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ON THE BURTON METHOD OF PROGRESSIVE CONTRACTIONS FOR VOLTERRA INTEGRAL EQUATIONS

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Abstract. In the paper [4] the author give a new method to study the existence and uniqueness of a solution on $[0,\infty[$ of a scalar integral equation

$$x(t) = g(t, x(t)) + \int_0^t A(t-s)f(t, s, x(s))ds,$$

where $u, v \in \mathbb{R},\ t \in [0,\infty[$ imply that there exists $0 < l < 1$ with

$$|g(t,u) - g(t,v)| \leq l\,|u - v|$$

and for each $b > 0$ there exists $L_b > 0$ such that

$$|f(t,u) - f(t,v)| \leq L_b\,|u - v|\,,\ \forall t \in [0,b],\ \forall u, v \in \mathbb{R}.$$

In this paper we extend the Burton method to the case where instead of scalar equations we consider an equation in a Banach space.

Key Words and Phrases: Progressive contractions, fixed points, existence, uniqueness, integro-differential equations.
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1. Introduction

The purpose of this paper is to present an existence and uniqueness result for integral equations with sum of two operators. The approach is based on proving the existence of a solution on a short interval, then the equation is translated to a new starting time so that the solution on another short interval is fitted onto the first solution and so on.

Our result is connected to some recent papers of T.A. Burton [1]-[4] where it is introduced the technique named progressive contraction. This technique is suited to integral equations and shows that when the equation is defined by the sum of a contraction and a Lipschitz operator one can prove first existence on arbitrary interval $[0, b]$ and then one can parlay that into a solution on $[0, \infty)$. In our paper we combine the above technique with the classical method of Banach fixed point theorem (see [10]-[13]).

Regarding the integral equations that contain a sum of two operators, one can see the following papers [6]-[7], [12], [13].

Let $(\mathbb{B}, |\cdot|)$ be a Banach space and $K \in C(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{B}, \mathbb{B})$. We consider the Volterra integral equation corresponding to K

$$x(t) = \int_0^t K(t, s, x(s))ds,\ t \in \mathbb{R}_+,\ x \in C(\mathbb{R}_+, \mathbb{B}). \tag{1.1}$$

For $b > 0$, let us consider the same equation defined on $[0, b]$ as follows

$$x(t) = \int_0^t K(t, s, x(s))ds,\ t \in [0, b],\ x \in C([0, b], \mathbb{B}). \tag{1.2}$$

In what follows we consider the Bielecki norm $\|\cdot\|_\tau$, defined by

$$\|x\|_\tau = \max |x(t)|\, e^{-\tau t},\ \tau > 0,$$

the Chebyshev norm defined by

$$\|x\|_\infty = \sup_{t \in [0,b]} \{|x(t)|\}.$$

We consider the equation (1.2) with the following conditions

(C_1) $K \in (C([0, b] \times [0, b] \times \mathbb{B}, \mathbb{B}))$;
(C_2) for each $b > 0$ there exists $L_b > 0$ such that

$$|K(t, s, u) - K(t, s, v)| \le L_b\, |u - v|\,,\ \forall t, s \in [0, b], u, v \in \mathbb{B}.$$

The following result is well known.

Theorem 1.1. *If the conditions (C_1) and (C_2) are satisfied, then the equation (1.2) has a unique solution in $C([0, \infty[, \mathbb{B})$.*

Since we have a Volterra integral equation is sufficient to prove the existence and uniqueness of the solution in $C([0, b], \mathbb{B})$ for any positive b. To prove this we consider on $C([0, b], \mathbb{B})$ the Bielecki norm, with respect to which the operator

$$S : C([0, b], \mathbb{B}) \to C([0, b], \mathbb{B})$$

defined by

$$S(x)(t) = \int_0^t K(t, s, x(s))ds$$

is a $\frac{L_b}{\tau}$-contraction, with τ chosen sufficiently large.

The problem is if we can obtain this result using Chebyshev norm.

