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BOUNDARY VALUE PROBLEM FOR DIFFERENTIAL EQUATIONS WITH GENERALIZED HILFER-TYPE FRACTIONAL DERIVATIVE

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Abstract. In this paper, we establish the existence and uniqueness of solutions to boundary value problem for differential equations with generalized Hilfer type fractional derivative. The arguments are based upon the Banach contraction principle and Krasnoselskii's fixed point theorem. An example is included to show the applicability of our results.
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1. Introduction

Differential equations of fractional order occur more frequently on different research areas and engineering, such as physics, chemistry, economics. control of dynamical, etc. Naturally, such equations required to be solved. Analogues to the Cauchy and Dirichlet problems for differential equations of fractional order often arose in applications. There are numerous books and articles focused in this direction, that is, concerning the linear and nonlinear initial value problems for fractional differential equations involving different kinds of fractional derivatives, see for instance [2, 3, 4, 5, 6, 9]. Whereas there are less works for boundary value problems for fractional differential equations [11].

Fractional derivatives are generalizations for derivative of integral order. There are several kinds of fractional derivatives, such as, Riemann–Liouville fractional derivative, Marchaud fractional derivative, Caputo derivative, Grunwald–Letnikov fractional derivative, generalized Hilfer derivative etc. (see [1, 12, 13, 22, 25]). There have appeared a number of works, especially in the theory of viscoelasticity and in hereditary solid mechanics, where fractional derivatives are used for a better description of material properties. Mathematical modelling based on enhanced rheological models naturally leads to differential equations of fractional order and to the necessity of the formulation of initial conditions to such equations. In [24] the authors provide some properties of Caputo-type modification of the Erdélyi-Kober fractional derivative. More details on the Erdélyi-Kober fractional integral and fractional derivative are given in [8, 10, 14, 21, 22, 23].

In this paper, we establish existence and uniqueness results to the boundary value problem of the following generalized Hilfer type fractional differential equation:

$$\left({}^{\rho}D_{a^+}^{\alpha,\beta}y\right)(t) = f\left(t, y(t), \left({}^{\rho}D_{a^+}^{\alpha,\beta}y\right)(t)\right), \text{ for each }, \ t \in (a,b], \quad 0 < a < b < +\infty, \tag{1.1}$$

$$u\left({}^{\rho}I_{a^+}^{1-\gamma}y\right)(a^+) + v\left({}^{\rho}I_{a^+}^{1-\gamma}y\right)(b) = w, \tag{1.2}$$

where ${}^{\rho}D_{a^+}^{\alpha,\beta}, {}^{\rho}I_{a^+}^{1-\gamma}$ are the generalized Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized fractional integral of order $1-\gamma, (\gamma = \alpha + \beta - \alpha\beta)$ respectively, $f : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and u, v, w are real with $u + v \neq 0$.

The present paper is organized as follows. In Section 2, some notations are introduced and we recall some concepts of preliminaries about generalized Hilfer type fractional derivative and auxiliary results. In Section 3, two results for the problem (1.1)-(1.2) are presented: the first one is based on the Banach contraction principle, the second one on Krasnoselskii's fixed point theorem. Finally, in the last section, we give an example to illustrate the applicability of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $0 < a < b, J = [a,b]$. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

We consider the weighted spaces of continuous functions

$$C_{\gamma,\rho}(J) = \left\{ y : (a,b] \to \mathbb{R} : \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma y(t) \in C(J,\mathbb{R}) \right\}, 0 \leq \gamma < 1,$$

and

$$C^n_{\gamma,\rho}(J) = \left\{ y \in C^{n-1}(J) : y^{(n)} \in C_{\gamma,\rho}(J) \right\}, n \in \mathbb{N},$$
$$C^0_{\gamma,\rho}(J) = C_{\gamma,\rho}(J),$$

with the norms

$$\|y\|_{C_{\gamma,\rho}} = \sup_{t\in J}\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma y(t)\right|$$

and

$$\|y\|_{C^n_{\gamma,\rho}} = \sum_{k=0}^{n-1}\|y^{(k)}\|_\infty + \|y^{(n)}\|_{C_{\gamma,\rho}}.$$

Consider the space $X^p_c(a,b),\ (c\in\mathbb{R},\ 1\le p\le\infty)$ of those complex-valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{X^p_c}<\infty$, where the norm is defined by

$$\|f\|_{X^p_c} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}},\quad (1\le p<\infty, c\in\mathbb{R}).$$

In particular, when $c=\frac{1}{p}$, the space $X^p_c(a,b)$ coincides with the $L_p(a,b)$ space: $X^p_{\frac{1}{p}}(a,b) = L_p(a,b)$.

Definition 2.1. ([15, 20, 21]) (Generalized fractional integral).
Let $\alpha\in\mathbb{R}_+, c\in\mathbb{R}$ and $g\in X^p_c(a,b)$. The generalized fractional integral of order α is defined by

$$\left({}^\rho I^\alpha_{a+} g\right)(t) = \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{g(s)}{\Gamma(\alpha)}ds,\ t>a, \rho>0,$$

where $\Gamma(\cdot)$ is the Euler gamma function defined by $\Gamma(\alpha)=\displaystyle\int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha>0.$

Definition 2.2. ([15, 20, 21]) (Generalized fractional derivative).
Let $\alpha\in\mathbb{R}_+\setminus\mathbb{N}$ and $\rho>0$. The generalized fractional derivative ${}^\rho D^\alpha_{a+}$ of order α is defined by

$$\begin{aligned}\left({}^\rho D^\alpha_{a+} g\right)(t) &= \delta^n_\rho\left({}^\rho I^{n-\alpha}_{a+} g\right)(t)\\ &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{n-\alpha-1}\frac{g(s)}{\Gamma(n-\alpha)}ds,\ t>a, \rho>0,\end{aligned}$$

where $n=[\alpha]+1$ and $\delta^n_\rho = \left(t^{1-\rho}\dfrac{d}{dt}\right)^n$.

