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DOMAINS OF GLOBAL CONVERGENCE FOR THE SECANT METHOD

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Abstract. In this study, we introduce a new result based on auxiliary points to establish the domains of global convergence for Secant method. We assume generalized conditions on divided difference operator that cover differential as well as non-differentiable operators. In addition, we obtain semilocal and local convergence results as a particular case. Moreover, we implement our theoretical results on nonlinear integral Hammerstein type equations to show the applicability of our work. Furthermore, we have obtained a new fixed point result by using the Secant method.

Key Words and Phrases: Secant method, recurrence relations, global convergence, convergence balls, integral equations.
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1. Introduction

Many real life problems arising in the field of Applied Mathematics and Engineering can be expressed in the form

$$\mathcal{F}(u) = 0 \tag{1.1}$$

where, $\mathcal{F} : \Delta \subset \mathcal{U} \to \mathcal{V}$ be nonlinear operator in an open convex domain $\Delta \subset \mathcal{U}$ of a Banach space $\mathcal{U}$ to the Banach space $\mathcal{V}$. Generally, the solution of (1.1) cannot be obtained in a closed form, so iterative methods are commonly used for solving (1.1). Thus, starting with one or more initial guesses a sequence $\{u_k\}$ of approximations is constructed so that an improved approximation is obtained at each step.

Notice that if the operator $\mathcal{F}$ is non-differentiable, we have to select the iterative method that does not use the derivative of the operator. To obtain derivative-free methods, we approximate the derivative of the operator $\mathcal{F}$ by divided differences of first order. Note that an operator $[u, v, \mathcal{F}] \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ is a divided difference of first order of the operator $\mathcal{F}$ on u and v $(u \neq v)$ if

$$[u, v, \mathcal{F}](u - v) = \mathcal{F}(u) - \mathcal{F}(v)$$

holds, where $\mathcal{L}(\mathcal{U}, \mathcal{V})$ is the set of bounded linear operator from $\mathcal{U}$ to $\mathcal{V}$. Derivative-free methods are very special since their application becomes essential when the evaluation of the derivatives of the operator is very expensive or the derivative does not exist.

The well known derivative-free method with memory, that uses divided differences of first order, is the Secant method which is given by

$$\begin{cases} u_{-1}, u_0 \in \Delta, \\ u_{k+1} = u_k - \mathcal{A}_k^{-1}\mathcal{F}(u_k), \qquad k \geqslant 0, \end{cases} \tag{1.2}$$

where $\mathcal{A}_k = [u_{k-1}, u_k; \mathcal{F}]$.

The convergence analysis of the Secant method is generally performed in two ways: semilocal and local convergence. The semilocal convergence requires conditions on the operator and initial guess whereas the local convergence require the conditions on the operator and the solution of the equation. The biggest challenge with semilocal convergence result is locating good starting points for the Secant method whereas with local convergence result is that the solution u^* of (1.1) is unknown.

Many authors investigated the convergence of the Secant method for differentiable operators (see [1, 2, 6, 4, 12, 14, 15]) and nondifferntiable operators (see [3, 13]) under different continuity conditions. Recently, researchers have used the conditions on an auxiliary point and the operator to obtain the domains of global convergence for point-to-point methods (see [9]). Ezquerro and Hernández [8] established the global convergence for Newton's method under Lipschitz condition on first Fréchet derivative. Further, they have extended their results for third order iterative methods (see [10], [11]).

To the best of our knowledge, the study of domains of global convergence for methods with memory in Banach spaces has not been carried out. Thus, the main novelty of our work and, therefore, the main objective of our study is to obtain domains of global convergence for the Secant method, which is a iterative method with memory. To do this, we introduce a variant on the auxiliary point technique used to obtain domains of global convergence for point-to-point methods. In the case of the Secant method we will introduce two auxiliary points, this scheme will allow us to obtain domains of global convergence for the Secant method, which is our main objective. In addition, following the mentioned study, we obtain semilocal and local convergence results as a particular case.

