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A NEW CLASS OF CONTRACTIONS ON GRAPHICAL FUZZY METRIC SPACES AND FIXED POINT THEOREMS

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Abstract. The aim of this paper is to introduce a new class of mappings called (H, H')-fuzzy graphical ξ-contractions in graphical fuzzy metric spaces. Some fixed point results for the mappings of this class are proved. This class of mappings and the fixed point results of this paper generalize and extend several fuzzy analogues of Banach contraction principle in graphical fuzzy metric spaces. Some examples and several consequences of the new concepts and results are given.
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1. Introduction

In 1965, Zadeh introduced fuzzy sets which are useful in understanding and explaining the systems with uncertainty and vagueness present in their nature. Kramosil and Michálek [12] used this approach in extending the idea of usual distance functions to the fuzzy distance functions and consequently, they introduced the notion of fuzzy distance or fuzzy metric space. Afterwards, by considering the fuzzy distance of two points as a measure of degree of nearness of those points with respect to a parameter $t \in (0, +\infty)$, George and Veeramani [3] gave some modifications to the definition of fuzzy metric spaces due to Kramosil and Michálek [12]. The existence of fixed points of mappings of contractive nature in fuzzy metric spaces was considered by Grabiec [4] which has become of interest to many authors. The contractive condition used by Grabiec [4] is directly associated with the parameter t. For the existence of fixed point, the contractive condition of Grabiec [4] demands an additional condition that the fuzzy distance between any two points of spaces must tend to 1 as the parameter t tends to $+\infty$. Gregori and Sapena [6] introduced another class of contractive mappings which uses the contractive constant independent of the parameter t and showed the existence of fixed points for such mappings. Miheţ [14, 15] introduced fuzzy ψ-contractive mappings in fuzzy metric spaces (in the sense of Kramosil and Michálek [12]) and generalized the class of contractive mappings introduced by Gregori and Sapena [6]. The results discussed above are followed by several extensions and generalizations, see e.g., [25, 19, 20] and the references therein.

The role of triangular inequality plays an important role in the structure of metric and fuzzy metric spaces. Shukla et al. [22] introduced the concept of graphical metric spaces and weakened the triangular inequality by using a graphical structure associated with the space. Recently, Saleem et al. [17] extended the idea of Shukla et al. [22] to its fuzzy analogue and introduced graphical fuzzy metric spaces and proved some fixed point results in such spaces. These results use the contractive condition directly associated with the parameter, and so, the results of Saleem et al. [17] are an extension of Grabiec [4]. In the paper [23] authors improved some notions and results of Saleem et al. [17]. The purpose of this paper is to introduce a new class of contraction mappings in graphical fuzzy metric spaces, called (H, H')-fuzzy graphical ξ-contraction, and to prove some fixed point results for the mappings of this new class. Our results retrieve the fixed point results of Miheţ [14], Gregori and Sapena [6], Wardowski [25], and Tirado [24] in a more general context of graphical fuzzy metric spaces. For illustrations and justification, some examples are provided.

The remainder of the paper is organized as follows. Section 2 is devoted to compile all basics need in our study. Then, in Section 3 are provided our main results. Finally, in Section 4 we set the conclusions on the study carried out and propose two open problems.

2. Preliminaries

We recall some known definitions and the properties of fuzzy metric spaces, graphs, and graphical metric spaces.

Definition 2.1 (Schweizer and Sklar [18]). A binary operation $*\colon [0,1]\times[0,1] \to [0,1]$ is called a t-norm if the following conditions are satisfied:

(T1) $a * b = b * a$;
(T2) $a * b \le c * d$ for $a \le c, b \le d$;
(T3) $(a * b) * c = a * (b * c)$;
(T4) $a * 0 = 0, a * 1 = a$; for all $a, b, c, d \in [0,1]$.

For $a_1, a_2, ..., a_n \in [0,1]$ and $n \in \mathbb{N}$, the expression $a_1 * a_2 * \cdots * a_n$ will be denoted by $\prod_{i=1}^{n} a_i$. In further discussion, by $*_m$, $*_p$ and $*_L$ we denote the minimum, product and Lukasiewicz t-norms respectively, i.e., $a *_m b = \min\{a, b\}$, $a *_p b = ab$ and $a *_L b = \max\{a + b - 1, 0\}$ for all $a, b \in [0,1]$. For the details concerning t-norms the reader is referred to [11, 7].

For a nonempty set X, let $\Delta = \{(x,x) \colon x \in X\}$ (see, [8, 22]) and consider a directed graph H, such that $V(H) = X$, $E(H) \supseteq \Delta$ and H is without parallel edges. In this case, the set X is said to be endowed with the graph $H = (V(H), E(H))$. By H^{-1}, we define a graph such that $V(H^{-1}) = V(H)$, $E(H^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(H)\}$. The graph H induces an undirected graph H_I such that $V(H_I) = V(H)$ and $E(H_I) = E(H) \cup E(H^{-1})$. For two vertices u and v in H, a path from u to v of length $n \in \mathbb{N}$ in H is a sequence $\{x_i\}_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = u, x_n = v$ and $(x_{i-1}, x_i) \in E(H)$ for $i = 1, 2, \ldots, n$. A graph H is called connected if, there is a path between any two vertices. H is weakly connected if the induced undirected graph H_I is connected. We define a relation P on X by:

$$P = \{(u, v) \in X \times X \colon \text{ there is a directed path from } u \text{ to } v \text{ in } H\}.$$

We write $(uPv)_{\mathrm{H}}$ if $(u, v) \in P$. We say that a vertex w falls on some directed path joining u and v, and we write $w \in (uPv)_{\mathrm{H}}$, if w is contained in some directed path from u to v in H. For a natural number n, we denote

$$[u]_H^n = \{v \in X \colon \text{ there is a directed path from } u \text{ to } v \text{ of length } n\}.$$

If for a sequence $\{x_n\}$ in X we have $(x_n P x_{n+1})_{\mathrm{H}}$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ is said to be an H-termwise connected sequence. For a mapping $T \colon X \to X$, a sequence $\{x_n\}$ in X is called a T-Picard sequence (or a Picard sequence generated by T) with initial value $x_0 \in X$ if $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. By $\mathrm{Fix}(T)$, we denote the set of all fixed points of T. Also we denote $X_T = \{x \in X : (x, Tx) \in E(H)\}$. The graph H is called transitive if $(x, y), (y, z) \in E(H)$ implies $(x, z) \in E(H)$.