2. Burton method in the case of Volterra integral equation

Theorem 2.1. *If the condition* (C_1) *and* (C_2) *are satisfied and* $bL_b < 1$*, then the equation (1.2) has a unique solution in* $C([0,b],\mathbb{B})$.

Proof. Following the idea of T.A. Burton [4], [1], [2] and using

$$[0,b]=\bigcup_{k=0}^{m-1}\left[\frac{kb}{m},\frac{(k+1)b}{m}\right],\ m\in\mathbb{N}^*,$$

we divide the interval $[0,b]$ into m equal parts, denoting the end points by

$$0,\frac{b}{m},\frac{2b}{m},\ldots,b.$$

Step 1. Let $(M_1,\|\cdot\|_1)$ be the complete metric space of continuous functions $x:[0,\frac{b}{m}]\to\mathbb{R}$ with the Chebyshev metric $\|\cdot\|_1$, where

$$\|x(t)\|_i=\max_{t\in\left[0,\frac{ib}{m}\right]}|x(t)|\,,\ i=\overline{1,m-1}.$$

We define the following mapping $A_1:M_1\to M_1$ with $x\in M_1$

$$A_1(x)(t)=\int_0^t K(t,s,x(s))ds,\ t\in[0,\frac{b}{m}].$$

Then for $x,y\in M_1$ and $0\le t\le\frac{b}{m}$ we have

$$\begin{aligned}|A_1(x)(t)-A_1(y)(t)|&\le\int_0^t|f(t,s,x(s))-f(t,s,y(s))|\,ds\\&\le\frac{L_{b,1}b}{m}\max_{t\in[0,\frac{b}{m}]}|x(t)-y(t)|\\&\le\frac{L_{b,1}b}{m}\|x-y\|_1\,.\end{aligned}$$

So,

$$\max_{t\in[0,\frac{b}{m}]}|A_1(x)(t)-A_1(y)(t)|\le\frac{L_{b,1}b}{m}\|x-y\|_1\,.$$

Thus

$$\|A_1(x)-A_1(y)\|_1\le\frac{L_{b,1}b}{m}\|x-y\|_1\,.$$

The mapping A_1 is a contraction with a unique fixed point x_1^* on $[0,\frac{b}{m}]$ with

$$(A_1x_1^*)(t)=x_1^*(t)=\int_0^t K(t,s,x_1^*(s))ds,\ 0\le t\le\frac{b}{m}. \tag{2.1}$$

Step 2. Let $(M_2,\|\cdot\|_2)$ be the complete metric space of continuous functions $x:[0,\frac{2b}{m}]\to\mathbb{R}$ with the Chebyshev metric and

$$x(t)=x_1^*(t)\text{ on }\left[0,\frac{b}{m}\right].$$

We define the mapping $A_2 : M_2 \to M_2$ with $x \in M_2$

$$A_2(x)(t) = \int_0^t K(t, s, x(s))ds.$$

Notice that for $0 \le t \le \frac{b}{m}$ and $x \in M_2$ then $x = x_1^*$ which is a fixed point and from (2.1) we have

$$(A_2 x)(t) = \begin{cases} x_1^*(t), \ t \in \left[0, \dfrac{b}{m}\right] \\ \displaystyle\int_0^t K(t, s, x(s))ds, \ t \in \left[\dfrac{b}{m}, \dfrac{2b}{m}\right] \end{cases}$$

$$= \begin{cases} x_1^*(t), \ t \in \left[0, \dfrac{b}{m}\right] \\ \displaystyle\int_0^{\frac{b}{m}} K(t, s, x_1^*(s))ds + \int_{\frac{b}{m}}^t K(t, s, x(s))ds, \ t \in \left[\dfrac{b}{m}, \dfrac{2b}{m}\right] \end{cases}$$