On the other hand, from the study of the global convergence domains, obtained for the Secant method, we will be able to obtain a fixed point type result restricted to the domain Δ of the Banach space $\mathcal{U}$. This study will be developed in Section 4.

Furthermore, we denote

$$B(u, \varrho) = \{v \in \mathcal{U}; \|v - u\| < \varrho\} \text{ and } \overline{B(u, \varrho)} = \{v \in \mathcal{U}; \|v - u\| \leq \varrho\}$$

for the open and closed balls with center in u and radius $\varrho > 0$ respectively.

2. Global Convergence of the Secant Method

Throughout the work we suppose that there exists at least one divided difference of first order defined in the Banach space $\mathcal{U}$ (see [5]).

On the other hand, let us note that $u_{k-1} \neq u_k$, for all $k \geqslant 1$, since that if $u_{k-1} = u_k$ for some $k \geqslant 1$, then the sequence $\{u_n\}$ given in (1.2) converges to u^*, a solution of (1.1), with $u^* = u_n = u_{k-1} = u_k$ for all $n \geqslant k+1$. Therefore, the operators $[u_{k-1}, u_k; \mathcal{F}]$ are always well defined.

We analyse the global convergence of (1.2) using recurrence relations under the following conditions:

(B_1) For some $\tilde{u}, \tilde{v} \in \Delta$ such that $\|\tilde{u} - \tilde{v}\| = \delta > 0$ and there exists $\tilde{\mathcal{A}}^{-1} = [\tilde{v}, \tilde{u}; \mathcal{F}]^{-1} \in L(\mathcal{V}, \mathcal{U})$ with $\|\tilde{\mathcal{A}}^{-1}\| \leq \beta$ and $\|\tilde{\mathcal{A}}^{-1}\mathcal{F}(\tilde{u})\| \leq \eta$.

(B_2) $\|[u, v, \mathcal{F}] - [p, q; \mathcal{F}]\| \leq L + K(\|u - p\| + \|v - q\|)$, for all $u, v, p, q \in \Delta$, $u \neq v, p \neq q$.

Notice that, from (B_2) condition, it follows that

$$\|[u, v, \mathcal{F}] - [\tilde{v}, \tilde{u}; \mathcal{F}]\| \leq L + \tilde{K}(\|u - \tilde{v}\| + \|v - \tilde{u}\|), \tag{2.1}$$

for all $u, v \in \Delta$, with $u \neq v$, and $\tilde{K} \leq K$.

After that, we give two technical results which are used later.

Lemma 2.1 *Let $\{u_k\}$ be the sequence defined by* (1.2)*, then for $u_0 \neq \tilde{u}$ we have*

(i) $\mathcal{F}(u_0) = \mathcal{F}(\tilde{u}) + [\tilde{v}, \tilde{u}; \mathcal{F}](u_0 - \tilde{u}) + ([u_0, \tilde{u}; \mathcal{F}] - [\tilde{v}, \tilde{u}; \mathcal{F}])(u_0 - \tilde{u})$.

(ii) $\mathcal{F}(u_k) = ([u_k, u_{k-1}; \mathcal{F}] - [u_{k-2}, u_{k-1}; \mathcal{F}])(u_k - u_{k-1})$.

Proof. The proof of Lemma is immediate by using (1.2) and divided difference properties hence details are omitted.

Lemma 2.2 *If $\beta(L + \tilde{K}(2\varrho + \delta)) < 1$, then for each $u, v \in B(\tilde{u}, \varrho) \subset \Delta$, with $u \neq v$, we have that there exists $[u, v; \mathcal{F}]^{-1}$ and*

(i) $\|[u, v; \mathcal{F}]^{-1}\| \leq \dfrac{\beta}{1 - \beta(L + \tilde{K}(2\varrho + \delta))}$,

(ii) $\|[u, v; \mathcal{F}]^{-1}[\tilde{v}, \tilde{u}; \mathcal{F}]\| \leq \dfrac{1}{1 - \beta(L + \tilde{K}(2\varrho + \delta))}$.