All the graphs considered in this paper are directed and with nonempty sets of vertices and edges.

Definition 2.2 (Shukla et al. [22])**.** Let X be a nonempty set endowed with a graph H and $d_{\mathrm{H}} : X \times X \to \mathbb{R}$ be a function satisfying the following conditions:

- (GM_1) $d_{\mathrm{H}}(x, y) \geq 0$ for all $x, y \in X$;
- (GM_2) $d_{\mathrm{H}}(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
- (GM_3) $d_{\mathrm{H}}(x, y) = d_{\mathrm{H}}(y, x)$ for all $x, y \in X$;
- (GM_4) $(xPy)_{\mathrm{H}}, z \in (xPy)_{\mathrm{H}}$ implies $d_{\mathrm{H}}(x, y) \leq d_{\mathrm{H}}(x, z) + d_{\mathrm{H}}(z, y)$ for all $x, y, z \in X$.

Then, the mapping d_{H} is called a graphical metric on X, and the pair (X, d_{H}) is called a graphical metric space. For examples and properties of graphical metric spaces, we refer to [21, 22].

Definition 2.3 (Saleem et al. [17])**.** Let X be a nonempty set endowed with a graph H, $*$ be a continuous t-norm and $M_{\mathrm{H}} \colon X \times X \times (0, +\infty) \to [0, 1]$ is a fuzzy set. Then the triplet $(X, M_{\mathrm{H}}, *)$ is called a graphical fuzzy metric space and M_{H} is called a graphical fuzzy metric on X if the following conditions are satisfied:

- (GFM1) $M_{\mathrm{H}}(x, y, t) > 0$;
- (GFM2) $M_{\mathrm{H}}(x, y, t) = 1$ if and only if $x = y$;
- (GFM3) $M_{\mathrm{H}}(x, y, t) = M_{\mathrm{H}}(y, x, t)$;
- (GFM4) $(xPy)_{\mathrm{H}}, z \in (xPy)_{\mathrm{H}}$ implies $M_{\mathrm{H}}(x, y, t+s) \geq M_{\mathrm{H}}(x, z, t) * M_{\mathrm{H}}(z, y, s)$;
- (GFM5) $M_{\mathrm{H}}(x, y, \cdot) : (0, +\infty) \to [0, 1]$ is a continuous mapping;

for all $x, y, z \in X$ and $s, t > 0$.

If the graph H is considered as the cross product $X \times X$, then the corresponding graphical fuzzy metric space reduces to a fuzzy metric space in the sense of George and Veeramani [3], hence, graphical fuzzy metric spaces are a generalization of fuzzy metric spaces. For examples and topological properties of a graphical fuzzy metric space, we refer to [23].

The following definition of "M_{H}-convergence" in the context of a graphical fuzzy metric will be used in developing the fixed point theory in graphical fuzzy metric spaces, and it is similar to the definition of "convergence" used in [17].

Definition 2.4 (Shukla et al. [23])**.** Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space. A sequence $\{x_n\}$ in X will be called M_{H}-convergent to $x \in X$ if for each $r \in (0,1)$ and $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $1 - M_{\mathrm{H}}(x_n, x, t) < \varepsilon$ for all $n > n_0$ (i.e. $\lim_{n\to+\infty} M_{\mathrm{H}}(x_n, x, t) = 1$ for all $t > 0$). In such a case, as usual, we will say that $\{x_n\}$ is M_{H}-convergent in X.

As customary, we define the concepts of Cauchy sequences and completeness of graphical fuzzy metric spaces in two ways.

Definition 2.5 (Shukla et al. [23])**.** Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space and $\{x_n\}$ be a sequence in X. Then:

(I) $\{x_n\}$ is called a G-Cauchy sequence if for all $t > 0$ and each $p \in \mathbb{N}$ we have $\lim_{n\to+\infty} M_{\mathrm{H}}(x_{n+p}, x_n, t) = 1$. Or equivalently, the sequence $\{x_n\}$ is G-Cauchy if for all $t > 0$, $0 < \varepsilon < 1$ and each $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $1 - M_{\mathrm{H}}(x_n, x_{n+p}, t) < \varepsilon$ for all $n > n_0$. A G-complete graphical fuzzy metric space is a graphical fuzzy metric space in which every G-Cauchy sequence is M_{H}-convergent.

(II) $\{x_n\}$ is called an M-Cauchy sequence if for all $t > 0$ we have $\lim_{n,m\to+\infty} M_{\mathrm{H}}(x_n, x_m, t) = 1$. Or equivalently, the sequence $\{x_n\}$ is M-Cauchy if for all $t > 0$ and $0 < \varepsilon < 1$ there exists $n_0 \in \mathbb{N}$ such that $1 - M_{\mathrm{H}}(x_n, x_m, t) < \varepsilon$ for all $n, m > n_0$. An M-complete graphical fuzzy metric space is a graphical fuzzy metric space in which every M-Cauchy sequence is M_{H}-convergent.

Every M-Cauchy sequence is a G-Cauchy sequence in graphical fuzzy metric spaces, and every G-complete graphical fuzzy metric space is an M-complete graphical fuzzy metric space, but the converse is not true in general, as every fuzzy metric space is a graphical fuzzy metric space.

Definition 2.6 (Shukla et al. [23])**.** Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space and let H' be a subgraph of H such that $E(H') \supseteq \Delta$. Then, the space $(X, M_{\mathrm{H}}, *)$ is called H'-G-complete (H'-M-complete respectively) if every H'-termwise connected G-Cauchy sequence (M-Cauchy sequence respectively) in X is M_{H}-convergent to some point in X.