For $x, y \in M_2$ we have

$$\begin{aligned} |A_2(x)(t) - A_2(y)(t)| &\le \int_{\frac{b}{m}}^t |K(t, s, x(s)) - K(t, s, y(s))| \, ds \\ &\le \int_{\frac{b}{m}}^t L_{b,2} |x(s) - y(s)| \, ds \\ &\text{(and since } x(t) = y(t) = x_1^*(t) \text{ on } [0, \frac{b}{m}]) \\ &\le \int_{\frac{b}{m}}^t L_{b,2} |x(s) - y(s)| \, ds \\ &\le L_{b,2} \frac{b}{m} \max_{t \in [\frac{b}{m}, \frac{2b}{m}]} |x(t) - y(t)| \\ &\le L_{b,2} \frac{b}{m} \max_{t \in [0, \frac{2b}{m}]} |x(t) - y(t)| \, . \end{aligned}$$

So,

$$\max_{t \in [0, \frac{2b}{m}]} |A_2(x)(t) - A_2(y)(t)| \le L_{b,2} \frac{b}{m} \max_{t \in [0, \frac{2b}{m}]} |x(t) - y(t)| \, .$$

Thus

$$\|A_2(x) - A_2(y)\|_2 \le L_{b,2} \frac{b}{m} \|x - y\|_2 \, .$$

The mapping A_2 is a contraction with a unique fixed point x_2^* on $[0, \frac{2b}{m}]$. Clearly x_2^* is a unique continuous solution of (2.1) with $x_2^*(t) = x_1^*(t)$ on $[0, \frac{b}{m}]$.

Step 3. We define the complete metric space $(M_3, \|\cdot\|_3)$ of continuous functions $x : [0, \frac{3b}{m}] \to \mathbb{R}$ with $x(t) = x_2^*$ on $[0, \frac{2b}{m}]$. But x_2^* is a fixed point and so A_3 is well defined. Analogously we obtain a continuous solution x_3^* on $[0, \frac{3b}{m}]$.

As follows we get that A_m is a contraction and thus we obtain a unique continuous solution on $[0, b]$, using the induction method.

For $2 < i < m-1$ let x_{i-1}^* be the unique solution of (2.1) on $[0, \frac{(i-1)b}{m}]$. Let $(M_i, \|\cdot\|_i)$ be the complete metric space of continuous functions $x : [0, \frac{ib}{m}] \to \mathbb{R}$ with the supremum metric and $x(t) = x_{i-1}^*(t)$ on $[0, \frac{(i-1)b}{m}]$. We define $A_i : M_i \to M_i$ by $x \in M_i$ imply

$$\begin{aligned}
(A_i x)(t) &= \int_0^t K(t,s,x(s))ds \\
&= \begin{cases} x_{i-1}^*(t), \ t \in \left[0, \dfrac{(i-1)b}{m}\right] \\ \displaystyle\int_0^t K(t,s,x(s))ds, \ t \in \left[\dfrac{(i-1)b}{m}, \dfrac{ib}{m}\right] \end{cases} \\
&= \begin{cases} x_{i-1}^*(t), \ t \in \left[0, \dfrac{(i-1)b}{m}\right] \\ \displaystyle\int_0^{\frac{(i-1)b}{m}} K(t,s,x_{i-1}^*(s))ds + \int_{\frac{(i-1)b}{m}}^t K(t,s,x(s))ds, \ t \in \left[\dfrac{(i-1)b}{m}, \dfrac{ib}{m}\right] \end{cases}
\end{aligned}$$

To prove that A_i is a contraction, let $x, y \in M_i$ and $0 \le t \le \frac{ib}{m}$ so that

$$\begin{aligned}
|A_i(x)(t) - A_i(y)(t)| &\le \int_{\frac{(i-1)b}{m}}^t |K(t,s,x(s)) - K(t,s,y(s))| \, ds \\
&\le \int_{\frac{(i-1)T}{m}}^t L_{b,i} |x(s) - y(s)| \, ds \\
&\text{(and since } x(t) = y(t) = x_{i-1}^*(t) \text{ on } [0, \frac{(i-1)b}{m}]) \\
&\le \int_{\frac{(i-1)b}{m}}^t L_{b,i} |x(s) - y(s)| \, ds \\
&\le L_{b,i} \frac{b}{m} \max_{t \in [\frac{(i-1)b}{m}, \frac{ib}{m}]} |x(t) - y(t)| \\
&\le L_{b,i} \frac{b}{m} \max_{t \in [0, \frac{ib}{m}]} |x(t) - y(t)|
\end{aligned}$$

