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PERIODIC SOLUTIONS FOR A SINGULAR DIFFERENTIAL EQUATION WITH INDEFINITE WEIGHTS

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Abstract. As a consequence of the main result of this paper efficient conditions guaranteeing the existence of at least one periodic solution to a second-order singular equation with indefinite weights. Our proof is based as Schauder's fixed point theorem and the positivity of the associated Green function. Besides, the results are applicable to weak as well as strong singularities.
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1. Introduction

The main purpose of this paper is to consider the existence of a periodic solution for the following singular equation with indefinite weights

$$x'' + a(t)x = \frac{h(t)}{x^\rho} + e(t), \tag{1.1}$$

where $a,\ h,\ e \in L^1(\mathbb{R}/T\mathbb{Z})$, ρ is a real constant and $\rho > 0$, T is a real positive constant. In eq. (1.1), the sign of the functions $a(t),\ h(t),\ e(t)$ are allowed to change. This means that the singularity associated to $\frac{h(t)}{x^\rho}$ at $x = 0$ is an indefinite type.

More recently, the method of lower and upper solutions [3, 12], Schauder's fixed point theorem [13, 15], Leray-Schauder alternative principle [7, 11], Krasnoselskii's fixed point theorem in cones [6, 16], and fixed point theorem [4, 5] have been employed to investigate the existence of periodic solutions of eq. (1.1) with singularity of attractive type (i.e. $h(t) < 0$ for all $t \in [0, T]$) or singularity of repulsive type (i.e. $h(t) > 0$ for all $t \in [0, T]$), where $a \in L^1(\mathbb{R}/T\mathbb{Z})$ is positive.

At the same time, there are several interesting results on indefinite singular equations [1, 2, 5, 8, 9, 10, 14, 17], which are interesting from the practice as well as from the theory. For examples, Hakl and Zamora [10] in 2017 discussed the existence of periodic solutions for a special form of eq. (1.1) (Emden-Fowler equation),

$$x'' = \frac{h(t)}{x^{\rho}}, \tag{1.2}$$

where $\rho \geq 1$ (i.e. indefinite strong singularity). After that, Godoy and Zamora [8, 9] in 2019 proved that eq. (1.2) had at least one periodic solution if $\rho \in (0,1)$ (i.e. indefinite weak singularity). Their proof were based on the Leray-Schauder degree theory. Recently, applying fixed point theorem in cones, Cheng and Cui [2] in 2021 studied the existence of at least one positive periodic solution for (1.1), where $a,\ e \in C(\mathbb{R}/T\mathbb{Z})$ are positive and $h \in C(\mathbb{R}/T\mathbb{Z})$. After that, Cheng et al. [5] considered the existence of periodic solutions for a special form of (1.1) (L_p-Minkowski problem), where $a(t) \equiv 1,\ e(t) \equiv 0$ and $\rho > 0$.

We are mainly motivated by the recent works [2, 8, 9, 10, 17] and focus on the existence of a periodic solution of (1.1) with indefinite weights. Our proof is based as the Schauder's fixed point theorem and the positivity of the associated Green function. This technique has been used before in related problems (see [15]), but in this case the consideration of weights $a(t)$ and $h(t)$ of indefinite sign supposes the main novelty and a considerable technical difficulty as well.

2. Preliminaries and notations

Our proof is based on the following Schauder's fixed point theorem, which can be found in [18, P. 61].

Lemma 2.1 *A compact operator $\Phi : \Omega \to \Omega$ has a fixed point provided Ω is a bounded, closed, convex, nonempty subset of a Banach space X over $\mathbb{R}$.*

A second important tool in our proofs is the concept of Green function. In what follows, we give positivity of the associated Green functions.