We now state our main results.

3. Main Results

In this section, we define a new class of mappings called (H, H')-fuzzy graphical ξ-contractions and prove some fixed point results for the mappings of this new class in graphical fuzzy metric spaces. These results generalize and extend the results of Tirado [24], Gregori and Sapena [6], Miheţ [14], Wardowski [25], and Shukla et al. [22] in graphical fuzzy metric spaces.

Let Ξ be the class of all functions $\xi\colon (0,1] \to (0,1]$ satisfying the following properties:

(ξ1): ξ is nondecreasing, i.e., $a \leq b$ implies $\xi(a) \leq \xi(b)$ for all $a, b \in (0,1]$;
(ξ2): $\lim_{n\to+\infty} \xi^n(a) = 1$ for all $a \in (0,1]$.

Remark 3.1. If $a \in (0,1)$ then $\xi(a) > a$, and if $a = 1$, then $\xi(a) = 1$.

Example 3.2. The following functions $\xi\colon (0,1] \to (0,1]$, belong to the class Ξ:
(a) $\xi(a) = 1$ for all $a \in (0,1]$;
(b) $\xi(a) = a^k$ for all $a \in (0,1]$, where $k \in (0,1)$;
(c) $\xi(a) = 1 - k(1-a)$ for all $a \in (0,1]$, where $k \in (0,1)$;
(d) $\xi(a) = \dfrac{a}{k + (1-k)a}$ for all $a \in (0,1]$, where $k \in (0,1)$;
(e) $\xi(a) = e^{a-1}$ for all $a \in (0,1]$;
(f) $\xi(a) = e^{\chi(a)-1}$ for all $a \in (0,1]$, where $\chi \in \Xi$.
(g) $\xi(a) = \dfrac{\chi(a) + f(a)}{2}$ for all $a \in (0,1]$, where $\chi \in \Xi$ and $f\colon (0,1] \to (0,1]$ is any nondecreasing function such that $\chi(a) \leq f(a)$ for all $a \in (0,1]$.
(h) $\xi(a) = \sqrt{\chi(a)f(a)}$ for all $a \in (0,1]$, where $\chi \in \Xi$ and $f\colon (0,1] \to (0,1]$ is any nondecreasing function such that $\chi(a) \leq f(a)$ for all $a \in (0,1]$.

Example 3.3. Let Ψ be the class of all functions $\psi\colon (0,1] \to (0,1]$ such that ψ is continuous, nondecreasing and $\psi(t) > t$ for all $t \in (0,1]$ (see, Miheţ [14]). Then for each $\psi \in \Psi$ we have $\psi \in \Xi$, i.e., $\Psi \subset \Xi$ and this inclusion is proper (see Example 3.14 of this paper).

The concept of class Ξ is motivated by Matkowski [13], who introduced the functions with such properties involved in the contractive conditions in metric spaces. Several authors used this idea in various spaces, see, e.g., [2, 9, 19] and the references therein.

Definition 3.4. Let X be a nonempty set endowed with a graph H. Then, a mapping $T\colon X \to X$ is said to be:

(I) H-edge preserving if $(x,y) \in E(H)$ implies $(Tx,Ty) \in E(H)$.
(II) H-path preserving if $y \in [x]_H^q$, $q \in \mathbb{N}$ implies $Ty \in [Tx]_H^r$ for some $r \in \mathbb{N}$.

Remark 3.5. If T is H-edge preserving, then obviously it is H-path preserving. The converse of this assertion is not always true, e.g., consider the set $X = \{z \in \mathbb{Z} : 0 \leq z \leq 5\}$ endowed with graph H, where $E(H) = \{(0,1),(1,2),(5,4),(4,3)\} \cup \Delta$ and consider a mapping $T\colon X \to X$ defined by $T0 = 5, T1 = T2 = 3, T3 = T4 = T5 = 2$. Then, it is easy to see that T is H-path preserving but not H-edge preserving. Indeed,

$(0,1) \in E(H)$, but $(T0,T1) = (5,3) \notin E(H)$. We point out that if a graph H is transitive, then the concepts (I) and (II) are equivalent.

Definition 3.6. Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space and H' be a subgraph of H such that $E(H') \supseteq \Delta$. A mapping $T\colon X \to X$ is called an (H,H')-fuzzy graphical ξ-contraction if there exists $\xi \in \Xi$ such that the following condition holds:

(FGξ) $M_{\mathrm{H}}(Tx,Ty,t) \geq \xi(M_{\mathrm{H}}(x,y,t))$ for all $x,y \in X, t > 0$ with $(x,y) \in E(H')$.

Definition 3.7. Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space and let H' be a subgraph of H such that $E(H') \supseteq \Delta$. A sequence $\{x_n\}$ in X will be called $E(H'_I)$-convergent to $x \in X$ if for each $0 < \varepsilon < 1$ and $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $(x, x_n) \in E(H'_I)$ and $M_{\mathrm{H}}(x_n, x, t) > 1 - \varepsilon$ for all $n > n_0$. Again, in such a case, we will say that $\{x_n\}$ is $E(H'_I)$-convergent in X.

Remark 3.8. Observe that an $E(H'_I)$-convergent sequence $\{x_n\}$ to $x \in X$, which is, in addition, an H'-termwise connected sequence, satisfies the following more restrictive condition: for each $0 < \varepsilon < 1$ and $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $(x, x_n) \in E(H')$ for all $n > n_0$ or $(x_n, x) \in E(H')$ for all $n > n_0$, and $M_{\mathrm{H}}(x_n, x, t) > 1 - \varepsilon$ for all $n > n_0$.

Throughout the paper, we assume that H' is a subgraph of H such that $E(H') \supseteq \Delta$.

We now prove a convergence result for the Picard sequence generated by an (H,H')-fuzzy graphical ξ-contraction in an H'-M-complete graphical fuzzy metric space.