Theorem 2.3. [21] *Let $\alpha>0, \beta>0, 1\le p\le\infty, 0<a<b<\infty$ and $\rho, c\in\mathbb{R}, \rho\ge c$. Then, for $g\in X^p_c(a,b)$ the semigroup property is valid, i.e.*

$$\left({}^\rho I^\alpha_{a+}\ {}^\rho I^\beta_{a+} g\right)(t) = \left({}^\rho I^{\alpha+\beta}_{a+} g\right)(t).$$

Lemma 2.4. [20, 21, 26] *Let $\alpha>0$, and $0\le\gamma<1$. Then, ${}^\rho I^\alpha_{a+}$ is bounded from $C_{\gamma,\rho}(J)$ into $C_{\gamma,\rho}(J)$.*

Lemma 2.5. [26] *Let $0<a<b<\infty, \alpha>0, 0\le\gamma<1$ and $y\in C_{\gamma,\rho}(J)$. If $\alpha>\gamma$, then ${}^\rho I^\alpha_{a+} y$ is continuous on J and*

$$\left({}^\rho I^\alpha_{a+} y\right)(a) = \lim_{t\to a^+}\left({}^\rho I^\alpha_{a+} y\right)(t) = 0$$

Lemma 2.6. [7] *Let $x > a$. Then, for $\alpha \geq 0$ and $\beta > 0$, we have*

$$\begin{aligned} \left[{}^\rho I^\alpha_{a^+} \left(\frac{s^\rho - a^\rho}{\rho} \right)^{\beta-1} \right](t) &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha+\beta-1} \\ \left[{}^\rho D^\alpha_{a^+} \left(\frac{s^\rho - a^\rho}{\rho} \right)^{\alpha-1} \right](t) &= 0, \quad 0 < \alpha < 1. \end{aligned}$$

Lemma 2.7. [26] *Let $\alpha > 0, 0 \leq \gamma < 1$ and $g \in C_\gamma[a,b]$. Then,*

$$\left({}^\rho D^\alpha_{a^+} \, {}^\rho I^\alpha_{a^+} g \right)(t) = g(t), \quad \textit{for all} \quad t \in (a,b].$$

Lemma 2.8. [26] *Let $0 < \alpha < 1, 0 \leq \gamma < 1$. If $g \in C_{\gamma,\rho}[a,b]$ and ${}^\rho I^{1-\alpha}_{a^+} g \in C^1_{\gamma,\rho}[a,b]$, then*

$$\left({}^\rho I^\alpha_{a^+} \, {}^\rho D^\alpha_{a^+} g \right)(t) = g(t) - \frac{\left({}^\rho I^{1-\alpha}_{a^+} g \right)(a)}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1}, \quad \textit{for all} \quad t \in (a,b].$$

Definition 2.9. ([26]) Let order α and type β satisfy $n-1 < \alpha < n$ and $0 \leq \beta \leq 1$, with $n \in \mathbb{N}$. The generalized Hilfer type fractional derivative to t, with $\rho > 0$ of a function $g \in C_{1-\gamma,\rho}[a,b]$, is defined by

$$\begin{aligned} \left({}^\rho D^{\alpha,\beta}_{a^+} g \right)(t) &= \left({}^\rho I^{\beta(n-\alpha)}_{a^+} \left(t^{\rho-1} \frac{d}{dt} \right)^n \, {}^\rho I^{(1-\beta)(n-\alpha)}_{a^+} g \right)(t) \\ &= \left({}^\rho I^{\beta(n-\alpha)}_{a^+} \delta^n_\rho \, {}^\rho I^{(1-\beta)(n-\alpha)}_{a^+} g \right)(t). \end{aligned}$$

In this paper we consider the case $n = 1$ only, because $0 < \alpha < 1$.

Property 2.10. ([26]) The operator ${}^\rho D^{\alpha,\beta}_{a^+}$ can be written as

$${}^\rho D^{\alpha,\beta}_{a^+} = {}^\rho I^{\beta(1-\alpha)}_{a^+} \delta_\rho \, {}^\rho I^{1-\gamma}_{a^+} = {}^\rho I^{\beta(1-\alpha)}_{a^+} \, {}^\rho D^\gamma_{a^+}, \quad \gamma = \alpha + \beta - \alpha\beta.$$

Property 2.11. The fractional derivative ${}^\rho D^{\alpha,\beta}_{\alpha^+}$ is an interpolator of the following fractional derivatives: Hilfer ($\rho \to 1$) [18], Hilfer–Hadamard ($\rho \to 0^+$) [20], Caputo–type ($\beta = 1$) [26], Riemann–Liouville ($\beta = 0, \rho \to 1$) [21], Hadamard ($\beta = 0, \rho \to 0^+$) [21], Caputo ($\beta = 1, \rho \to 1$) [21], Caputo–Hadamard ($\beta = 1, \rho \to 0^+$) [16], Liouville ($\beta = 0, \rho \to 1, a = 0$) [21] and Weyl ($\beta = 0, \rho \to 1, a = -\infty$)[19].

