{n-1}\|+\tfrac{4M^2}{\tau-\delta}\cdot\tfrac{\theta_n}{\alpha_n}].
\end{aligned} \tag{2.24}$$

By Lemma 1.2, it suffices to show that $\limsup_{n\to\infty}\langle(f-\rho F)x^*,z_n-x^*\rangle\leq 0$. From (2.21), $x_n-x_{n+1}\to 0$, $\alpha_n\to 0$, $\theta_n\to 0$ and $\{\beta_n\}\subset[a,b]\subset(0,1)$, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}(1-\alpha_n\tau)(1-b)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]\\
&\leq \limsup_{n\to\infty}[\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_nM_4]\\
&\leq \limsup_{n\to\infty}(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|=0.
\end{aligned}$$

This immediately implies that

$$\lim_{n\to\infty}\|w_n-y_n\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|u_n-y_n\|=0. \tag{2.25}$$

Obviously, the assumption $\|x_n-y_n\|\to 0$ together with (2.25), guarantees that $\|w_n-x_n\|\leq\|w_n-y_n\|+\|y_n-x_n\|\to 0$ $(n\to\infty)$. It follows that

$$\begin{aligned}
\|T^nx_n-x_n\| &=\|w_n-x_n-\sigma_n(T^nx_n-T^nx_{n-1})\|\\
&\leq \|w_n-x_n\|+\sigma_n(1+\theta_n)\|x_n-x_{n-1}\|\to 0 \quad (n\to\infty).
\end{aligned} \tag{2.26}$$

Since $z_n=\alpha_nf(x_n)+(I-\alpha_n\rho F)T^nu_n$ with $u_n:=P_{C_n}(w_n-\tau_nAy_n)$, from (2.25), (2.26) and the boundedness of $\{x_n\},\{T^nu_n\}$, we conclude that as $n\to\infty$,

$$\begin{aligned}
&\|z_n-x_n\|=\|\alpha_nf(x_n)-\alpha_n\rho FT^nu_n+T^nu_n-x_n\|\\
&\leq \alpha_n(\|f(x_n)\|+\|\rho FT^nu_n\|)+\|T^nu_n-x_n\|\\
&\leq \alpha_n(\|f(x_n)\|+\|\rho FT^nu_n\|)+(1+\theta_n)(\|u_n-y_n\|+\|y_n-x_n\|)+\|T^nx_n-x_n\|\\
&\to 0
\end{aligned} \tag{2.27}$$

(due to the assumption $\|x_n-y_n\|\to 0$). Obviously, the limit $\lim_{n\to\infty}\|w_n-x_n\|=0$ together with (2.27), guarantees that $\|w_n-z_n\|\leq\|w_n-x_n\|+\|x_n-z_n\|\to 0$ $(n\to\infty)$. From the boundedness of $\{z_n\}$, it follows that there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle(f-\rho F)x^*,z_n-x^*\rangle=\lim_{k\to\infty}\langle(f-\rho F)x^*,z_{n_k}-x^*\rangle. \tag{2.28}$$

Since H is reflexive and $\{z_n\}$ is bounded, we may assume, without loss of generality, that $z_{n_k} \rightharpoonup \tilde{z}$. Hence from (2.28) we get

$$\limsup_{n\to\infty}\langle (f-\rho F)x^*, z_n - x^*\rangle = \lim_{k\to\infty}\langle (f-\rho F)x^*, z_{n_k} - x^*\rangle = \langle (f-\rho F)x^*, \tilde{z} - x^*\rangle. \tag{2.29}$$

It is easy to see from $w_n - z_n \to 0$ and $z_{n_k} \rightharpoonup \tilde{z}$ that $w_{n_k} \rightharpoonup \tilde{z}$.
Since $T^n x_n - T^{n+1}x_n \to 0,\ x_n - x_{n+1} \to 0,\ w_n - x_n \to 0,\ w_n - z_n \to 0$ and $w_{n_k} \rightharpoonup \tilde{z}$, by Lemma 2.3 we infer that $\tilde{z} \in \Omega$. Therefore, from (2.12) and (2.29) we conclude that

$$\limsup_{n\to\infty}\langle (f-\rho F)x^*, z_n - x^*\rangle = \langle (f-\rho F)x^*, \tilde{z} - x^*\rangle \le 0. \tag{2.30}$$