Theorem 3.9. *Let $(X, M_H, *)$ be an H'-M-complete graphical fuzzy metric space and $T\colon X \to X$ be an H'-edge preserving (H,H')-fuzzy graphical ξ-contraction. Suppose, the following conditions hold:*

(i) there exists $x_0 \in X$ such that $Tx_0 \in [x_0]^q_{H'}$ for some $q \in \mathbb{N}$ and $\inf_{k \geq 1} M_H(x_0, T^k x_0, t) > 0$ for all $t > 0$;
(ii) the graph H' is transitive;
(iii) if an H'-termwise connected T-Picard sequence is M_H-convergent in X, then it is $E(H'_I)$-convergent in X.

Then, there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$ with initial value $x_0 \in X$ is H'-termwise connected and M_H-convergent to both $x^\star$ and $Tx^\star$.

Proof. Let $x_0 \in X$ be the given point such that $Tx_0 \in [x_0]^q_{H'}$ and $q \in \mathbb{N}$. Since H' is transitive, we must have $(x_0, Tx_0) \in E(H')$. Consider the T-Picard sequence $\{x_n\}$ with initial value x_0. Then by the definition, we have $(x_0, x_1) \in E(H')$. As, T is H'-edge preserving, we have $(x_1, x_2) = (Tx_0, Tx_1) \in E(H')$. Similarly, one can easily deduce that $(x_{n-1}, x_n) \in E(H')$ for all $n \in \mathbb{N}$. By transitivity of H', we obtain $(x_n, x_m) \in E(H')$ for all $m > n$.

Suppose, $x_n = x_m$, where $m > n$. Since $(x_n, x_m) \in E(H')$ for all $n, m \in \mathbb{N}$ with $m > n$, T is H'-edge preserving and $\{x_n\}$ is a T-Picard sequence, by successive use of (FGξ) we obtain:

$$M_{\mathrm{H}}(x_n, x_{n+1}, t) = M_{\mathrm{H}}(x_m, x_{m+1}, t) \geq \xi^{m-n}(M_{\mathrm{H}}(x_n, x_{n+1}, t)).$$

If $x_n = x_{n+1}$, then x_n is a fixed point of T and $x_m = x_n$ for all $m \geq n$. Therefore, the T-Picard sequence $\{x_n\}$ is H'-termwise connected and converges to both x_n and Tx_n and we are done. Otherwise, the above inequality with Remark 3.1 yields a contradiction.

Now suppose that $x_n \neq x_m$ and so, $M_{\mathrm{H}}(x_n, x_m, t) < 1$ for all $n, m \in \mathbb{N}$ such that $m \neq n$.

For fixed $t > 0$ and for every $n \in \mathbb{N} \cup \{0\}$, define a set $\mathcal{M}_n$ by:

$$\mathcal{M}_n = \{M_{\mathrm{H}}(x_n, x_m, t) \colon m > n\}.$$

Then, $\mathcal{M}_n \subseteq (0, 1]$, and so, its infimum exists. Let

$$I_n = \inf \mathcal{M}_n = \inf_{m>n} M_{\mathrm{H}}(x_n, x_m, t).$$

We prove the M_H-convergence of $\{x_n\}$ with the help of the following claims:

Claim I: The sequence $\{I_n\}$ is a nondecreasing sequence.

Proof. Since $(x_n, x_m) \in E(H')$ for all $m > n \in \mathbb{N}$, hence by (FGξ) and Remark 3.1 we have

$$\begin{aligned} M_{\mathrm{H}}(x_n, x_m, t) &= M_{\mathrm{H}}(Tx_{n-1}, Tx_{m-1}, t) \\ &\geq \xi(M_{\mathrm{H}}(x_{n-1}, x_{m-1}, t)) \\ &> M_{\mathrm{H}}(x_{n-1}, x_{m-1}, t). \end{aligned}$$

Taking infimium over $m(> n)$ yields:

$$\begin{aligned} \inf_{m>n} M_{\mathrm{H}}(x_n, x_m, t) &\geq \inf_{m>n} M_{\mathrm{H}}(x_{n-1}, x_{m-1}, t) \\ &= \inf_{m>n-1} M_{\mathrm{H}}(x_{n-1}, x_m, t). \end{aligned}$$

This shows that $I_n \geq I_{n-1}$ for all $n \in \mathbb{N}$.

Claim II: There exists $N \in \mathbb{N}$ such that $I_n > 0$ for all $n > N$.

Proof. Since the sequence $\{I_n\}$ is nondecreasing, it is sufficient to show that there exists $N \in \mathbb{N}$ such that $I_N > 0$. We prove the claim by contradiction. So, suppose that $I_n = 0$ for all $n \in \mathbb{N}$, then using ($\xi 1$) and (FGξ) successively we obtain: for all $m, n \in \mathbb{N}$ with $m > n$

$$\begin{aligned} M_{\mathrm{H}}(x_n, x_m, t) &\geq \xi(M_{\mathrm{H}}(x_{n-1}, x_{m-1}, t)) \\ &\geq \xi^2(M_{\mathrm{H}}(x_{n-2}, x_{m-2}, t)) \\ &\vdots \\ &\geq \xi^n(M_{\mathrm{H}}(x_0, x_{m-n}, t)). \end{aligned} \tag{3.1}$$

Taking infimium over $m(> n)$ we obtain

$$\begin{aligned} 0 &\geq \inf_{m>n} \xi^n(M_{\mathrm{H}}(x_0, x_{m-n}, t)) \\ &= \inf_{k\geq 1} \xi^n(M_{\mathrm{H}}(x_0, x_k, t)). \end{aligned}$$

Hence, we must have $\inf_{k\geq 1} \xi^n(M_{\mathrm{H}}(x_0, x_k, t)) = 0$ for all $n \in \mathbb{N}$. So, by the definition of infimum for every $n \in \mathbb{N}$ and for every $0 < \alpha < 1$ there exists $r_n \in \mathbb{N}$ such that

$$\xi^n(M_{\mathrm{H}}(x_0, x_{r_n}, t)) < \alpha \text{ for all } n \in \mathbb{N}.$$