Consider the following parameters α, β, γ satisfying

$$\gamma = \alpha + \beta - \alpha\beta, \quad 0 < \alpha, \beta, \gamma < 1.$$

Thus, we define the spaces

$$C^{\alpha,\beta}_{1-\gamma,\rho}(J) = \left\{ y \in C_{1-\gamma,\rho}(J), \; {}^\rho D^{\alpha,\beta}_{a^+} y \in C_{1-\gamma,\rho}(J) \right\}$$

and

$$C^{\gamma}_{1-\gamma,\rho}(J) = \left\{ y \in C_{1-\gamma,\rho}(J), \; {}^\rho D^{\gamma}_{a^+} y \in C_{1-\gamma,\rho}(J) \right\}.$$

Since ${}^\rho D^{\alpha,\beta}_{a^+} y = {}^\rho I^{\gamma(1-\alpha)}_{a^+} \, {}^\rho D^\gamma_{a^+} y$, it follows from Lemma 2.4 that

$$C^{\gamma}_{1-\gamma,\rho}(J) \subset C^{\alpha,\beta}_{1-\gamma,\rho}(J) \subset C_{1-\gamma,\rho}(J).$$

Lemma 2.12. [26] *Let* $0 < \alpha < 1, 0 \leq \beta \leq 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *If* $y \in C^{\gamma}_{1-\gamma,\rho}(J)$, *then*

$$^{\rho}I^{\gamma}_{a^+}\ {}^{\rho}D^{\gamma}_{a^+}y = {}^{\rho}I^{\alpha}_{a^+}\ {}^{\rho}D^{\alpha,\beta}_{a^+}y$$

and

$$^{\rho}D^{\gamma}_{a^+}\ {}^{\rho}I^{\alpha}_{a^+}y = {}^{\rho}D^{\beta(1-\alpha)}_{a^+}y.$$

Lemma 2.13. *(Theorem 4.1,* [26]*). Let* $f : J \times \mathbb{R} \to \mathbb{R}$ *be a function such that* $f(\cdot, y(\cdot)) \in C_{1-\gamma,\rho}(J)$, *for any* $y \in C_{1-\gamma,\rho}(J)$. *Then* $y \in C^{\gamma}_{1-\gamma,\rho}(J)$ *is a solution of the differential equation:*

$$\left({}^{\rho}D^{\alpha,\beta}_{a^+}y\right)(t) = f(t, y(t)),\ \textit{for each}\ ,\ t \in (a, b],\ 0 < \alpha < 1,\ 0 \leq \beta \leq 1,$$

if and only if y *satisfies the following Volterra integral equation:*

$$y(t) = \frac{\left({}^{\rho}I^{1-\gamma}_{a^+}y\right)(a^+)}{\Gamma(\gamma)}\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s, y(s))ds,$$

where $\gamma = \alpha + \beta - \alpha\beta$.

Theorem 2.14. [27]*($C_{1-\gamma}$ type Arzela–Ascoli Theorem) Let* $A \subset C_{1-\gamma}(J, \mathbb{R})$. *A is relatively compact (i.e* $\overline{A}$ *is compact) if:*
1) *A is uniformly bounded i.e, there exists* $M > 0$ *such that*

$$|f(x)| < M \textit{ for every } f \in A \textit{ and } x \in J.$$

2) *A is equicontinuous i.e, for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for each* $x, \overline{x} \in J$, $|x - \overline{x}| \leq \delta$ *implies* $|f(x) - f(\overline{x})| \leq \epsilon$, *for every* $f \in A$.

Theorem 2.15. *(*[17]*) (Banach's fixed point theorem). Let C be a non-empty closed subset of a Banach space E, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 2.16. *(*[17]*) (Krasnoselskii's fixed point theorem). Let M be a closed, convex, and nonempty subset of a Banach space X, and A, B the operators such that*
1) $Ax + By \in M$ *for all* $x, y \in M$;
2) *A is compact and continuous;*
3) *B is a contraction mapping.*
Then there exists $z \in M$ *such that* $z = Az + Bz$.

3. Main results

We consider the following linear fractional differential equation

$$\left({}^{\rho}D^{\alpha,\beta}_{a^+}y\right)(t) = \varphi(t), \quad t \in (a, b], \tag{3.1}$$

where $0 < \alpha < 1, 0 \leq \beta \leq 1, \rho > 0$, with the boundary condition

$$u\left({}^{\rho}I^{1-\gamma}_{a^+}y\right)(a^+) + v\left({}^{\rho}I^{1-\gamma}_{a^+}y\right)(b) = w, \tag{3.2}$$

where $\gamma = \alpha + \beta - \alpha\beta$, and $u, v, w \in \mathbb{R}$ with $u + v \neq 0$. The following theorem shows that the problem (3.1)–(3.2) has a unique solution given by

$$y(t) = \frac{1}{(u+v)\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \left[w - \frac{v}{\Gamma(1-\gamma+\alpha)} \int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1}\varphi(s)ds\right] + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\varphi(s)ds. \tag{3.3}$$

Theorem 3.1. *Let $\gamma = \alpha + \beta - \alpha\beta$, where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. If $\varphi : (a, b] \to \mathbb{R}$ is a function such that $\varphi(\cdot) \in C_{1-\gamma,\rho}(J)$, then $y \in C^\gamma_{1-\gamma,\rho}(J)$ satisfies the problem (3.1)–(3.2) if and only if it satisfies (3.3).*

Proof. ($\Rightarrow$) By Lemma 2.13, we have the solution of (3.1) can be written as

$$y(t) = \frac{\left({}^\rho I^{1-\gamma}_{a^+} y\right)(a^+)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\varphi(s)ds. \tag{3.4}$$