Proof. Indeed, from

$$\|I - [\widetilde{v}, \widetilde{u}; \mathcal{F}]^{-1}[u, v; \mathcal{F}]\| \leq \|[\widetilde{v}, \widetilde{u}; \mathcal{F}]^{-1}\|\|[\widetilde{v}, \widetilde{u}; \mathcal{F}] - [u, v; \mathcal{F}]\|\beta(L + \tilde{K}(2\varrho + \delta)) < 1$$

and the Banach lemma on invertible operators, it follows that the operator $[u, v; \mathcal{F}]^{-1}$ exists and items (i) and (ii) hold.

Let us study the behavior of the Secant method in its first iterations. So, let any two distinct points $u_{-1}, u_0 \in \overline{B(\tilde{u}, \varrho)}$ and using (1.2), we have

$$\|u_1 - u_0\| = \|[u_{-1}, u_0; \mathcal{F}]^{-1}\mathcal{F}(u_0)\| \leq \|[u_{-1}, u_0; \mathcal{F}]^{-1}[\tilde{v}, \tilde{u}, \mathcal{F}]\|\|[\tilde{v}, \tilde{u}, \mathcal{F}]^{-1}\mathcal{F}(u_0)\|. \tag{2.2}$$

Using Lemma 2 and Lemma 2, we get

$$\|[u_{-1}, u_0; \mathcal{F}]^{-1}[\tilde{v}, \tilde{u}; \mathcal{F}]\| \leq \frac{1}{1 - \beta(L + \tilde{K}(2\varrho + \delta))},$$

and, for $u_0 \neq \tilde{u}$, we have

$$\begin{aligned}\|[\tilde{v},\tilde{u},\mathcal{F}]^{-1}\mathcal{F}(u_0)\| &\leq \|[\tilde{v},\tilde{u},\mathcal{F}]^{-1}\mathcal{F}(\tilde{u})\| + \|(u_0-\tilde{u})\| \\ &\quad + \|[\tilde{v},\tilde{u},\mathcal{F}]^{-1}\|(\|[u_0,\tilde{u};\mathcal{F}]-[\tilde{v},\tilde{u};\mathcal{F}]\|)\|(u_0-\tilde{u})\| \\ &\leq \eta+\varrho+\beta(L+\tilde{K}(\varrho+\delta))\varrho.\end{aligned}$$

Therefore,

$$\|u_1-u_0\| \leq \frac{\eta+\varrho+\beta(L+\tilde{K}(\varrho+\delta))\varrho}{1-\beta(L+\tilde{K}(2\varrho+\delta))}, \tag{2.3}$$

and, as $u_0 \neq \tilde{u}$ and $u_{-1} \neq u_0$, it follows

$$\begin{aligned}\|u_1-\tilde{u}\| &\leq \big\|[u_{-1},u_0;\mathcal{F}]^{-1}\big(\mathcal{F}(u_0)-\mathcal{F}(\tilde{u})-[u_{-1},u_0;\mathcal{F}](u_0-\tilde{u})+\mathcal{F}(\tilde{u})\big)\big\| \\ &\leq \big\|[u_{-1},u_0;\mathcal{F}]^{-1}\big([u_0,\tilde{u};\mathcal{F}]-[u_{-1},u_0;\mathcal{F}]\big)(u_0-\tilde{u})\big\| + \big\|[u_{-1},u_0;\mathcal{F}]^{-1}\mathcal{F}(\tilde{u})\big\| \\ &\leq \frac{\eta+\beta(L+3K\varrho)\varrho}{1-\beta(L+\tilde{K}(2\varrho+\delta))}.\end{aligned} \tag{2.4}$$