Lemma 2.2 (see [16, Corollary 2.3]) *Assume that $a \in L^p(\mathbb{R}/T\mathbb{Z})$ for some $1 \leq p \leq \infty$ and $a(t) \geq 0$ for almost every $t \in [0,T]$, then for any $f \in L^1(\mathbb{R}/T\mathbb{Z})$ the equation*

$$x'' + a(t)x = f(t)$$

has a unique T-periodic solution given by

$$x(t) = \int_0^T G_1(t,s) f(s) ds,$$

where $G_1(t,s)$ is called Green's function. Further, define

$$\mathcal{Q}(\alpha) = \begin{cases} \frac{2\pi}{\alpha T^{1+2/\alpha}} \left(\frac{2}{2+\alpha}\right)^{1-2/\alpha} \left(\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{\alpha}\right)}\right)^2, & \text{if } 1 \leq \alpha < \infty, \\ \frac{4}{T}, & \text{if } \alpha = \infty, \end{cases}$$

where Γ is the Gamma function, i.e., $\Gamma(t) = \int_0^{+\infty} x^{t-1} e^{-x} dx$. If

$$\|a\|_p := \left(\int_0^T |a(t)|^p dt\right)^{\frac{1}{p}} < \mathcal{Q}(2q), \tag{2.1}$$

where $q = \frac{p}{p-1}$ *if* $1 \le p < \infty$ *and* $q = 1$ *if* $p = +\infty$, *then* $G_1(t,s) > 0$ *for all* $(t,s) \in [0,T] \times [0,T]$.

Lemma 2.3 (see [16, Corollary 2.2]) *Assume that* $b \in L^1(\mathbb{R}/T\mathbb{Z})$ *and* $b(t) \ge 0$ *for almost every* $t \in [0,T]$, *then for any* $f \in L^1(\mathbb{R}/T\mathbb{Z})$ *the equation*

$$-x'' + b(t)x = f(t)$$

has a unique T*-periodic solution given by*

$$x(t) = \int_0^T G_2(t,s) f(s) ds.$$

Then $G_2(t,s) > 0$ *for all* $(t,s) \in [0,T] \times [0,T]$.

We denote from Lemmas 2.2 and 2.3 that

$$\begin{aligned} A_i &:= \min_{0 \le s,t \le T} G_i(t,s), \ B_i := \max_{0 \le s,t \le T} G_i(t,s), \\ \sigma_i &:= \frac{A_i}{B_i}, \ \gamma_i(t) = \int_0^T G_i(t,s) e(s) ds, \ i = 1,2. \end{aligned} \tag{2.2}$$

It is clear that $0 < A_1 \le B_1$ and $0 < \sigma_1 \le 1$ from (2.1) of Lemma 2.2, $0 < A_2 \le B_2$ and $0 < \sigma_2 \le 1$ from Lemma 2.3.

Finally, for a given periodic functions $h(t)$ and $e(t)$, we denote

$$h^+(t) := \max\{0, h(t)\}, \ h^-(t) := -\min\{0, h(t)\}, \ \overline{h} := \frac{1}{T}\int_0^T h(t)dt,$$

$$e^* := \max_{t \in [0,T]} e(t), \ e_* := \min_{t \in [0,T]} e(t).$$

3. Main results

In this sections, we study the existence of a periodic solution for eq. (1.1) with indefinite weights by using the Schauder's fixed point theorem (see Lemma 2.1). According to the sign of functions $\gamma_1(t)$ and $\gamma_2(t)$, we consider the following three cases.

Case (I) $\gamma_{1*} > 0$

Theorem 3.1 *Assume that* $\|a^+\|_p < \mathcal{Q}(2q)$ *and* $B_1 T \overline{a^-} < 1$ *hold.*
If $\gamma_{1*} > \left(\frac{\overline{h^-}}{\sigma_1 \overline{a^-}} \right)^{\frac{1}{1+\rho}}$, *then eq.* (1.1) *admits at least one* T*-periodic solution.*

Proof. We write eq. (1.1) as

$$x'' + a^+(t)x = \frac{h(t)}{x^\rho} + a^-(t)x + e(t), \tag{3.1}$$

a T-periodic solution of eq. (3.1) is just a fixed point of the map Φ defined by

$$(\Phi x)(t) := \int_0^T G_1(t,s) \left(\frac{h(s)}{x^\rho(s)} + a^-(s)x(s) + e(s) \right) ds, \tag{3.2}$$

and we know that $G_1(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$ by lemma 2.2.