Applying ${}^\rho I^{1-\gamma}_{a^+}$ on both sides of (3.4), using Lemma 2.6 and taking $t = b$, we obtain

$$\left({}^\rho I^{1-\gamma}_{a^+} y\right)(b) = \left({}^\rho I^{1-\gamma}_{a^+} y\right)(a^+) + \frac{1}{\Gamma(1-\gamma+\alpha)} \int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1}\varphi(s)ds, \tag{3.5}$$

multiplying both sides of (3.5) by v, we get

$$v\left({}^\rho I^{1-\gamma}_{a^+} y\right)(b) = v\left({}^\rho I^{1-\gamma}_{a^+} y\right)(a^+) + \frac{v}{\Gamma(1-\gamma+\alpha)} \int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1}\varphi(s)ds.$$

Using condition (3.2), we obtain

$$v\left({}^\rho I^{1-\gamma}_{a^+} y\right)(b) = w - u\left({}^\rho I^{1-\gamma}_{a^+} y\right)(a^+).$$

Thus

$$w - u\left({}^\rho I^{1-\gamma}_{a^+} y\right)(a^+) = v\left({}^\rho I^{1-\gamma}_{a^+} y\right)(a^+) + \frac{v}{\Gamma(1-\gamma+\alpha)} \int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1}\varphi(s)ds,$$

which implies that

$$\left({}^\rho I^{1-\gamma}_{a^+} y\right)(a^+) = \frac{w}{u+v} - \frac{v}{(u+v)\Gamma(1-\gamma+\alpha)} \int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1}\varphi(s)ds, \tag{3.6}$$

Substituting (3.6) into (3.4), we obtain (3.3).
($\Leftarrow$) Applying ${}^\rho I^{1-\gamma}_{a^+}$ on both sides of (3.3) and using Lemma 2.6 and Theorem 2.3, we get

$$\left({}^\rho I^{1-\gamma}_{a^+} y\right)(t) = \frac{w}{u+v} - \frac{v}{(u+v)} \left({}^\rho I^{1-\gamma+\alpha}_{a^+} \varphi\right)(b) + \left({}^\rho I^{1-\gamma+\alpha}_{a^+} \varphi\right)(t). \tag{3.7}$$

Next, taking the limit $t \to a^+$ of (3.7) and using Lemma 2.5, with $1 - \gamma < 1 - \gamma + \alpha$, we obtain

$$\left({}^\rho I^{1-\gamma}_{a^+} y\right)(a^+) = \frac{w}{u+v} - \frac{v}{(u+v)} \left({}^\rho I^{1-\gamma+\alpha}_{a^+} \varphi\right)(b). \tag{3.8}$$

Now, taking $t = b$ in (3.7), we get

$$\left({}^{\rho}I_{a^+}^{1-\gamma}y\right)(b) = \frac{w}{u+v} - \frac{v}{(u+v)}\left({}^{\rho}I_{a^+}^{1-\gamma+\alpha}\varphi\right)(b) + \left({}^{\rho}I_{a^+}^{1-\gamma+\alpha}\varphi\right)(b). \tag{3.9}$$

From (3.8) and (3.9), we find that

$$\begin{aligned} & u\left({}^{\rho}I_{a^+}^{1-\gamma}y\right)(a^+) + v\left({}^{\rho}I_{a^+}^{1-\gamma}y\right)(b) \\ = & \frac{uw}{u+v} - \frac{uv}{u+v}\left({}^{\rho}I_{a^+}^{1-\gamma+\alpha}\varphi\right)(b) + \frac{vw}{u+v} \\ - & \frac{v^2}{u+v}\left({}^{\rho}I_{a^+}^{1-\gamma+\alpha}\varphi\right)(b) + v\left({}^{\rho}I_{a^+}^{1-\gamma+\alpha}\varphi\right)(b) \\ = & w + \left(v - \frac{uv}{u+v} - \frac{v^2}{u+v}\right)\left({}^{\rho}I_{a^+}^{1-\gamma+\alpha}\varphi\right)(b) = w, \end{aligned}$$

which shows that the boundary condition $u\left({}^{\rho}I_{a^+}^{1-\gamma+\alpha}\varphi\right)(a^+)+v\left({}^{\rho}I_{a^+}^{1-\gamma+\alpha}\varphi\right)(b) = w$, is satisfied. Next, apply operator ${}^{\rho}D_{a^+}^{\gamma}$ on both sides of (3.3). Then, from Lemma 2.6 and Lemma 2.12 we obtain

$$({}^{\rho}D_{a^+}^{\gamma}y)(t) = \left({}^{\rho}D_{a^+}^{\beta(1-\alpha)}\varphi\right)(t). \tag{3.10}$$

Since $y \in C_{1-\gamma,\rho}^{\gamma}(J)$ and by definition of $C_{1-\gamma,\rho}^{\gamma}(J)$, we have ${}^{\rho}D_{a^+}^{\gamma}y \in C_{1-\gamma,\rho}(J)$, then, (3.10) implies that

$$({}^{\rho}D_{a^+}^{\gamma}y)(t) = \left(\delta_{\rho}\ {}^{\rho}I_{a^+}^{1-\beta(1-\alpha)}\varphi\right)(t) = \left({}^{\rho}D_{a^+}^{\beta(1-\alpha)}\varphi\right)(t) \in C_{1-\gamma,\rho}(J). \tag{3.11}$$

As $\varphi(\cdot) \in C_{1-\gamma,\rho}(J)$ and from Lemma 2.4, follows

$$\left({}^{\rho}I_{a^+}^{1-\beta(1-\alpha)}\varphi\right) \in C_{1-\gamma,\rho}(J). \tag{3.12}$$