Thus, $u_1 \in \overline{B(\tilde{u},\varrho)}$ if $\dfrac{\eta+\beta(L+3K\varrho)\varrho}{1-\beta(L+\tilde{K}(2\varrho+\delta))} \leq \varrho$. Now, from Lemma 2, we have

$$\|[u_0,u_1\mathcal{F}]^{-1}[\tilde{v},\tilde{u};\mathcal{F}]\| \leq \frac{1}{1-\beta(L+\tilde{K}(2\varrho+\delta))}$$

and

$$\|\mathcal{F}(u_1)\| \leq \|[u_1,u_0;\mathcal{F}]-[u_{-1},u_0;\mathcal{F}]\|\|u_1-u_0\| \leq (L+2K\varrho)\|u_1-u_0\|.$$

Therefore,

$$\|u_2-u_1\| \leq \|[u_0,u_1;\mathcal{F}]^{-1}\|\|\mathcal{F}(u_1)\| \leq \frac{\beta(L+2K\varrho)}{1-\beta(L+\tilde{K}(2\varrho+\delta))}\|u_1-u_0\| = a\|u_1-u_0\|, \tag{2.5}$$

where, $a = \dfrac{\beta(L+2K\varrho)}{1-\beta(L+\tilde{K}(2\varrho+\delta))}$. Notice that $a \leq \dfrac{\beta(L+2K\varrho)\varrho}{\eta+\beta(L+3K\varrho)\varrho} < 1$.

On the other hand, as $u_1 \in \overline{B(\tilde{u},\varrho)}$, there can be two cases: $u_1 = \tilde{u}$ or $u_1 \neq \tilde{u}$.

If $u_1 = \tilde{u}$, then $u_0 \neq \tilde{u}$, since $u_1 = u_0$, and $[u_0,\tilde{u};\mathcal{F}]$ is well defined. In addition, we get

$$\begin{aligned}\|u_2-\tilde{u}\| &\leq \big\|u_1-[u_0,u_1;\mathcal{F}]^{-1}\mathcal{F}(u_1)-\tilde{u}\big\| \\ &\leq \big\|[u_0,\tilde{u};\mathcal{F}]^{-1}[\tilde{v},\tilde{u};\mathcal{F}]\big\|\big\|[\tilde{v},\tilde{u};\mathcal{F}]^{-1}\mathcal{F}(\tilde{u})\big\| \\ &\leq \frac{\eta}{1-\beta(L+\tilde{K}(2\varrho+\delta))} \leq \varrho,\end{aligned}$$

so that $u_1 \in \overline{B(\tilde{u},\varrho)}$.

If, on the contrary, $u_1 \neq \tilde{u}$, then $[u_1, \tilde{u}; \mathcal{F}]$ exists and

$$\begin{aligned}\|u_2 - \tilde{u}\| &\leq \big\|[u_0, u_1; \mathcal{F}]^{-1}\big([u_1, \tilde{u}; \mathcal{F}] - [u_0, u_1; \mathcal{F}]\big)(u_1 - \tilde{u})\big\| + \big\|[u_0, u_1; \mathcal{F}]^{-1}\mathcal{F}(\tilde{u})\big\| \\ &\leq \frac{\eta + \beta(L + 3K\varrho)\varrho}{1 - \beta(L + \tilde{K}(2\varrho + \delta))} \leq \varrho.\end{aligned}$$

Thus, $u_2 \in \overline{B(\tilde{u}, \varrho)}$.

Then, by mathematical induction, we can easily prove the recurrence relations, for the Secant method, which are shown in the following items for $k \geq 1$:

(i) $\|\mathcal{F}(u_k)\| \leq (L + 2K\varrho)\|u_k - u_{k-1}\|$.
(ii) $\|u_{k+1} - u_k\| \leq a\|u_k - u_{k-1}\| \leq a^2\|u_{k-1} - u_{k-2}\| \leq \cdots \leq a^n\|u_1 - u_0\|$.
(iii) $\|u_{k+1} - \tilde{u}\| \leq \dfrac{\eta + \beta(L + 3K\varrho)\varrho}{1 - \beta(L + \tilde{K}(2\varrho + \delta))} \leq \varrho$.