Define

$$\Omega_1 := \{x \in C_T : r_1 \le x(t) \le R_1, \ \text{for all } t \in \mathbb{R}\},$$

where $C_T := \{x \in C(\mathbb{R}, \mathbb{R}) :\ x(t+T) = x(t),\ \text{for all } t \in \mathbb{R}\}$. It is clear that Ω_1 is a closed convex set. From $\gamma_{1*} > \left(\frac{\overline{h^-}}{\sigma_1 \overline{\overline{a^-}}}\right)^{\frac{1}{1+\rho}}$, the positive constants r_1 and R_1 can be fixed such that

$$R_1 > r_1 = \gamma_{1*} > \left(\frac{\overline{h^-}}{\sigma_1 \overline{\overline{a^-}}}\right)^{\frac{1}{1+\rho}}.$$

In the following, we prove that $\Phi(\Omega_1) \subset \Omega_1$. In fact, for any $x \in \Omega_1$, we get

$$\begin{aligned}(\Phi x)(t) &= \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} + a^-(s)x(s) + e(s)\right)ds \\ &= \int_0^T G_1(t,s)\left(\frac{h^+(s)}{x^\rho(s)} - \frac{h^-(s)}{x^\rho(s)} + a^-(s)x(s)\right)ds + \gamma_1(t) \\ &\geq -\frac{B_1 T\overline{h^-}}{r_1^\rho} + A_1 T\overline{a^-} r_1 + \gamma_{1*} \\ &\geq \gamma_{1*} = r_1,\end{aligned} \tag{3.3}$$

where $-\frac{B_1 T\overline{h^-}}{r_1^\rho} + A_1 T\overline{a^-} r_1 > 0$ holds because $r_1 > \left(\frac{\overline{h^-}}{\sigma_1 \overline{\overline{a^-}}}\right)^{\frac{1}{1+\rho}}$.

On the other hand, we have

$$\begin{aligned}(\Phi x)(t) &= \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} + a^-(s)x(s) + e(s)\right)ds \\ &= \int_0^T G_1(t,s)\left(\frac{h^+(s)}{x^\rho(s)} - \frac{h^-(s)}{x^\rho(s)} + a^-(s)x(s)\right)ds + \gamma_1(t) \\ &\leq \frac{B_1 T\overline{h^+}}{r_1^\rho} + B_1 T\overline{a^-} R_1 + \gamma_1^* \\ &= \frac{B_1 T\overline{h^+}}{\gamma_{1*}^\rho} + B_1 T\overline{a^-} R_1 + \gamma_1^* \\ &\leq R_1,\end{aligned} \tag{3.4}$$

where the last inequality holds because R_1 large appropriate and $B_1 T\overline{a^-} < 1$.

Therefore, (3.3) and (3.4) help us to get $\Phi(\Omega_1) \subset \Omega_1$. The proof is completed. □

Case (II) $\gamma_{1*} = 0$

Theorem 3.2 *Assume that* $\|a^+\|_p < \mathcal{Q}(2q)$, $BT\overline{a^-}\left(\frac{\sigma_1 \overline{h^+}}{\overline{\overline{h^-}}} + 1\right) < 1$ *hold. Further, suppose the follow condition is satisfied:*

$$\overline{h^+} > \frac{\overline{h^-}}{\gamma_1^*}\left(\frac{\overline{h^-}}{\sigma_1 \overline{\overline{a^-}}}\right)^{\frac{1}{1+\rho}} \quad \text{and } \overline{h^-} > \gamma_1^* \left(\frac{(\overline{h^+})^\rho}{A_1 T}\right)^{\frac{1}{1+\rho}}. \tag{3.5}$$

If $\gamma_{1_*} = 0$, *then eq.* (1.1) *admits at least one positive* T*-periodic solution if* $0 < \rho < 1$.

Proof. Define

$$\Omega_2 := \{x \in C_T : r_2 \leq x(t) \leq R_2,\ \text{for all } t \in \mathbb{R}\}.$$

It is clear that Ω_2 is a closed convex set.