Let $\beta = \inf_{k\geq 1} M_{\mathrm{H}}(x_0, x_k, t)$, then by condition (i) we have $0 < \beta \leq 1$. Therefore, we have $\beta \leq M_{\mathrm{H}}(x_0, x_{r_n}, t)$ for all $n \in \mathbb{N}$. Then, by ($\xi 1$) we obtain

$$\xi^n(\beta) \leq \xi^n(M_{\mathrm{H}}(x_0, x_{r_n}, t)) < \alpha \text{ for all } n \in \mathbb{N}.$$

Therefore, by ($\xi 2$) and the above inequalities we have: there exists $n_1 \in \mathbb{N}$ such that

$$\alpha < \xi^n(\beta) < \alpha \text{ for all } n > n_1.$$

This contradiction proves the claim.

Claim III: The sequence $\{x_n\}$ is M-Cauchy.
Proof. Let $\varepsilon > 0$ be given such that $0 < \varepsilon < 1$ and let $n > N$, i.e., $I_n > 0$ for all $n > N$. By (3.1) we have:

$$I_n \geq \inf_{k\geq 1} \xi^n(M_{\mathrm{H}}(x_0, x_k, t)).$$

Again, by the definition of infimum, for every $n > N$ there exists $p_n \in \mathbb{N}$ such that

$$\inf_{k\geq 1} \xi^n(M_{\mathrm{H}}(x_0, x_k, t)) > \xi^n(M_{\mathrm{H}}(x_0, x_{p_n}, t)) - \frac{\varepsilon}{2}.$$

Let β be the same as it was in the proof of Claim II. Then, again, we have

$$\xi^n(\beta) \leq \xi^n(M_{\mathrm{H}}(x_0, x_{p_n}, t)) \text{ for all } n > N.$$

Therefore

$$\inf_{k\geq 1} \xi^n(M_{\mathrm{H}}(x_0, x_k, t)) > \xi^n(\beta) - \frac{\varepsilon}{2}.$$

By ($\xi 2$), there exists $n_2 \in \mathbb{N}$ such that $n_2 > N$ and $|\xi^n(\beta) - 1| < \dfrac{\varepsilon}{2}$ for all $n > n_2$, i.e.

$$\xi^n(\beta) > 1 - \frac{\varepsilon}{2} \text{ for all } n > n_2.$$

This shows that

$$I_n > 1 - \varepsilon \text{ for all } n > n_2.$$

Therefore

$$M_{\mathrm{H}}(x_n, x_m, t) \geq I_n > 1 - \varepsilon \text{ for all } m > n > n_2.$$

Thus, $\{x_n\}$ is an H'-termwise connected M-Cauchy sequence, and the claim is proved.

By the H'-M-completeness of X, the sequence $\{x_n\}$ is M_{H}-convergent in X. Moreover, by the condition (iii) and Remark 3.8 we have that, there exists $x^\star \in X$ such that for each $0 < \delta < 1$ and $t > 0$, there exist $n_0 \in \mathbb{N}$ satisfying $(x_n, x^\star) \in E(H')$ for all $n > n_0$ or $(x^\star, x_n) \in E(H')$ for all $n > n_0$, and

$$M_{\mathrm{H}}(x_n, x^\star, t) > 1 - \delta \text{ for all } n > n_0. \tag{3.2}$$

Obviously, the sequence $\{x_n\}$ is M_{H}-convergent to $x^\star \in X$. We shall show that $\{x_n\}$ is M_{H}-convergent to $Tx^\star$ as well.

Fixed $0 < \delta < 1$ and $t > 0$. Consider n_0 satisfying the aforementioned properties. Suppose that $(x_n, x^\star) \in E(H')$ for all $n > n_0$. Let $\mathbb{N}_1$ be the set of all natural numbers n such that $x_n = x^\star$, and $\mathbb{N}_2 = \mathbb{N} \setminus \mathbb{N}_1$.

Since $x_n \neq x_m$ for all $n \neq m$, therefore $\mathbb{N}_1$ contains at most one element, hence $\mathbb{N}_2$ must be infinite. Hence, we can choose $n_3 \in \mathbb{N}_2$ such that $n_3 > n_0$ and $x_n \neq x^\star$ for all $n > n_3$. Then using (FGξ) and Remark 3.1 we have

$$M_{\mathrm{H}}(x_{n+1}, Tx^\star, t) \geq \xi(M_{\mathrm{H}}(x_n, x^\star, t)) > M_{\mathrm{H}}(x_n, x^\star, t) \text{ for all } n \in \mathbb{N}_2 \text{ with } n > n_3.$$

Using (3.2) in the above inequality, we get

$$M_{\mathrm{H}}(x_{n+1}, Tx^\star, t) > 1 - \delta \text{ for all } n \in \mathbb{N}_2 \text{ with } n > n_3.$$

Since $x_n \neq x^\star$ for all $n > n_3$, it follows from the above inequality that

$$M_{\mathrm{H}}(x_{n+1}, Tx^\star, t) > 1 - \delta \text{ for all } n \in \mathbb{N} \text{ with } n > n_3.$$

This shows that the sequence $\{x_n\}$ is M_{H}-convergent to $Tx^\star \in X$. Similarly, if $(x^\star, x_n) \in E(H')$ for all $n > n_0$, we reach to the same conclusion. Thus, the sequence $\{x_n\}$ is M_{H}-convergent to both $x^\star$ and $Tx^\star$. □

The conditions used in the above theorem ensure the convergence of T-Picard sequence with initial value x_0 to a point $x^\star$ and its image $Tx^\star$, but these conditions do not ensure the existence of a fixed point of the mapping T. The following example verifies this fact.