Next, using all of the above, we obtain the following global convergence result for the Secant method.

Theorem 2.3 *Under the conditions* $(B_1)-(B_2)$*, assume that there exists* $\varrho > 0$ *which satisfies* $\dfrac{\eta + \beta(L + 3K\varrho)\varrho}{1 - \beta(L + \tilde{K}(2\varrho + \delta))} \leq \varrho$*, we consider* $\delta < \dfrac{1 - \beta(L + 2\tilde{K}\varrho)}{\beta\tilde{K}}$ *and suppose that* $\overline{B(\tilde{u}, \varrho)} \subset \Delta$*, then the Secant method* (1.2) *is well-defined and converges to a solution* u^* *of* $\mathcal{F}(u) = 0$*, with* $u^* \in \overline{B(\tilde{u}, \varrho)}$*, from any two distinct points* u_{-1}*,* $u_0 \in \overline{B(\tilde{u}, \varrho)}$.

Proof. For the convergence, we have to prove that $\{u_k\}$ is Cauchy sequence. For this, consider

$$\begin{aligned}\|u_{k+m} - u_k\| &\leq \|u_{k+m} - u_{k+m-1}\| + \|u_{k+m-1} - u_{k+m-2}\| + \cdots + \|u_{k-1} - u_k\| \\ &\leq a^k(a^{m-1} + a^{m-2} + \cdots + 1)\|u_1 - u_0\| \leq \frac{a^k(1 - a^m)}{1 - a}\|u_1 - u_0\| \\ &< \frac{a^k}{1 - a}\|u_1 - u_0\|.\end{aligned}$$

As $k \to \infty$, $\{u_k\} \to u^*$ as $a < 1$ and $\|\mathcal{F}(u_k)\| \leq (L + 2K\varrho)\,\|u_k - u_{k-1}\|$. Thus, $\|u_k - u_{k-1}\| \to 0$ as $k \to \infty$, we obtain $\mathcal{F}(u^*) = 0$.

Lemma 2.4 *Under the conditions of the previous theorem, the solution* u^* *is unique in* $B(\tilde{u}, \zeta)$*, where* ζ *is given by*

$$\zeta = \frac{1}{\tilde{K}}\Big(\frac{1}{\beta} - L\Big) - (\delta + \varrho). \tag{2.6}$$

Proof. We assume that $v^* \in B(\tilde{u}, \zeta) \cap \Delta$ be the another solution of $\mathcal{F}(u) = 0$ such that $u^* \neq v^*$. Consider

$$\mathcal{F}(u^*) - \mathcal{F}(v^*) = [u^*, v^*, \mathcal{F}](u^* - v^*) = H(u^* - v^*) = 0,$$

where, $H = [u^*, v^*, \mathcal{F}]$. If H is invertible, then we get $u^* = v^*$. From

$$\begin{aligned} \|I - [\tilde{v}, \tilde{u}, \mathcal{F}]^{-1}[u^*, v^*, \mathcal{F}]\| &\leq \|[\tilde{v}, \tilde{u}, \mathcal{F}]^{-1}\| \|[[\tilde{v}, \tilde{u}, \mathcal{F}] - [u^*, v^*, \mathcal{F}]]\| \|u^* - v^*\| \\ &\leq \beta(L + \tilde{K}(\|\tilde{v} - u^*\| + \|\tilde{u} - v^*\|)) \\ &\leq \beta(L + \tilde{K}(\delta + \varrho + \zeta)) = 1. \end{aligned}$$

Hence, the result is proved.

Corollary 2.5 *Under the conditions of Theorem 2, the solution u^* of $\mathcal{F}(u) = 0$ is unique in $\overline{B(\tilde{u}, \varrho)}$.*

Proof. Notice that, taking into account (2.6), $\zeta > \varrho$ if and only if $\beta(L+\tilde{K}(2\varrho+\delta)) < 1$. But this condition hold under the conditions of Theorem 2.