Note that $\{\beta_n\} \subset [a,b] \subset (0,1)$, $\{\frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}\} \subset [0,1]$, $\sum_{n=1}^{\infty} \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2} = \infty$, and

$$\limsup_{n\to\infty}\left[\frac{4}{\tau-\delta}\langle (f-\rho F)x^*, z_n - x^*\rangle + \frac{4M}{\tau-\delta}\cdot\frac{\sigma_n}{\alpha_n}\cdot\|x_n - x_{n-1}\| + \frac{4M^2}{\tau-\delta}\cdot\frac{\theta_n}{\alpha_n}\right] \le 0. \tag{2.31}$$

Consequently, applying Lemma 1.2 to (2.24), we have $\lim_{n\to 0}\|x_n - x^*\| = 0$. This completes the proof.

Next, we introduce another asymptotic inertial subgradient extragradient algorithm with line-search process.

Algorithm 2.2.
Initialization: Given $\gamma > 0,\ l \in (0,1),\ \mu \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Set $w_n = T^n x_n + \sigma_n(T^n x_n - T^n x_{n-1})$ and compute $y_n = P_C(w_n - \tau_n A w_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\tau\|Aw_n - Ay_n\| \le \mu\|w_n - y_n\|. \tag{2.32}$$

Step 2. Compute $z_n = \alpha_n f(x_n) + (I - \alpha_n \rho F)T^n P_{C_n}(w_n - \tau_n A y_n)$ with

$$C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \le 0\}.$$

Step 3. Compute

$$x_{n+1} = \beta_n w_n + \gamma_n z_n + \delta_n S z_n. \tag{2.33}$$

Again set $n := n+1$ and go to Step 1.

It is worth pointing out that Lemmas 2.1, 2.2 and 2.3 are still valid for Algorithm 2.2.

Theorem 2.2. *Let the sequence $\{x_n\}$ be generated by Algorithm 2.2. Assume that $T^n x_n - T^{n+1}x_n \to 0$. Then*

$$x_n \to x^* \in \Omega \ \Leftrightarrow \ \begin{cases} x_n - x_{n+1} \to 0, \\ x_n - y_n \to 0 \end{cases}$$

where $x^* \in \Omega$ is a unique solution to the VIP: $\langle (\rho F - f)x^*, p - x^*\rangle \ge 0\ \forall p \in \Omega$.

Proof. Utilizing the same arguments as in the proof of Theorem 2.1, we deduce that there exists a unique solution $x^* \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ to the VIP (2.12), and that the necessity of the theorem is valid.
Next we show the sufficiency of the theorem. To the aim, we assume

$$\lim_{n\to\infty}(\|x_n - x_{n+1}\| + \|x_n - y_n\|) = 0$$

and divide the proof of the sufficiency into several steps.
Step 1. We show that $\{x_n\}$ is bounded. Indeed, utilizing the same arguments as in Step 1 of the proof of Theorem 2.1, we obtain that inequalities (2.13)-(2.17) hold. Taking into account $\lim_{n\to\infty} \frac{\theta_n(2+\theta_n)}{\alpha_n(1-\beta_n)} = 0$, we know that

$$\theta_n(2+\theta_n) \le \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2},\ \forall n \ge n_0$$

for some $n_0 \ge 1$. Hence we deduce that for all $n \ge n_0$,

$$\begin{array}{l} \quad \alpha_n(1-\beta_n)\delta + [1-\alpha_n(1-\beta_n)\tau](1+\theta_n)^2 \\ = 1-\alpha_n(1-\beta_n)(\tau-\delta) + [1-\alpha_n(1-\beta_n)\tau]\theta_n(2+\theta_n) \\ \le 1 - \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}. \end{array}$$

Also, from Algorithm 2.2, Lemma 1.4 and (2.17), it follows that

$$\begin{array}{rl} \|z_n - p\| & \le \alpha_n\delta\|x_n - p\| + (1-\alpha_n\tau)(1+\theta_n)\|u_n - p\| + \alpha_n\|(f-\rho F)p\| \\ & \le \alpha_n\delta\|x_n - p\| + (1-\alpha_n\tau)(1+\theta_n)\|w_n - p\| + \alpha_n\|(f-\rho F)p\|, \end{array}$$

which together with Lemma 1.6 and $(\gamma_n + \delta_n)\zeta \le \gamma_n$, implies that for all $n \ge n_0$,