Example 3.10. Let $n \in \mathbb{N}$ and denote by $||\cdot||$ the usual norm on $\mathbb{R}^n$, i.e. $||\mathbf{x}|| = \sqrt{x_1^2 + \cdots + x_n^2}$ for each $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Consider $X = \mathbb{R}^n$ endowed with graph H, where $E(H) = \{(\mathbf{x}, \mathbf{y}) \in X \times X \colon 0 < ||\mathbf{y}|| \leq ||\mathbf{x}|| \leq 1\}$. Suppose $H' = H$ and define the fuzzy set $M_{\mathrm{H}} : X \times X \times (0, +\infty) \to [0, 1]$ by

$$M_{\mathrm{H}}(\mathbf{x}, \mathbf{y}, t) = \begin{cases} 1, & \text{if } \mathbf{x} = \mathbf{y}; \\ \dfrac{1}{1 + ||\mathbf{x}|| \cdot ||\mathbf{y}||}, & \text{if } \mathbf{x}, \mathbf{y} \in X \setminus \{\mathbf{0}\}, \mathbf{x} \neq \mathbf{y} \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

for all $t > 0$, where $\mathbf{0} = (0, \ldots, 0)$. Then $(X, M_{\mathrm{H}}, *_m)$ is an H'-M-complete graphical fuzzy metric space, but not a fuzzy metric space. Define the mappings $T \colon X \to X$ and $\xi \colon (0, 1] \to (0, 1]$ as follows:

$$T\mathbf{x} = \begin{cases} \dfrac{\mathbf{x}}{2}, & \text{if } \mathbf{x} \neq \mathbf{0}; \\ 1, & \text{if } \mathbf{x} = \mathbf{0} \end{cases}$$

and $\xi(a) = \sqrt{a}$ for all $a \in (0, 1]$. Then, T is an H'-edge preserving (H, H')-fuzzy graphical ξ-contraction. For any $\mathbf{x}_0 \in X \setminus \{\mathbf{0}\}$ we have $T\mathbf{x}_0 \in [\mathbf{x}_0]^1_{H'}$, and H' is a transitive. Note that, any T-Picard sequence in X is M_{H}-convergent to $\mathbf{x}$ for all $\mathbf{x} \in X$, hence any such sequence will be $E(H'_I)$-convergent to $\mathbf{u}$, where $\mathbf{u} = (1, \ldots, 1)$. Hence, all the conditions of Theorem 3.9 are satisfied. We observe that for every $\mathbf{x}_0 \in \{\mathbf{x} \in \mathbb{R}^n : 0 < ||\mathbf{x}|| \leq 1\}$, the T-Picard sequence with initial value $\mathbf{x}_0$ is H'-termwise connected and converges to both $\mathbf{x}^\star$ and $T\mathbf{x}^\star$ for all $\mathbf{x}^\star \in X \setminus \{\mathbf{0}\}$, but T has no fixed point in X.

To ensure the existence of a fixed point, we use an additional condition called *Property (S)* which is associated with the structure of a graphical fuzzy metric space and the mapping T.

Property (S): Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space, H' be a subgraph of graph H and $T\colon X \to X$ be a mapping. Then, the five-tuple $(X, M_{\mathrm{H}}, *, H', T)$ is said to have the property (S) (see [17]) if:

$$\begin{gathered}\text{whenever an } H'\text{-termwise connected } T\text{-Picard sequence } \{x_n\} \text{ has two limits}\\ x^\star \text{ and } y^\star, \text{ where } x^\star \in X, y^\star \in T(X), \text{ then } x^\star = y^\star.\end{gathered} \tag{S}$$

Theorem 3.11. *Let $(X, M_H, *)$ be an H'-M-complete graphical fuzzy metric space and $T\colon X \to X$ be an H'-edge preserving (H, H')-fuzzy graphical ξ-contraction. Suppose, all the conditions of Theorem 3.9 hold. In addition, if the five-tuple $(X, M_H, *, H', T)$ has the property (S), then T has a fixed point in X.*

Proof. It follows from Theorem 3.9, that the T-Picard sequence $\{x_n\}$ with initial value x_0 is M_{H}-convergent to both $x^\star$ and $Tx^\star$. Since $x^\star \in X$ and $Tx^\star \in T(X)$, therefore by the property (S) we must have $Tx^\star = x^\star$. Thus, $x^\star$ is a fixed point of T. □

Theorem 3.12. *Let $(X, M_H, *)$ be an H'-M-complete graphical fuzzy metric space and $T\colon X \to X$ be an H'-edge preserving (H, H')-fuzzy graphical ξ-contraction. Suppose, all the conditions of Theorem 3.11 are satisfied. In addition, if X_T is connected (as a subgraph of H'), then T has a unique fixed point.*

Proof. The existence of a fixed point of T follows from Theorem 3.11. Suppose that X_T is connected (as a subgraph of H') and $x^\star, y^\star$ are two distinct fixed points of T, i.e., $M_{\mathrm{H}}(x^\star, y^\star, t) < 1$ for all $t > 0$. Since $E(H') \supseteq \Delta$, therefore $\mathrm{Fix}(T) \subseteq X_T$ and so $x^\star, y^\star \in X_T$. Again, due to X_T is connected and H' is transitive, we have $(x^\star, y^\star) \in E(H')$. Since $M_{\mathrm{H}}(x^\star, y^\star, t) < 1$ for all $t > 0$, therefore, by (FGξ) and Remark 3.1 we obtain

$$\begin{aligned} M_{\mathrm{H}}(x^\star, y^\star, t) &= M_{\mathrm{H}}(Tx^\star, Ty^\star, t)\\ &\geq \xi(M_{\mathrm{H}}(x^\star, y^\star, t))\\ &> M_{\mathrm{H}}(x^\star, y^\star, t).\end{aligned}$$

This contradiction proves the uniqueness of the fixed point. □

Remark 3.13. In the above theorem, for the uniqueness of the fixed point of the mapping T we have imposed the condition of connectedness of X_T. It is easy to see that if we replace this condition by the condition of connectedness of the set of all fixed points of T, i.e., $\mathrm{Fix}(T)$, then the uniqueness of the fixed point of T remains unaffected.