By (3.5), the positive constants r_2 and R_2 can be fixed such that

$$\left(\frac{A_1 T\overline{h^-}}{\gamma_1^*}\right)^{\frac{1}{\rho}} > R_2 > \frac{\gamma_1^*\overline{h^+}}{\overline{h^-}} > r_2 > \left(\frac{\overline{h^-}}{\sigma_1\overline{\overline{a^-}}}\right)^{\frac{1}{1+\rho}}.$$

Next, we prove that $\Phi(\Omega_2) \subset \Omega_2$, here Φ is defined in (3.2). It follows from (3.3) that

$$\begin{aligned}(\Phi x)(t) \geq& \frac{A_1 T\overline{h^+}}{R_2^\rho} - \frac{B_1 T\overline{h^-}}{r_2} + A_1 T\overline{a^-} r_2 + \gamma_{1*}\\ \geq& \frac{A_1 T\overline{h^+}}{R_2^\rho}\end{aligned} \tag{3.6}$$

since $\gamma_{1*} = 0$ and $-\frac{B_1 T\overline{h^-}}{r_2} + A_1 T\overline{a^-} r_2 > 0$ from $r_2 > \left(\frac{\overline{h^-}}{\sigma\overline{\overline{a^-}}}\right)^{\frac{1}{1+\rho}}$.

On the other hand, it follows from (3.4) that

$$\begin{aligned}(\Phi x)(t) \leq& \frac{B_1 T\overline{h^+}}{r_2^\rho} - \frac{A_1 T\overline{h^-}}{R_2^\rho} + B_1 T\overline{a^-} R_2 + \gamma_1^*\\ \leq& \frac{B_1 T\overline{h^+}}{r_2^\rho} + B_1 T\overline{a^-} R_2,\end{aligned} \tag{3.7}$$

where $-\frac{A_1 T\overline{h^-}}{R_2^\rho} + \gamma_1^* < 0$ holds because $R_2 < \left(\frac{A_1 T\overline{h^-}}{\gamma_1^*}\right)^{\frac{1}{\rho}}$. From (3.6) and (3.7), some easy computations show that it is sufficient to find R_2 and r_2 such that

$$(\Phi x)(t) \geq \frac{A_1 T\overline{h^+}}{R_2^\rho} \geq r_2, \tag{3.8}$$

and

$$(\Phi x)(t) \leq \frac{B_1 T\overline{h^+}}{r_2^\rho} + B_1 T\overline{a^-} R_2 \leq R_2. \tag{3.9}$$

Since $R_2 < \left(\frac{A_1 T\overline{h^-}}{\gamma_1^*}\right)^{\frac{1}{\rho}}$ and $r_2 < \frac{\gamma_1^*\overline{h^+}}{\overline{h^-}}$, we give

$$(\Phi x)(t) \geq \frac{A_1 T\overline{h^+}}{R_2^\rho} \geq \frac{\gamma_1^*\overline{h^+}}{\overline{h^-}} > r_2.$$

Then (3.8) holds. Further, from $r_2 > \left(\frac{\overline{h^-}}{\sigma_1\overline{\overline{a^-}}}\right)^{\frac{1}{1+\rho}}$, we have

$$\frac{B_1 T\overline{h^+}}{r_2^\rho} + B_1 T\overline{a^-} R_2 = \frac{B_1 T\overline{h^+} R_2}{r_2^{1+\rho}} + B_1 T\overline{a^-} R_2 < \left(\frac{A_1 T\overline{a^- h^+}}{\overline{h^-}}\right) R_2 + B_1 T\overline{a^-} R_2 < R_2$$

since $B_1 T\overline{a^-}\left(\frac{\sigma_1\overline{h^+}}{\overline{h^-}} + 1\right) < 1$. Hence, (3.9) holds. The proof is completed. □

Case III $\gamma_2^* < 0$

Theorem 3.3 *Assume that $B_2 T\overline{a^+} < 1$ holds. If $-\left(\frac{\overline{h^+}}{\sigma_2\overline{\overline{a^+}}}\right)^{\frac{1}{1+\rho}} < \gamma_2^* < 0$, then eq.* (1.1) *admits at least one positive T-periodic solution.*

Proof. We write eq. (1.1) as

$$-x'' + a^-(t)x = -\frac{h(t)}{x^\rho} + a^+(t)x - e(t), \tag{3.10}$$

a T-periodic solution of eq. (3.10) is just a fixed point of the map Φ defined by

$$(\Psi x)(t) := \int_0^T G_2(t,s)\left(-\frac{h(s)}{x^\rho(s)} + a^+(s)x(s) - e(s)\right) ds,$$

and we know that $G_2(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$ by lemma 2.3.

From this point, the proof follows the same steps as Theorem 2.1. □

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