$$\begin{array}{l} \quad \|x_{n+1} - p\| \le \beta_n\|w_n - p\| + (1-\beta_n)\|\frac{1}{1-\beta_n}[\gamma_n(z_n - p) + \delta_n(Tz_n - p)]\| \\ \le \beta_n\|w_n - p\| + (1-\beta_n)\|z_n - p\| \\ \le [\alpha_n(1-\beta_n)\delta + (1-\alpha_n(1-\beta_n)\tau)(1+\theta_n)^2](\|x_n - p\| + \alpha_n M_1) \\ \ +\alpha_n(1-\beta_n)\|(f-\rho F)p\| \\ \le [1 - \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - p\| + \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2} \cdot \frac{2(\frac{M_1}{1-\beta_n} + \|(f-\rho F)p\|)}{\tau-\delta} \\ \le \max\left\{\|x_n - p\|, \frac{2(\frac{M_1}{1-b} + \|(f-\rho F)p\|)}{\tau-\delta}\right\}. \end{array}$$

By induction, we obtain

$$\|x_n - p\| \le \max\left\{\|x_{n_0} - p\|, \frac{2(\frac{M_1}{1-b} + \|(f-\rho F)p\|)}{\tau-\delta}\right\},\ \forall n \ge n_0.$$

Thus, $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{Sz_n\}$, $\{T^n u_n\}$ and $\{T^n x_n\}$.
Step 2. We show that for all $n \ge n_0$,

$$\begin{array}{l} \quad (1-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \\ \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4, \end{array}$$

with constant $M_4 > 0$. Indeed, utilizing Lemma 1.6, Lemma 2.2 and the convexity of $\|\cdot\|^2$, from $(\gamma_n + \delta_n)\zeta \le \gamma_n$ we get

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\le \beta_n\|w_n - p\|^2 + (1 - \beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n - p) + \delta_n(Sz_n - p)]\|^2 \\ &\le \beta_n\|w_n - p\|^2 + (1 - \beta_n)\{\alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)(1 + \theta_n)\|w_n - p\|^2 \\ &\quad -(1 - \alpha_n\tau)(1 + \theta_n)(1 - \mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] + \alpha_n M_2\},\end{aligned} \tag{2.34}$$

where $\sup_{n\ge1} 2\|(f - \rho F)p\|\|z_n - p\| \le M_2$ for some $M_2 > 0$. Also, from (2.17) we have

$$\begin{aligned}\|w_n - p\|^2 &\le \|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2) \\ &\quad +\theta_n(2 + \theta_n)[\|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2)] \\ &\le \|x_n - p\|^2 + \alpha_n M_3,\end{aligned} \tag{2.35}$$

where

$$\sup_{n\ge1}\{2M_1\|x_n - p\| + \alpha_n M_1^2 + \frac{\theta_n}{\alpha_n}(2 + \theta_n)[\|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2)]\} \le M_3$$

for some $M_3 > 0$. Note that

$$\alpha_n\delta + (1 - \alpha_n\tau)(1 + \theta_n) \le 1 - \frac{\alpha_n(\tau - \delta)}{2}$$

for all $n \ge n_0$. From (2.34) and (2.35), we obtain that for all $n \ge n_0$,

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\le \beta_n\|w_n - p\|^2 + (1 - \beta_n)\{\alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)(1 + \theta_n)[\|x_n - p\|^2 \\ &\quad +\alpha_n M_3] - (1 - \alpha_n\tau)(1 + \theta_n)(1 - \mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] + \alpha_n M_2\} \\ &\le [1 - \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - p\|^2 + \beta_n\alpha_n M_3 + (1 - \beta_n)(1 - \alpha_n\tau)(1 + \theta_n)\alpha_n M_3 \\ &\quad -(1 - \alpha_n\tau)(1 - \beta_n)(1 + \theta_n)(1 - \mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] + (1 - \beta_n)\alpha_n M_2 \\ &\le \|x_n - p\|^2 - (1 - \alpha_n\tau)(1 - \beta_n)(1 + \theta_n)(1 - \mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \\ &\quad +\alpha_n M_4,\end{aligned} \tag{2.36}$$

where $\sup_{n\ge1}(M_2 + (1 + \theta_n)M_3) \le M_4$ for some $M_4 > 0$. This immediately implies that for all $n \ge n_0$,

$$\begin{aligned}&(1 - \alpha_n\tau)(1 - \beta_n)(1 + \theta_n)(1 - \mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4.\end{aligned} \tag{2.37}$$