Example 3.14. Let $X = \mathbb{R}$ and $\{x_n\}$ be an increasing sequence in $(0, 1)$ such that $x_n \to 1$ as $n \to +\infty$ (with respect to usual topology of $\mathbb{R}$). Let $A = \{x_n \colon n \in \mathbb{N}\}$, $k \in \mathbb{N}$ be fixed and H, H' be two graphs defined by $V(H) = V(H') = X$ and

$$E(H) = E(H') = \Delta \cup \{(x, y) \in A \times A \colon x = x_{nk}, y = x_{mk}, m > n\} \cup \{(x_{nk}, 1) \colon n \in \mathbb{N}\}.$$

Then, H' is transitive. Define a fuzzy set $M_{\mathrm{H}} : X \times X \times (0, +\infty) \to [0, 1]$ by

$$M_{\mathrm{H}}(x, y, t) = \begin{cases} 1, & \text{if} \quad x = y; \\ \min\{x, y\}, & \text{if } x, y \in A \cup \{1\}, x \neq y \\ \frac{1}{1+|x-y|}, & \text{otherwise.} \end{cases}$$

for all $t > 0$. Then $(X, M_{\mathrm{H}}, *_m)$ is an H'-M-complete graphical fuzzy metric space. Define mappings $T\colon X \to X$ and $\xi\colon (0, 1] \to (0, 1]$ as follows:

$$Tx = \begin{cases} x_{n+k}, & \text{if } x = x_n,\ n \in \mathbb{N}; \\ 1, & \text{if } x = 1; \\ 2x, & \text{otherwise} \end{cases}$$

and

$$\xi(a) = \begin{cases} x_k, & \text{if } 0 < a < x_k; \\ x_{(n+1)k}, & \text{if } x_{nk} \leq a < x_{(n+1)k},\ n \in \mathbb{N}; \\ 1, & \text{if } a = 1 \end{cases}$$

Note that, T is H'-edge preserving (H, H')-fuzzy ξ-contraction and for all $x = x_{nk}$ we have $Tx \in [x]^1_{H'}$ and $\inf_{r \geq 1} M_{\mathrm{H}}(x, T^r x, t) > 0$ for all $t > 0, n \in \mathbb{N}$. Finally, any H'-termwise connected T-Picard sequence in X is either a constant sequence (with each term equal to 1) or a sequence $\{x_{nk}\}_{n \in \mathbb{N}}$ and such sequence has a unique limit, namely, 1. Hence, the condition (iii) of Theorem 3.9 and the property (S) are satisfied. Therefore, by Theorem 3.11 and Remark 3.13, the mapping T has a unique fixed point, namely, 1.

Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space and H' be a subgraph of H such that $E(H') \supseteq \Delta$. Then, a mapping $T\colon X \to X$ is said to be an:

(I) (H, H')-fuzzy graphical contraction if for all $t > 0$, $x, y \in X$ with $(x, y) \in E(H')$ the following condition is satisfied:

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k \left[\frac{1}{M(x, y, t)} - 1\right] \text{ for some } k \in (0, 1).$$

(II) (H, H')-fuzzy graphical Tirado contraction if for all $t > 0$, $x, y \in X$ with $(x, y) \in E(H')$ the following condition is satisfied:

$$1 - M(Tx, Ty, t) \leq k\,[1 - M(x, y, t)] \text{ for some } k \in (0, 1).$$

(III) (H, H')-fuzzy graphical ψ-contraction if for all $t > 0$, $x, y \in X$ with $(x, y) \in E(H')$ the following condition is satisfied:

$$M(Tx, Ty, t) \geq \psi\,(M(x, y, t)) \text{ for some } \psi \in \Psi.$$

(IV) (H, H')-fuzzy graphical Wardowski contraction if for all $t > 0$, $x, y \in X$ with $(x, y) \in E(H')$ the following condition is satisfied:

$$\eta(M(Tx, Ty, t)) \leq k\eta(M(x, y, t))$$

where $\eta\colon (0, 1] \to [0, +\infty)$ is a strictly decreasing function which transform $(0, 1]$ onto $[0, +\infty)$.

The following corollary provides a fixed point result for the contractions given by Gregori and Sapena [6], Tirado [24], Miheţ [14], and Wardowski[25], respectively, in the context of graphical fuzzy metric spaces.

Corollary 3.15. *Let $(X, M_H, *)$ be an H'-M-complete graphical fuzzy metric space and $T\colon X \to X$ be an H'-edge preserving mapping of at least one of the following types:*

(I): *(H, H')-fuzzy graphical contraction;*
(II): *(H, H')-fuzzy graphical Tirado contraction;*
(III): *(H, H')-fuzzy graphical ψ-contraction;*
(IV): *(H, H')-fuzzy graphical Wardowski contraction.*

Suppose, the conditions (i), (ii), and (iii) of Theorem 3.9 hold. Then, there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$ with initial value $x_0 \in X$ is H'-termwise connected and M_H-convergent to both $x^\star$ and $Tx^\star$. In addition, if the five-tuple $(X, M_H, *, H', T)$ has the property (S), then T has a fixed point in X. Furthermore, if X_T is connected (as a subgraph of H'), then the fixed point of T is unique.

Proof. **(I):** If T is of the type (I), then the proof follows from Theorem 3.12 with $\xi(a) = \dfrac{a}{k + (1-k)a}$ for all $a \in (0, 1]$.

(II): If T is of the type (II), then the proof follows from Theorem 3.12 with $\xi(a) = 1 - k(1-a)$ for all $a \in (0, 1]$.

(III): If T is of the type (III), then the proof follows from Theorem 3.12 with $\xi(a) = \psi(a)$ for all $a \in (0, 1]$.