2.1. **Particular cases.** In this Section, we consider the semilocal and local convergence analysis for the Secant method (1.2) which are the particular cases of global convergence.

Theorem 2.1.1 *Under the conditions*

- (C_1) *For some $u_0, u_{-1} \in \Delta$ such that $\|u_{-1} - u_0\| = \delta_1 > 0$ and there exists $\mathcal{A}^{-1} = [u_{-1}, u_0; \mathcal{F}]^{-1}$ with $\|\mathcal{A}^{-1}\| \leq \beta$ and $\|\mathcal{A}^{-1}\mathcal{F}(u_0)\| \leq \eta_1$*
- (C_2) *$\|[u, v, \mathcal{F}] - [p, q; \mathcal{F}]\| \leq L + K(\|u - p\| + \|v - q\|)$, for all $u, v, p, q \in \Delta$, $u \neq v, p \neq q$*

and $u_{-1}, u_0 \in \Delta$, if the equation

$$(1 - \beta_1(2L + K(4t + \delta_1)))t - (1 - \beta_1(L + K(2t + \delta_1)))\eta_1 = 0$$

has at least one positive root and the smallest root denoted by ϱ_1 which satisfies $\beta_1(L+K(4\varrho_1+\delta_1)) < 1$, then the sequence $\{u_k\}$ defined in (1.2) is well defined and converges to the solution u^* of the equation $\mathcal{F}(u) = 0$ in the domain $\overline{B(u_0, \varrho_1)}$.

Proof. If we consider $\tilde{v} = u_{-1}$ and $\tilde{u} = u_0$, the proof is same as Theorem 2, hence the details are omitted.

Theorem 2.1.2 *Under the conditions*

- (D_1) *For some $\tilde{v}, u^* \in \Delta$ such that $\|u^* - \tilde{v}\| = \delta_2 > 0$ and there exists $\mathcal{A}^{*-1} = [\tilde{v}, u^*; \mathcal{F}]^{-1}$ with $\|\mathcal{A}^{*-1}\| \leq \beta$.*
- (D_2) *$\|[u, v, \mathcal{F}] - [p, q; \mathcal{F}]\| \leq L + K(\|u - p\| + \|v - q\|)$, for all $u, v, p, q \in \Delta$, $u \neq v, p \neq q$.*

if the equation

$$(1 - \beta_2(2L + K(4t + \delta_2))) = 0$$

has at least one positive root and the smallest root denoted by ϱ_2 such that

$$\beta(L + K(2\varrho_2 + \delta_2)) < 1,$$

then the iterative sequence $\{u_k\}$ defined in (1.2) is well defined and converges to the solution u^* of the equation $\mathcal{F}(u) = 0$ for all distinct point $u_{-1}, u_0 \in \overline{B(u^*, \varrho_2)}$.

Proof. If we consider $\tilde{u} = u^*$, the proof is same as Theorem 2, hence the details are omitted.

Note that, in the last two situations, we have considered $K = \widetilde{K}$.

3. Numerical Example

To show the applicability of our work, we consider a nonlinear integral equation in which non-differentiable term is involved. Consider the nonlinear integral equation of mixed Hammerstein type

$$u(t) = g(t) + \int_c^d \mathcal{H}(t,s)K(s,u(s))ds \qquad t \in [c,d] \tag{3.1}$$

where $-\infty < c < d < \infty, g, K$ are the known function, $\mathcal{H}(t,s)$ is the Green's function defined by

$$\mathcal{H}(t,s) = \begin{cases} \frac{(d-t)(s-c)}{d-c}, & s \le t \\ \frac{(t-c)(d-s)}{d-c}, & t \le s. \end{cases}$$

and u is the unknown function which is to be determined. Solving (3.1) is equivalent to solve $\mathcal{F}(u) = 0$, where

$$\mathcal{F}(u(t)) = u(t) - g(t) - \int_c^d \mathcal{H}(t,s)K(s,u(s))ds \qquad t \in [c,d] \tag{3.2}$$