Step 3. We show that for all $n \ge n_0$,

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\le [1 - \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - p\|^2 + \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}[\tfrac{4}{\tau-\delta}\langle(f - \rho F)p, z_n - p\rangle \\ &\quad + \tfrac{4M}{(\tau-\delta)(1-b)} \cdot \tfrac{\sigma_n}{\alpha_n} \cdot \|x_n - x_{n-1}\| + \tfrac{4M^2}{(\tau-\delta)(1-b)} \cdot \tfrac{\theta_n}{\alpha_n}],\end{aligned}$$

with constant $M > 0$. Indeed, we have

$$\|w_n - p\|^2 \le \|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|M + \theta_n M^2, \tag{2.38}$$

where $\sup_{n\ge1}(2 + \theta_n)(\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|) \le M$ for some $M > 0$. Note that

$$\alpha_n\delta + (1 - \alpha_n\tau)(1 + \theta_n) \le 1 - \frac{\alpha_n(\tau - \delta)}{2}$$

for all $n \geq n_0$. Thus, combining (2.34) and (2.38), we have that for all $n \geq n_0$,

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \beta_n\|w_n - p\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)[\|x_n - p\|^2 \\
&+\sigma_n\|x_n - x_{n-1}\|M + \theta_n M^2] + 2\alpha_n\langle(f-\rho F)p, z_n - p\rangle\} \\
&\leq [1 - \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - p\|^2 + (1+\theta_n)[\sigma_n\|x_n - x_{n-1}\|M + \theta_n M^2] \\
&+2\alpha_n(1-\beta_n)\langle(f-\rho F)p, z_n - p\rangle \\
&= [1 - \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - p\|^2 + \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}[\tfrac{4}{\tau-\delta}\langle(f-\rho F)p, z_n - p\rangle \\
&+\tfrac{4M}{(\tau-\delta)(1-b)}\cdot\tfrac{\sigma_n}{\alpha_n}\cdot\|x_n - x_{n-1}\| + \tfrac{4M^2}{(\tau-\delta)(1-b)}\cdot\tfrac{\theta_n}{\alpha_n}].
\end{aligned} \tag{2.39}$$

Step 4. We show that $\{x_n\}$ converges strongly to a unique solution $x^* \in \Omega$ to the VIP (2.12). Indeed, utilizing the same argument as in Step 4 of the proof of Theorem 2.1, we obtain the desired assertion. This completes the proof.

It is remarkable that our results improve and extend the results in Kraikaew and Saejung [10], Thong and Hieu [16, 15] and Yao et al. [20]. In what follows, our results are applied to solve the VIP and CFPP in an illustrated example. The initial point $x_0 = x_1$ is randomly chosen in $\mathbf{R}$.
Take $f(x) = F(x) = \frac{1}{2}x$, $\gamma = l = \mu = \frac{1}{2}$, $\sigma_n = \alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{3}$, $\gamma_n = \frac{1}{2}$, $\delta_n = \frac{1}{6}$ and $\rho = 2$. Then we know that $\delta = \kappa = \eta = \frac{1}{2}$, and

$$\tau = 1 - \sqrt{1-\rho(2\eta-\rho\kappa^2)} = 1 - \sqrt{1 - 2(2\cdot\frac{1}{2} - 2(\frac{1}{2})^2)} = 1 \in (0,1].$$

We first provide an example of Lipschitz continuous and pseudomonotone mapping A, asymptotically nonexpansive mapping T and strictly pseudocontractive mapping S with $\Omega = \mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A) \neq \emptyset$. Let $C = [-1,1]$ and $H = \mathbf{R}$ with the inner product $\langle a, b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. Let $A, T, S : H \to H$ be defined as $Ax := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|}$, $Tx := \frac{2}{3}\sin x$ and $Sx := \frac{3}{8}x + \frac{1}{2}\sin x$ for all $x \in H$. Now, we first show that A is pseudomonotone and Lipschitz continuous with $L = 2$. Indeed, for all $x, y \in H$ we have