So,

$$\max_{t \in [0, \frac{ib}{m}]} |A_i(x)(t) - A_i(y)(t)| \le L_{b,i} \frac{b}{m} \max_{t \in [0, \frac{ib}{m}]} |x(t) - y(t)| \, .$$

Thus

$$\|A_i(x) - A_i(y)\|_i \le L_{b,i} \frac{b}{m} \|x - y\|_i \, .$$

We obtain that A_i is a contraction with the unique fixed point x_i^* on $[0, \frac{ib}{m}]$. □

3. Applications of Burton method to functional integral equations

We consider the following integral equation

$$x(t) = g(t, x(t)) + \int_0^t f(t, s, x(s))ds, \ t \in [0, b) \tag{3.1}$$

where the functions $g \in C([0, b) \times \mathbb{R}, \mathbb{R}), \ f \in C([0, b) \times [0, b) \times \mathbb{R}, \mathbb{R})$ are given. We search the solution in the set $C([0, b], \mathbb{R})$ for which we consider the Chebyshev norm. We divide the interval $[0, b]$ in m equal parts such that

$$[0, b] = \bigcup_{k=0}^{m-1} \left[\frac{kb}{m}, \frac{(k+1)b}{m}\right], \ m \in \mathbb{N}^*. \tag{3.2}$$

We consider following hypothesis:

(H$_1$) there exists $L_g \in (0, 1)$ such that

$$|g(t, u) - g(t, v)| \leq L_g |u - v|, \ \forall u, v \in \mathbb{R}, \ 0 \leq t < b;$$

(H$_2$) for each $b > 0$ there exists $L_{f,k} > 0$ such that

$$|f(t, s, u) - f(t, s, v)| \leq L_{f,k}(b) |u - v|, \ \forall u, v \in \mathbb{R}, \ 0 \leq t \leq b;$$

Theorem 3.1. *In the conditions (H$_1$) and (H$_2$) the equation (3.1) has a unique solution on $C([0, b) \times \mathbb{B}, \mathbb{B})$.*

Proof. Following the same steps as in the proof of Theorem 2 and using (3.2) we divide the interval $[0, b]$ into m equal parts, denoting the end points by $0, \frac{b}{m}, \frac{2b}{m}, \ldots, b$.

Step 1. Let $(M_1, \|\cdot\|_1)$ be the complete metric space of continuous functions $x : [0, \frac{b}{m}] \to \mathbb{R}$ with the Chebyshev metric $\|\cdot\|_1$, where

$$\|x(t)\|_i = \max_{t \in \left[0, \frac{ib}{m}\right]} |x(t)|, \ i = \overline{1, m-1}.$$

We define the following mapping $A_1 : M_1 \to M_1$ with $x \in M_1$

$$A_1(x)(t) = g(t, x(t)) + \int_0^t f(t, s, x(s))ds, \ t \in [0, \frac{b}{m}].$$

Then for $x, y \in M_1$ and $0 \leq t \leq \frac{b}{m}$ we have

$$\begin{aligned} |A_1(x)(t) - A_1(y)(t)| &\leq L_g |x(t) - y(t)| + \int_0^t |f(t, s, x(s)) - f(t, s, y(s))| ds \\ &\leq L_g \|x - y\|_1 + \frac{L_{f,1}(b)b}{m} \|x - y\|_1 . \end{aligned}$$

Thus

$$\|A_1(x) - A_1(y)\|_1 \leq \left(L_g + L_{f,1}(b)\frac{b}{m}\right) \|x - y\|_1 .$$