Now, to approximate the numerical solution, we approach the integral by Gauss Legendre formula with p nodes

$$\int_c^d \mathcal{P}(t)dt \simeq \sum_{j=1}^{p} \omega_j \mathcal{P}(t_j)$$

where ω_j and t_j are weights and nodes of quadrature formula. Thus, we obtain the following nonlinear system

$$\mathcal{F}(u) = u - g - BY = 0$$

where $\mathcal{F} : \mathcal{R}^p \to \mathcal{R}^p$, $u = (u_1, u_2, \cdots, u_p)^t$, $g = (g_1, g_2, \cdots, g_p)^t, B = (b_{ij})_{i,j=1}^p$, $Y = (K(s_1,u_1), K(s_2,u_2), \cdots, K(s_p,u_p))$. The component of $\mathcal{F}$ are given by

$$\mathcal{F}_k = u_k - g_k - \sum_{j=1}^{p} b_{jk} K(s_j, u_j), \qquad k = 1, 2, \cdots, p. \tag{3.3}$$

and

$$b_{jk} = \omega_k \mathcal{H}(s_j, s_k) = \begin{cases} \omega_k \dfrac{(d - v_j)(v_k - c)}{d - c}, & k \le j \\ \omega_k \dfrac{(v_j - c)(d - v_k)}{d - c}, & j \le k. \end{cases}$$

Consider $K(s,u(s)) = \lambda u(s)^2 + \mu|u(s)|$, g=(1,1,...,1) and 8 nodes. Thus (3.3) becomes

$$\mathcal{F}_k = u_k - 1 - \sum_{j=1}^{8} b_{jk}(\lambda u_j^2 + \mu|u_j|), \qquad k = 1, 2, \cdots, 8. \tag{3.4}$$

For $u, v \in \mathcal{R}^8$, we consider the first-order divided difference given by

$$[u,v;\mathcal{F}]_{jk} = \frac{1}{u_k - v_k}(\mathcal{F}_j(u_1, u_2, \ldots, u_k, v_{k+1}, \ldots, v_8) - \mathcal{F}_j(u_1, u_2, \ldots, u_{k-1}, v_k, \ldots, v_8)).$$

(λ, μ)	Existence radius	Global convergence radius	Uniqueness radius
$(1/2, 1/2)$	0.2427625	1.608037	12.378563
$(1/2, 1/4)$	0.132844	2.213581	14.853755
$(1/4, 1/4)$	0.078996	5.0024041	30.0942041
$(1/4, 1/6)$	0.061624	5.351597	30.778242

TABLE 1. Radius obtained for different value of λ, μ and $\tilde{u} = (1, 1, \ldots, 1)$, $\tilde{v} = (0.8, 0.8, \ldots, 0.8)$.

So, we obtain

$$[u, v; \mathcal{F}]_{jk} = I - (\lambda U + \mu V),$$

where

$$u = (u_1, u_2, \ldots, u_8),\ \ v = (v_1, v_2, \ldots, v_8),$$

$$U = (U_{jk})_{j,k=1}^{8} = b_{jk}(u_k + v_k)$$

and

$$V = (V_{jk})_{j,k=1}^{8} = B_{jk}\frac{|u_k| - |v_k|}{u_k - v_k}.$$

Thus, we obtain

$$[u, v; \mathcal{F}] - [p, q; \mathcal{F}] \leq \|B\|(\lambda(\|u - p\| + \|v - q\|) + 2\mu)$$

By comparing with condition (B_2) and (2.1), we get $L = 2\mu\|B\|, K = \tilde{K} = \lambda\|B\|$.

For different value of λ, μ and δ we can find the radius of existence ball, global convergence ball and uniqueness ball for the solution. As we can observe that, for smaller value of λ and μ we obtain better solution localization (smaller radius of existence ball), greater accessibility (larger radius of global convergence ball) and better separation from other solutions (larger radius of uniqueness ball), as shown in Table 1 and Table 2.