From (3.11), (3.12) and by the Definition of the space $C_{1-\gamma,\rho}^{n}(J)$, we obtain

$$\left({}^{\rho}I_{a^+}^{1-\beta(1-\alpha)}\varphi\right) \in C_{1-\gamma,\rho}^{1}(J).$$

Applying operator ${}^{\rho}I_{a^+}^{\beta(1-\alpha)}$ on both sides of (3.10) and using Lemma 2.8, Lemma 2.5 and Property 2.10, we have

$$\begin{aligned} \left({}^{\rho}D_{a^+}^{\alpha,\beta}y\right)(t) &= {}^{\rho}I_{a^+}^{\beta(1-\alpha)}\ \left({}^{\rho}D_{a^+}^{\gamma}y\right)(t) \\ &= \varphi(t) + \frac{\left({}^{\rho}I_{a^+}^{1-\beta(1-\alpha)}\varphi(t)\right)(a)}{\Gamma(\beta(1-\alpha))}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\beta(1-\alpha)-1} \\ &= \varphi(t), \end{aligned}$$

that is, (3.1) holds. This completes the proof.

As a consequence of Theorem 3.1, we have the following result

Theorem 3.2. *Let $\gamma = \alpha+\beta-\alpha\beta$ where $0 < \alpha < 1$ and $0 \le \beta \le 1$, let $f : (a, b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, y(\cdot), z(\cdot)) \in C_{1-\gamma,\rho}(J)$ for any $y, z \in C_{1-\gamma,\rho}(J)$.*

If $y \in C^{\gamma}_{1-\gamma,\rho}(J)$, then y satisfies the problem (1.1) − (1.2) if and only if y is the fixed point of the operator $N : C_{1-\gamma,\rho}(J) \to C_{1-\gamma,\rho}(J)$ defined by

$$Ny(t) = \frac{1}{(u+v)\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\left[w - \frac{v}{\Gamma(1-\gamma+\alpha)}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1} g(s)ds\right] + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s)ds, \ t \in (a,b], \tag{3.13}$$

where $g : (0,b] \to \mathbb{R}$ be a function satisfying the functional equation

$$g(t) = f(t, y(t), g(t)).$$

Clearly, $g \in C_{1-\gamma,\rho}(J)$. Also, by Lemma 2.4, $Ny \in C_{1-\gamma,\rho}(J)$.

Assume that the function $f : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the conditions:

(*H1*) The function $f : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be such that

$$f(\cdot, y(\cdot), z(\cdot)) \in C^{\beta(1-\alpha)}_{1-\gamma,\rho} \quad \text{for any } y, z \in C_{1-\gamma,\rho}(J).$$

(H2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$|f(t,y,z) - f(t,\bar{y},\bar{z})| \le K|y - \bar{y}| + L|z - \bar{z}|$$

for any $y, z, \bar{y}, \bar{z} \in \mathbb{R}$ and $t \in (a,b]$.

We are now in a position to state and prove our existence result for the problem (1.1)−(1.2) based on Banach's fixed point.

Theorem 3.3. *Assume ($H1$) and ($H2$) hold. If*

$$\frac{K}{1-L}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\left[\frac{|v|}{|u+v|\Gamma(\alpha+1)} + \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\right] < 1, \tag{3.14}$$

then the problem (1.1)–(1.2) has unique solution in $C^{\gamma}_{1-\gamma,\rho}(J) \subset C^{\alpha,\beta}_{1-\gamma,\rho}(J)$.

Proof. The proof will be given in two steps.

Step 1: We show that the operator N defined in (3.13) has a unique fixed point y^* in $C_{1-\gamma,\rho}(J)$. Let $y, z \in C_{1-\gamma,\rho}(J)$ and $t \in (a,b]$, then, we have

$$\begin{aligned} &|Ny(t) - Nz(t)| \\ &\le \frac{|v|}{|u+v|\Gamma(\gamma)\Gamma(1-\gamma+\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1}|g(s) - h(s)|ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}|g(s) - h(s)|ds, \end{aligned}$$

where $g, h \in C_{1-\gamma,\rho}(J)$ such that

$$g(t) = f(t, y(t), g(t)),$$
$$h(t) = f(t, z(t), h(t)).$$

By (H2), we have

$$
\begin{aligned}
|g(t)-h(t)| &= |f(t,y(t),g(t))-f(t,z(t),h(t))| \\
&\leq K|y(t)-z(t)| + L|g(t)-h(t)|.
\end{aligned}
$$

Then,

$$
|g(t)-h(t)| \leq \frac{K}{1-L}|y(t)-z(t)|.
$$

Therefore, for each $t \in (a,b]$

$$
\begin{aligned}
&|Ny(t)-Nz(t)| \\
&\leq \frac{K|v|}{(1-L)|u+v|\Gamma(\gamma)\Gamma(1-\gamma+\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\int_a^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-\gamma}s^{\rho-1}|y(s)-z(s)|ds \\
&+\frac{K}{(1-L)\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|y(s)-z(s)|ds, \\
&\leq \frac{K|v|}{(1-L)|u+v|\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\|y-z\|_{C_{1-\gamma,\rho}}\left({}^\rho I_{a^+}^{1-\gamma+\alpha}\left(\frac{s^\rho-a^\rho}{\rho}\right)^{\gamma-1}\right)(b) \\
&+\frac{K}{(1-L)}\left(I_{a^+}^\alpha\left(\frac{s^\rho-a^\rho}{\rho}\right)^{\gamma-1}\right)(t)\|y-z\|_{C_{1-\gamma,\rho}}.
\end{aligned}
$$