The mapping A_1 is a contraction with a unique fixed point x_1^* on $[0, \frac{b}{m}]$ with

$$(A_1 x_1^*)(t) = x_1^*(t) = g(t, x_1^*(t)) + \int_0^t f(t, s, x_1^*(s))ds, \ 0 \leq t \leq \frac{b}{m}. \tag{3.3}$$

Step 2. Let $(M_2, \|\cdot\|_2)$ be the complete metric space of continuous functions $x : [0, \frac{2b}{m}] \to \mathbb{R}$ with the Chebyshev metric and

$$x(t) = x_1^*(t) \text{ on } \left[0, \frac{b}{m}\right].$$

We define the mapping $A_2 : M_2 \to M_2$ with $x \in M_2$

$$A_2(x)(t) = g(t, x(t)) + \int_0^t f(t, s, x(s))ds.$$

Notice that for $0 \le t \le \frac{b}{m}$ and $x \in M_2$ then $x = x_1^*$ which is a fixed point and from (3.3) we have

$$(A_2x)(t) = \begin{cases} x_1^*(t), \ t \in \left[0, \dfrac{b}{m}\right] \\ g(t, x(t)) + \displaystyle\int_0^t f(t, s, x(s))ds, \ t \in \left[\dfrac{b}{m}, \dfrac{2b}{m}\right] \end{cases}$$
$$= \begin{cases} x_1^*(t), \ t \in \left[0, \dfrac{b}{m}\right] \\ g(t, x(t)) + \displaystyle\int_0^{\frac{T}{m}} f(t, s, x_1^*(s))ds + \int_{\frac{T}{m}}^t f(t, s, x(s))ds, \ t \in \left[\dfrac{b}{m}, \dfrac{2b}{m}\right] \end{cases}$$

For $x, y \in M_2$ we have

$$\begin{aligned} |A_2(x)(t) - A_2(y)(t)| &\le L_g |x(t) - y(t)| + \int_{\frac{b}{m}}^t |f(t, s, x(s)) - f(t, s, y(s))| \, ds \\ &\le \left(L_g + L_{f,2}(b)\frac{b}{m}\right) \max_{t \in [0, \frac{2b}{m}]} |x(t) - y(t)| \, . \end{aligned}$$

Thus

$$\|A_1(x) - A_1(y)\|_2 \le \left(L_g + L_{f,2}(b)\frac{b}{m}\right) \|x - y\|_2 \, .$$

The mapping A_2 is a contraction with a unique fixed point x_2^* on $[0, \frac{2b}{m}]$. Clearly x_2^* is a unique continuous solution of (3.1) with $x_2^*(t) = x_1^*(t)$ on $[0, \frac{b}{m}]$.

Step 3. We define the complete metric space $(M_3, \|\cdot\|_3)$ of continuous functions $x : [0, \frac{3b}{m}] \to \mathbb{R}$ with $x(t) = x_2^*$ on $[0, \frac{2b}{m}]$. But x_2^* is a fixed point and so A_3 is well defined. Analogously we obtain a continuous solution x_3^* on $[0, \frac{3b}{m}]$.

By induction we get a unique continuous solution on $[0, b]$. We give below some induction details. For $2 < i < m - 1$ let x_{i-1}^* be the unique solution of (3.1) on $[0, \frac{(i-1)b}{m}]$. Let $(M_i, \|\cdot\|_i)$ be the complete metric space of continuous functions $x : [0, \frac{ib}{m}] \to \mathbb{R}$ with the supremum metric and $x(t) = x_{i-1}^*(t)$ on $[0, \frac{(i-1)b}{m}]$. We define $A_i : M_i \to M_i$