$$\begin{aligned}
\|Ax - Ay\| &= |\tfrac{1}{1+\|\sin x\|} - \tfrac{1}{1+\|x\|} - \tfrac{1}{1+\|\sin y\|} + \tfrac{1}{1+\|y\|}| \\
&\leq |\tfrac{1}{1+\|\sin x\|} - \tfrac{1}{1+\|\sin y\|}| + |\tfrac{1}{1+\|x\|} - \tfrac{1}{1+\|y\|}| \\
&= |\tfrac{1+\|\sin y\|-1-\|\sin x\|}{(1+\|\sin x\|)(1+\|\sin y\|)}| + |\tfrac{1+\|y\|-1-\|x\|}{(1+\|x\|)(1+\|y\|)}| \\
&= |\tfrac{\|\sin y\|-\|\sin x\|}{(1+\|\sin x\|)(1+\|\sin y\|)}| + |\tfrac{\|y\|-\|x\|}{(1+\|x\|)(1+\|y\|)}| \\
&\leq \tfrac{\|\sin x - \sin y\|}{(1+\|\sin x\|)(1+\|\sin y\|)} + \tfrac{\|x-y\|}{(1+\|x\|)(1+\|y\|)} \\
&\leq \|\sin x - \sin y\| + \|x - y\| \\
&\leq 2\|x - y\|.
\end{aligned}$$

This implies that A is Lipschitz continuous with $L = 2$. Next, we show that A is pseudomonotone. For any given $x, y \in H$, it is clear that the relation holds:

$$\langle Ax, y - x\rangle = \left(\frac{1}{1+|\sin x|} - \frac{1}{1+|x|}\right)(y - x) \geq 0$$

$$\Rightarrow \langle Ay, y - x\rangle = \left(\frac{1}{1+|\sin y|} - \frac{1}{1+|y|}\right)(y - x) \geq 0.$$

Furthermore, it is easy to see that T is asymptotically nonexpansive with

$$\theta_n = \left(\frac{2}{3}\right)^n, \ \forall n \geq 1,$$

such that $\|T^{n+1}x_n - T^n x_n\| \to 0$ as $n \to \infty$. Indeed, we observe that

$$\|T^n x - T^n y\| \leq \frac{2}{3}\|T^{n-1}x - T^{n-1}y\| \leq \cdots \leq \left(\frac{2}{3}\right)^n \|x-y\| \leq (1+\theta_n)\|x-y\|,$$

and

$$\begin{aligned}\|T^{n+1}x_n - T^n x_n\| &\leq \left(\frac{2}{3}\right)^{n-1} \|T^2 x_n - Tx_n\| = \left(\frac{2}{3}\right)^{n-1} \left\|\frac{2}{3}\sin(Tx_n) - \frac{2}{3}\sin x_n\right\| \\ &\leq 2\left(\frac{2}{3}\right)^n \to 0 \ (n \to \infty).\end{aligned}$$

It is clear that $\mathrm{Fix}(T) = \{0\}$ and

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = \lim_{n\to\infty} \frac{(2/3)^n}{1/(n+1)} = 0.$$

In addition, it is clear that S is strictly pseudocontractive with constant $\zeta = \frac{3}{4}$. Indeed, we observe that for all $x, y \in H$,

$$\|Sx - Sy\|^2 \leq [\frac{3}{8}\|x-y\| + \frac{1}{2}\|\sin x - \sin y\|]^2 \leq \|x-y\|^2 + \frac{3}{4}\|(I-S)x - (I-S)y\|^2.$$

It is clear that $(\gamma_n + \delta_n)\zeta = (\frac{1}{2} + \frac{1}{6}) \cdot \frac{3}{4} \leq \frac{1}{2} = \gamma_n$ for all $n \geq 1$. Therefore, $\Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{VI}(C, A) = \{0\} \neq \emptyset$. In this case, Algorithm 2.1 can be rewritten as follows:

$$\begin{cases} w_n = T^n x_n + \frac{1}{n+1}(T^n x_n - T^n x_{n-1}), \\ y_n = P_C(w_n - \tau_n A w_n), \\ z_n = \frac{1}{n+1} \cdot \frac{1}{2} x_n + \frac{n}{n+1} T^n P_{C_n}(w_n - \tau_n A y_n), \\ x_{n+1} = \frac{1}{3} x_n + \frac{1}{2} z_n + \frac{1}{6} S z_n \quad \forall n \geq 1, \end{cases} \tag{2.40}$$

where for each $n \geq 1$, C_n and τ_n are chosen as in Algorithm 2.1. Then, by Theorem 2.1, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ if and only if $|x_n - x_{n+1}| + |x_n - y_n| \to 0$ as $n \to \infty$.