By Lemma 2.6, we have

$$
\begin{aligned}
|Ny(t)-Nz(t)| \quad \leq \quad & \left[\frac{K|v|}{(1-L)|u+v|\Gamma(\alpha+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\right. \\
+ \quad & \left.\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\gamma-1}\right]\|y-z\|_{C_{1-\gamma,\rho}},
\end{aligned}
$$

hence

$$
\begin{aligned}
\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}(Ny(t)-Nz(t))\right| &\leq \left[\frac{K|v|}{(1-L)|u+v|\Gamma(\alpha+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\right. \\
&+\left.\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha}\right]\|y-z\|_{C_{1-\gamma,\rho}} \\
&\leq \frac{K}{1-L}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\left[\frac{|v|}{|u+v|\Gamma(\alpha+1)}\right. \\
&+\left.\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\right]\|y-z\|_{C_{1-\gamma,\rho}},
\end{aligned}
$$

which implies that

$$
\|Ny-Nu\|_{C_{1-\gamma,\rho}} \leq \frac{K}{1-L}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\left[\frac{|v|}{|u+v|\Gamma(\alpha+1)}+\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\right]\|y-z\|_{C_{1-\gamma,\rho}}.
$$

By (3.14), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point $y^* \in C_{1-\gamma,\rho}(J)$.

Step 2: We show that such a fixed point $y^* \in C_{1-\gamma,\rho}(J)$ is actually in $C^{\gamma}_{1-\gamma,\rho}(J)$. Since y^* is the unique fixed point of operator N in $C_{1-\gamma,\rho}(J)$, then, for each $t \in (a,b]$, we have

$$y^*(t) = \frac{1}{(u+v)\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\left[w - v\ \left({}^\rho I^{1-\gamma+\alpha}_{a^+} f(s,y^*(s),g(s))\right)(b)\right] + \left({}^\rho I^{\alpha}_{a^+} f(s,y^*(s),g(s))\right)(t).$$

Applying ${}^\rho D^{\gamma}_{a^+}$ to both sides and by Lemma 2.6, and Lemma 2.12, we have

$$\begin{aligned} {}^\rho D^{\gamma}_{a^+} y^*(t) &= \left({}^\rho D^{\gamma}_{a^+}\ {}^\rho I^{\alpha}_{a^+} f(s,y^*(s),g(s))\right)(t) \\ &= \left({}^\rho D^{\beta(1-\alpha)}_{a^+} f(s,y^*(s),g(s))\right)(t). \end{aligned}$$

Since $\gamma \geq \alpha$, by (H1), the right hand side is in $C_{1-\gamma,\rho}(J)$ and thus ${}^\rho D^{\gamma}_{a^+} y^* \in C_{1-\gamma,\rho}(J)$ which implies that $y^* \in C^{\gamma}_{1-\gamma,\rho}(J)$. As a consequence of Steps 1 and 2 together with Theorem 3.2, we can conclude that the problem (1.1) – (1.2) has a unique solution in $C^{\gamma}_{1-\gamma,\rho}(J)$.

Our second result is based on Krasnoselskii fixed point theorem.

Theorem 3.4. *Assume (H1) and (H2) hold. If*

$$\max\left\{\frac{K\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}, \frac{K|v|}{(1-L)|u+v|\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\right\} < 1. \tag{3.15}$$

Then the problem (1.1)-(1.2) has at least one solution in $C^{\gamma}_{1-\gamma,\rho}(J) \subset C^{\alpha,\beta}_{1-\gamma,\rho}(J)$.

Proof. Consider the set

$$B_{\eta^*} = \{y \in C_{1-\gamma,\rho}(J) : ||y||_{C_{1-\gamma,\rho}} \leq \eta^*\},$$

where

$$\eta^* \geq \frac{\dfrac{|w|}{|u+v|\Gamma(\gamma)} + \dfrac{|v|M\Gamma(\gamma)}{|u+v|\Gamma(\alpha+1)\Gamma(\gamma)}\left(\dfrac{b^\rho - a^\rho}{\rho}\right)^{\alpha} + \dfrac{f^*\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}\left(\dfrac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}}{1 - \dfrac{K\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}\left(\dfrac{b^\rho - a^\rho}{\rho}\right)^{\alpha}},$$

and $f^* = \sup_{t\in J}|f(t,0,0)|$.

We define the operators P and Q on B_{η^*} by

$$Py(t) = \frac{1}{(u+v)\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\left[w - \frac{v}{\Gamma(1-\gamma+\alpha)}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1} g(s)ds\right], \tag{3.16}$$

$$Qy(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s)ds. \tag{3.17}$$

Then the fractional integral equation (3.13) can be written as operator equation

$$Ny(t) = Py(t) + Qy(t), \quad y \in C_{1-\gamma,\rho}(J)$$

The proof will be given in several steps.