by $x \in M_i$ imply

$$
\begin{aligned}
(A_i x)(t) &= g(t,x(t)) + \int_0^t f(t,s,x(s))ds \\
&= \begin{cases} x^*_{i-1}(t), \ t \in \left[0, \dfrac{(i-1)b}{m}\right] \\ g(t,x(t)) + \displaystyle\int_0^t f(t,s,x(s))ds, \ t \in \left[\dfrac{(i-1)b}{m}, \dfrac{ib}{m}\right] \end{cases} \\
&= \begin{cases} x^*_{i-1}(t), \ t \in \left[0, \dfrac{(i-1)b}{m}\right] \\ g(t,x(t)) + \displaystyle\int_0^{\frac{(i-1)b}{m}} f(t,s,x^*_{i-1}(s))ds + \int_{\frac{(i-1)b}{m}}^t f(t,s,x(s))ds, \\ \qquad t \in \left[\dfrac{(i-1)b}{m}, \dfrac{ib}{m}\right]. \end{cases}
\end{aligned}
$$

To prove that A_i is a contraction, let $x, y \in M_i$ and $0 \le t \le \frac{ib}{m}$ so that

$$
\begin{aligned}
|A_i(x)(t) - A_i(y)(t)| &\le L_g \, |x(t) - y(t)| + \int_{\frac{(i-1)b}{m}}^t |f(t,s,x(s)) - f(t,s,y(s))| \, ds \\
&\le \left(L_{g,i} + L_{f,i}(b)\frac{b}{m}\right) \max_{t \in [0, \frac{ib}{m}]} |x(t) - y(t)| \,.
\end{aligned}
$$

Thus

$$
\|A_i(x) - A_i(y)\|_i \le \left(L_g + L_{f,i}(b)\frac{b}{m}\right) \|x - y\|_i \,.
$$

We obtain that A_i is a contraction with the unique fixed point x^*_i on $[0, \frac{ib}{m}]$. $\square$

Remark. Since on each subinterval we apply the contraction principle, the unique solution of the problem can be obtained on each subinterval using successive approximation method, see [9, 11].

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References

[1] T.A. Burton, *Integral equations, transformations, and a Krasnoselskii-Schaefer type fixed point theorem*, Electronic J. Qual. Theory Differ. Equ., 2016, no. 66, 1-13; DOI: 10.14232/ejqtde.2016.1.66.

[2] T.A. Burton, *Existence and uniqueness results by progressive contractions for integro- differential equations*, Nonlinear Dynamics and Systems Theory, **16**(4)(2016), 366-371.

[3] T.A. Burton, *An existence theorem for a fractional differential equation using progressive contractions*, J. Fractional Calculus and Applications, **8**(1)(2017), 168-172.

[4] T.A. Burton, *A note on existence and uniqueness for integral equations with sum of two operators: progressive contractions*, Fixed Point Theory, **20**(2019), no. 1, 107-112.

[5] V. Ilea, D. Otrocol, *An application of the Picard operator technique to functional integral equations*, J. Nonlinear Convex Anal., **18**(2017), no. 3, 405-413.

[6] N. Lungu, I.A. Rus, *On a functional Volterra-Fredholm integral equation, via Picard operators*, J. Math. Ineq., **3**(2009), no. 4, 519-527.
[7] D. Otrocol, V. Ilea, *On the qualitative properties of functional integral equations with abstract Volterra operators*, Res. Fixed Point Theory Appl., Vol. 2018, Art. ID 201813, 8 pages.
[8] I.A. Rus, *A class of nonlinear functional-integral equations, via weakly Picard operators*, Anal. Univ. Craiova, Ser. Mat-Inf., **28**(2001), 10-15.
[9] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, 2001.
[10] I.A. Rus, *Picard operators and applications*, Sci. Math. Jpn., **58**(2003), no. 1, 191-219.
[11] I.A. Rus, *Abstract models of step method which imply the convergence of successive approximations*, Fixed Point Theory, **9**(2008), 293-297.
[12] M.A. Şerban, *Data dependence for some functional-integral equations*, J. Appl. Math., **1**(2008), no. 1, 219-234.
[13] M.A. Şerban, I.A. Rus, A. Petruşel, *A class of abstract Volterra equations, via weakly Picard operators technique*, Math. Inequal. Appl., **13**(2010), 255-269.

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