On the other hand, Algorithm 2.2 can be rewritten as follows:

$$\begin{cases} w_n = T^n x_n + \frac{1}{n+1}(T^n x_n - T^n x_{n-1}), \\ y_n = P_C(w_n - \tau_n A w_n), \\ z_n = \frac{1}{n+1} \cdot \frac{1}{2} x_n + \frac{n}{n+1} T^n P_{C_n}(w_n - \tau_n A y_n), \\ x_{n+1} = \frac{1}{3} w_n + \frac{1}{2} z_n + \frac{1}{6} S z_n \quad \forall n \geq 1, \end{cases} \tag{2.41}$$

where for each $n \geq 1$, C_n and τ_n are chosen as in Algorithm 2.2. Then, by Theorem 2.2, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ if and only if $|x_n - x_{n+1}| + |x_n - y_n| \to 0$ as $n \to \infty$.

References

[1] Y. Censor, A. Gibali, S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space,* J. Optim. Theory Appl., **148**(2011), 318-335.
[2] S.S. Chang, Y.J. Cho, H. Zhou, *Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings*, J. Korean Math. Soc., **38**(2001), 1245-1260.
[3] C.E. Chidume, O.M. Romanus, U.V. Nnyaba, *An iterative algorithm for solving split equilibrium problems and split equality variational inclusions for a class of nonexpansive-type maps*, Optimization, **67**(2018), 1949-1962.
[4] S.Y. Cho, *Generalized mixed equilibrium and fixed point problems in a Banach space,* J. Nonlinear Sci. Appl., **9**(2016), 1083-1092.
[5] S.Y. Cho, *Strong convergence analysis of a hybrid algorithm for nonlinear operators in a Banach space*, J. Appl. Anal. Comput., **8**(2018), 19-31.
[6] S.Y. Cho, S.M. Kang, *Approximation of fixed points of pseudocontraction semigroups based on a viscosity iterative process*, Appl. Math. Lett., **24**(2011), 224-228.
[7] S.Y. Cho, W. Li, S.M. Kang, *Convergence analysis of an iterative algorithm for monotone operators*, J. Inequal. Appl., **2013**(2013), 199.
[8] R.W. Cottle, J.C. Yao, *Pseudo-monotone complementarity problems in Hilbert space,* J. Optim. Theory Appl., **75**(1992), 281-295.
[9] G.M. Korpelevich, *The extragradient method for finding saddle points and other problems,* Ekonomikai Matematicheskie Metody, **12**(1976), 747-756.
[10] R. Kraikaew, S. Saejung, *Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces,* J. Optim. Theory Appl., **163**(2014), 399-412.
[11] L. Liu, *A hybrid steepest descent method for solving split feasibility problems involving nonexpansive mappings*, J. Nonlinear Convex Anal., **20**(2019), 471-488.
[12] W. Takahashi, *The split feasibility problem in Banach spaces*, J. Nonlinear Convex Anal., **15** (2014), 1349–1335.
[13] S. Takahashi, W. Takahashi, *The split common null point problem and the shrinking projection method in Banach spaces*, Optimization, **65**(2016), 281-287.
[14] B. Tan, S. Xu, S. Li, *Inertial shrinking projection algorithms for solving hierarchical variational inequality problems*, J. Nonlinear Convex Anal., **21**(2020), 871-884.
[15] D.V. Thong, D.V. Hieu, *Modified subgradient extragradient method for variational inequality problems,* Numer. Algo., **79**(2018), 597-610.
[16] D.V. Thong, D.V. Hieu, *Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems,* Numer. Algo., **80**(2019), 1283-1307.
[17] N.T.T. Thuy, P.T. Hieu, *A hybrid method for solving variational inequalities over the common fixed point sets of infinite families of nonexpansive mappings in Banach spaces*, Optimization, **69**(2020), 2155-2176.
[18] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66**(2002), 240-256.
[19] Y. Yamada, *The hybrid steepest-descent method for variational inequalities problems over the intersection of the fixed point sets of nonexpansive mappings*, In: Butnariu, D., Censor, Y., Reich, S. (eds.), Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, pp. 473-504, North-Holland, Amsterdam, 2001.
[20] Y. Yao, Y.C. Liou, S.M. Kang, *Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method,* Comput. Math. Appl., **59**(2010), 3472-3480.
[21] H. Zhou, *Convergence theorems of common fixed points for a finite family of Lipschitz pseudocontractions in Banach spaces,* Nonlinear Anal., **68**(2008), 2977–2983.

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