Step 1: We prove that $Py+Qz \in B_{\eta^*}$ for any $y, z \in B_{\eta^*}$. For operator P, multiplying both sides of (3.16) by $\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}$, we have

$$\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} Py(t) = \frac{1}{(u+v)\Gamma(\gamma)}\left[w - \frac{v}{\Gamma(1-\gamma+\alpha)}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1} g(s)ds\right],$$

then

$$\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} Py(t)\right| \leq \frac{1}{|u+v|\Gamma(\gamma)}\left[|w| + \frac{|v|}{\Gamma(1-\gamma+\alpha)}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1} |g(s)|ds\right]. \tag{3.18}$$

By (H3), we have for each $t \in (a, b]$,

$$\begin{aligned} |g(t)| &= |f(t, y(t), g(t)) - f(t, 0, 0) + f(t, 0, 0)| \\ &\leq |f(t, y(t), g(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq K|y(t)| + L|g(t)| + f^*. \end{aligned}$$

Multiplying both sides of the above inequality by $\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}$, we get

$$\begin{aligned} \left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} g(t)\right| &\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} f^* + K\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} y(t)\right| \\ &+ L\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} g(t)\right| \\ &\leq \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} f^* + K\eta^* + L\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} g(t)\right|. \end{aligned}$$

Then, for each $t \in (a, b]$, we have

$$\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} g(t)\right| \leq \frac{\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} f^* + K\eta^*}{1-L} := M. \tag{3.19}$$

Thus, (3.18) and Lemma 2.6, imply

$$\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} Py(t)\right| \leq \frac{1}{|u+v|\Gamma(\gamma)}\left[|w| + \frac{|v|M\Gamma(\gamma)}{\Gamma(\alpha+1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha}\right].$$

This gives

$$||Py||_{C_{1-\gamma,\rho}} \leq \frac{1}{|u+v|\Gamma(\gamma)}\left[|w| + \frac{|v|M\Gamma(\gamma)}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\right]. \tag{3.20}$$

Using (3.19) and Lemma 2.6, we have

$$|Q(z)(t)| \leq \left[\frac{\Gamma(\gamma)f^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} + \frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\right]\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\gamma-1}.$$

Therefore

$$\begin{aligned}\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}Qz(t)\right| &\leq \left[\frac{\Gamma(\gamma)f^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma}\right.\\ &+ \left.\frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\right]\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha},\\ &\leq \frac{\Gamma(\gamma)f^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha}\\ &+ \frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}.\end{aligned}$$

Thus

$$\|Qz\|_{C_{1-\gamma,\rho}} \leq \frac{\Gamma(\gamma)f^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha} + \frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}. \tag{3.21}$$

Linking (3.20) and (3.21) for every $y, z \in B_{\eta^*}$ we obtain

$$\begin{aligned}\|Py+Qz\|_{C_{1-\gamma,\rho}} &\leq \|Py\|_{C_{1-\gamma,\rho}} + \|Qz\|_{C_{1-\gamma,\rho}}\\ &\leq \frac{|w|}{|u+v|\Gamma(\gamma)} + \frac{|v|M\Gamma(\gamma)}{|u+v|\Gamma(\alpha+1)\Gamma(\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\\ &+ \left[f^*\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma} + K\eta^*\right]\frac{\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\beta)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}.\end{aligned}$$

Since

$$\eta^* \geq \frac{\frac{|w|}{|u+v|\Gamma(\gamma)} + \frac{|v|M\Gamma(\gamma)}{|u+v|\Gamma(\alpha+1)\Gamma(\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha} + \frac{f^*\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha}}{1-\frac{K\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}},$$

we have

$$\|Py+Qz\|_{PC_{1-\gamma,\rho}} \leq \eta^*.$$

which infers that $Py+Qz \in B_{\eta^*}$.

Step 2: P is a contraction.
Let $y, z \in C_{1-\gamma,\rho}(J)$ and $t \in (a,b]$, then, we have

$$\begin{aligned}&|Py(t)-Pz(t)|\\ &\leq \frac{|v|}{|u+v|\Gamma(\gamma)\Gamma(1-\gamma+\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\int_a^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-\gamma}s^{\rho-1}|g(s)-h(s)|ds,\end{aligned}$$

where $g, h \in C_{1-\gamma,\rho}(J)$ such that

$$\begin{aligned}g(t) &= f(t, y(t), g(t)),\\ h(t) &= f(t, z(t), h(t)).\end{aligned}$$

By (H2), we have

$$\begin{aligned}|g(t)-h(t)| &= |f(t,y(t),g(t))-f(t,z(t),h(t))| \\ &\le K|y(t)-u(t)|+L|g(t)-h(t)|.\end{aligned}$$

Then,

$$|g(t)-h(t)| \le \frac{K}{1-L}|y(t)-z(t)|.$$

Therefore, for each $t \in (a,b]$

$$\begin{aligned}&|Py(t)-Pz(t)| \\ &\le \frac{K|v|}{(1-L)|u+v|\Gamma(\gamma)\Gamma(1-\gamma+\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\int_a^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1}|y(s)-z(s)|ds \\ &\le \frac{K|v|}{(1-L)|u+v|\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\|y-z\|_{C_{1-\gamma,\rho}}\left({}^\rho I_{a^+}^{1-\gamma+\alpha}\left(\frac{s^\rho-a^\rho}{\rho}\right)^{\gamma-1}\right)(b).\end{aligned}$$

By Lemma 2.6, we have

$$|Py(t)-Pz(t)| \le \frac{K|v|}{(1-L)|u+v|\Gamma(\alpha+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\|y-z\|_{C_{1-\gamma,\rho}},$$

hence

$$\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}(Py(t)-Pz(t))\right| \le \frac{K|v|}{(1-L)|u+v|\Gamma(\alpha+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\|y-z\|_{C_{1-\gamma,\rho}},$$

which implies that

$$\|Py-Pz\|_{C_{1-\gamma,\rho}} \le \frac{K|v|}{(1-L)|u+v|\Gamma(\alpha+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\|y-z\|_{C_{1-\gamma,\rho}}.$$

By (3.15), the operator P is a contraction.