(λ, μ)	Existence radius	Global convergence radius	Uniqueness radius
$(1/2, 1/2)$	0.245165	1.6004571	13.141404
$(1/2, 1/4)$	0.133851	2.208448	14.252749
$(1/4, 1/4)$	0.0792475	4.999568	29.493952
$(1/4, 1/6)$	0.061812	5.349168	30.178054

TABLE 2. Radius obtained for different value of λ, μ and $\tilde{u} = (1, 1, \ldots, 1)$, $\tilde{v} = (0.2, 0.2, \ldots, 0.2)$.

4. A fixed point result

In this Section we focus our study on locating a fixed point of an operator defined in a Banach space. Banach contraction principle is a very important tool in the theory of Banach spaces. The Banach contraction principle says [16]:

Theorem 4.1 *Let $\mathcal{U}$ be a Banach space. Let $T : \mathcal{U} \to \mathcal{U}$ be a contraction mapping, with Lipschitz constant $k < 1$. Then T has a unique fixed point $u^* \in \mathcal{U}$. Moreover, for every $u_0 \in \mathcal{U}$, the method of successive approximations, $u_{n+1} = T(u_n)$, $n \in \mathbb{N}$, converges to u^*.*

Notice that this result also holds for self operators T on a nonempty and closed subset of a Banach space.

Theorem 4.2 ([17]) *If $\mathcal{K}$ is a non-empty closed subset of a Banach space $\mathcal{U}$ and the operator $T : \mathcal{K} \to \mathcal{K}$ is a contraction, then T has a unique fixed point u^* in $\mathcal{K}$. Moreover, for every $u_0 \in \mathcal{K}$, the method of successive approximations, $u_{n+1} = T(u_n)$, $n \in \mathbb{N}$, converges to u^*.*

Since our aim is to approximate a fixed point of T, where $T : \mathcal{U} \to \mathcal{U}$, by the Secant method given in (1.2) and this algorithm approximates solutions of an equation of the form $\mathcal{F}(u) = 0$, we write $\mathcal{F}(u) = I(u) - \mathcal{T}(u)$, where $I : \mathcal{U} \to \mathcal{U}$ is the identity operator, so that $0 = \mathcal{F}(u^*) = u^* - \mathcal{T}(u^*)$ and therefore a solution u^* of $\mathcal{F}(u) = 0$ is a fixed point of T.

Next, from the previous analysis of global convergence of the Secant method we obtain a closed ball where the existence and uniqueness of fixed point for T is guaranteed.

Now, we can establish a fixed point result from the Secant method.

Theorem 4.3 *Under conditions of the Theorem 2, then there exists a unique fixed point u^* of the operator T in $\overline{B(\widetilde{u}, \varrho)}$. Moreover, the sequence given by the Secant method converges to the fixed point u^* starting at any two distinct points $u_0, u_{-1} \in \overline{B(\widetilde{u}, \varrho)}$.*

Therefore, we present in this Theorem a result that allows us to locate fixed points of the operator T in a given domain and approximate it with superlinear speed of convergence by the Secant method, which has order of convergence $\frac{1+\sqrt{5}}{2}$ (see [1, 7]).

5. Conclusions

An important problem in the application of iterative processes is the accessibility to a solution, that is, the location of starting points that ensure the convergence of the iterative process to a solution. For this reason, it is very interesting to obtain domains of global convergence for iterative processes, which allows us to easily select starting points. To date, domains of global convergence have been studied for point-to-point iterative processes but not for iterative processes with memory. In this work we have obtained domains of global convergence for a method with memory, the Secant method. The global convergence result obtained for the Secant method also allows us to obtain, as an immediate consequence, both local and semilocal convergence results for the Secant method. Furthermore, we have obtained a new fixed point result by using the Secant method.

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