Step 3: Q is compact and continuous.
The continuity of Q follows from the continuity of f. Next we prove that Q is uniformly bounded on B_{η^*}. Let any $z \in B_{\eta^*}$. Then by (3.21) we have

$$\|Qz\|_{C_{1-\gamma,\rho}} \le \frac{\Gamma(\gamma)f^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha} + \frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}.$$

This means that Q is uniformly bounded on B_{η^*}. Next, we show that QB_{η^*} is equicontinuous. Let any $y \in B_{\eta^*}$ and $0 < a < \tau_1 < \tau_2 \leq b$. Then

$$\begin{aligned}
&\left|\left(\frac{\tau_2^\rho - a^\rho}{\rho}\right)^{1-\gamma} Q(y)(\tau_2) - \left(\frac{\tau_1^\rho - a^\rho}{\rho}\right)^{1-\gamma} Q(y)(\tau_1)\right| \\
&\leq \frac{\left(\frac{\tau_2^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \left(\frac{\tau_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}|g(s)|ds \\
+ &\quad \frac{1}{\Gamma(\alpha)} \int_a^{\tau_1} \left|\left[\left(\frac{\tau_2^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\tau_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right.\right. \\
- &\quad \left.\left.\left(\frac{\tau_1^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\tau_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right]\right| |g(s)|ds \\
\leq &\quad \frac{M\Gamma(\gamma)\left(\frac{\tau_2^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha+\gamma)} \left(\frac{\tau_2^\rho - \tau_1^\rho}{\rho}\right)^{\alpha+\gamma-1} \\
+ &\quad \frac{M}{\Gamma(\alpha)} \int_a^{\tau_1} \left|\left[\left(\frac{\tau_2^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\tau_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right.\right. \\
- &\quad \left.\left.\left(\frac{\tau_1^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\tau_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right]\right| \left(\frac{s^\rho - a^\rho}{\rho}\right)^{\gamma-1} ds.
\end{aligned}$$

Note that

$$\left|\left(\frac{\tau_2^\rho - a^\rho}{\rho}\right)^{1-\gamma} Q(y)(\tau_2) - \left(\frac{\tau_1^\rho - a^\rho}{\rho}\right)^{1-\gamma} Q(y)(\tau_1)\right| \to 0 \quad \text{as} \quad \tau_2 \to \tau_1.$$

This shows that Q is equicontinuous on J. Therefore Q is relatively compact on B_{η^*}. By $C_{1-\gamma}$ type Arzela-Ascoli Theorem Q is compact on B_{η^*}.
As a consequence of Krasnoselskii's fixed point theorem, we deduce that N has at least a fixed point $y^* \in C_{1-\gamma,\rho}(J)$ and by the same way of the proof of Theorem 3.3, we can easily show that $y^* \in C^\gamma_{1-\gamma,\rho}(J)$. Using Lemma 3.2, we conclude that the problem $(1.1) - (1.2)$ has at least one solution in the space $C^\gamma_{1-\gamma,\rho}(J)$.

4. An example

Consider the following boundary value problem

$$^{\frac{1}{2}}D_{1^+}^{\frac{1}{2},0} y(t) = \frac{2 + |y(t)| + \left|^{\frac{1}{2}}D_{0^+}^{\frac{1}{2},0} y(t)\right|}{108e^{-t+3}\left(1 + |y(t)| + \left|^{\frac{1}{2}}D_{0^+}^{\frac{1}{2},0} y(t)\right|\right)} + \frac{\ln(\sqrt{t}+1)}{3\sqrt{\sqrt{t}-1}}, \quad t \in (1,2] \tag{4.1}$$

$$\left(^{\frac{1}{2}}I_{1^+}^{\frac{1}{2},0} y\right)(1) + \left(^{\frac{1}{2}}I_{1^+}^{\frac{1}{2},0} y\right)(2) = 0. \tag{4.2}$$

Set

$$f(t,y,z)=\frac{2+y+z}{108e^{-t+3}(1+y+z)}+\frac{\ln(\sqrt{t}+1)}{3\sqrt{t}},\ t\in(1,2],\ y,z\in[0,+\infty).$$

We have

$$C^{\beta(1-\alpha)}_{1-\gamma,\rho}([1,2])=C^{0}_{\frac12,\frac12}([1,2])=\left\{h:(1,2]\to\mathbb{R}:\sqrt{2}\left(\sqrt{t}-1\right)^{\frac12}h\in C([1,2])\right\},$$

with $\gamma=\alpha=\rho=\frac12$ and $\beta=0$. Clearly, the function $f\in C_{\frac12,\frac12}([1,2])$.
Hence condition (H1) is satisfied.
For each $y,\bar{y},z,\bar{z}\in\mathbb{R}$ and $t\in(1,2]$:

$$\begin{aligned}|f(t,y,z)-f(t,\bar{y},\bar{z})| &\le \frac{1}{108e^{-t+3}}(|y-\bar{y}|+|z-\bar{z}|)\\ &\le \frac{1}{108e}\left(|y-\bar{y}|+|z-\bar{z}|\right).\end{aligned}$$

Hence condition $(H2)$ is satisfied with $K=L=\dfrac{1}{108e}$.
The condition

$$\frac{K}{1-L}\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\left[\frac{|v|}{|u+v|\Gamma(\alpha+1)}+\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\right]\approx 0.0072<1,$$

is satisfied with with $b=2,a=1,u=v=1$ and $w=0$. It follows from Theorem 3.3 that the problem (4.1)-(4.2) has a unique solution in the space $C^{\frac12}_{\frac12,\frac12}([1,2])$.

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