(IV): If T is of the type (IV), then by following the process of [5] one can show that T is an (H, H')-fuzzy graphical ψ-contraction with $\psi(t) = \eta^{-1}(k\eta(t))$ for all $t \in (0, 1]$, where k is same as it was in the definition of (H, H')-fuzzy graphical Wardowski contraction. Now, the proof follows from the case (III). □

Remark 3.16. By taking $V(H) = V(H') = X$ and $E(H) = E(H') = X \times X$ in the above corollary one can derive the extensions of results of Gregori and Sapena [6], Tirado [24], Miheţ [14] and Wardowski[25], respectively in fuzzy metric spaces. Also, with suitable choice of the sets $E(H)$, $E(H')$ (see, e.g., [22]) and the function ξ, one can derive the extensions of results of Ran and Reurings [16], Kirk et al. [10] and Edelstein [1] in graphical fuzzy metric spaces.

Remark 3.17. If X is a nonempty set endowed with a graph H, then the relation $P = \{(u, v) \in X \times X\colon$ there is a directed path from u to v in $H\}$ on X, is a transitive relation.

Remark 3.18. In Theorem 3.9, Theorem 3.11 and Theorem 3.12, we have proved the existence and uniqueness of the fixed point of an (H, H')-fuzzy graphical ξ-contraction with the assumption that the property (FGξ) and the transitivity of the subgraph H' hold. If we replace the condition (FGξ) by the condition (FGξ1) then by using Remark 3.17, the mentioned theorems can be proved without the assumption that H' is transitive, where (FGξ1) is as follows:

$$\text{(FG}\xi 1)\ \ M_{\mathrm{H}}(Tx, Ty, t) \geq \xi(M_{\mathrm{H}}(x, y, t)) \text{ for all } x, y \in X,\ t > 0 \text{ with } y \in [x]^q_{H'}.$$

It is obvious that the condition (FGξ1) is stronger than the condition (FGξ).

In the previous theorems, the existence of $x_0 \in X$ such that $\inf_{k\geq 1} M_{\mathrm{H}}(x_0, T^k x_0, t) > 0$ for all $t > 0$, plays a crucial role. In the next proposition, for any given graphical fuzzy metric space, we construct a new graphical fuzzy metric space such that the property "$\inf_{x,y\in X} M_{\mathrm{H}}(x,y,t) > 0$ for all $t > 0$" holds in it.

Proposition 3.19. *Let $(X, M_H, *)$ be a graphical fuzzy metric space such that $a * b \geq a *_L b$ for all $a, b \in [0,1]$. Define a fuzzy set $M_H^p\colon X \times X \times (0,+\infty) \to [0,1]$ by*

$$M_H^p(x,y,t) = e^{M_H(x,y,t)-1} \text{ for all } x, y \in X, t > 0.$$

Then, $(X, M_H^p, *_p)$ is a graphical fuzzy metric space and $\inf_{x,y\in X} M_H^p(x,y,t) > 0$ for all $t > 0$. Furthermore, $(X, M_H, *)$ is H'-M-complete if and only if $(X, M_H^p, *_p)$ is H'-M-complete.

Proof. It is easy to see that the properties (GFM1)-(GFM3) and (GFM5) are satisfied by $(X, M_{\mathrm{H}}^{\mathrm{p}}, *_p)$. For (GFM4), suppose that $(xPy)_{\mathrm{H}}, z \in (xPy)_{\mathrm{H}}$, then we have: for all $t, s > 0$

$$\begin{aligned} M_{\mathrm{H}}^{\mathrm{p}}(x,y,t+s) &= e^{M_{\mathrm{H}}(x,y,t+s)-1} \\ &\geq e^{M_{\mathrm{H}}(x,z,t)*M_{\mathrm{H}}(z,y,s)-1} \\ &\geq e^{M_{\mathrm{H}}(x,z,t)*_L M_{\mathrm{H}}(z,y,s)-1} \\ &\geq e^{M_{\mathrm{H}}(x,z,t)+M_{\mathrm{H}}(z,y,s)-2} \\ &= M_{\mathrm{H}}^{\mathrm{p}}(x,z,t) *_p M_{\mathrm{H}}^{\mathrm{p}}(z,y,s). \end{aligned}$$

Hence, $(X, M_{\mathrm{H}}^{\mathrm{p}}, *_p)$ is a graphical fuzzy metric space. The claim $\inf_{x,y\in X} M_{\mathrm{H}}^{\mathrm{p}}(x,y,t) > 0$ for all $t > 0$ follows from the fact that $M_{\mathrm{H}}^{\mathrm{p}}(x,y,t) > e^{-1}$ for all $x, y \in X$ and $t > 0$.

It is easy to see that a sequence is M-Cauchy (M_{H}-convergent respectively) in $(X, M_{\mathrm{H}}, *)$ if and only if it is M-Cauchy ($M_{\mathrm{H}}^{\mathrm{p}}$-convergent respectively) in $(X, M_{\mathrm{H}}^{\mathrm{p}}, *_p)$. Therefore, $(X, M_{\mathrm{H}}, *)$ is H'-M-complete if and only if $(X, M_{\mathrm{H}}^{\mathrm{p}}, *_p)$ is H'-M-complete. $\square$

For every given constant $k \in (0,1)$, we have seen that if $\xi(a) = 1 - k(1-a)$ for all $a \in (0,1]$, then $\xi \in \Xi$ and $\xi(a) = ka - k + 1$ for all $a \in (0,1]$. Now, by Example 3.2 we can find infinitely many members (say χ) of the class Ξ with the property that $\chi(a) \geq \xi(a) \geq ka - k + 1$ for all $a \in (0,1]$. Hence, the subclass $\Xi_k = \{\chi \in \Xi\colon \chi(a) \geq ka - k + 1 \text{ for all } a \in (0,1]\}$ of Ξ is a rich class, also the (H,H')-fuzzy graphical ξ-contractions produced by this class ensure the fixed points in some particular type of spaces. This fact is the motivation for the following theorem.

Theorem 3.20. *Let $(X, M_H, *)$ be an H'-M-complete graphical fuzzy metric space such that $a * b \geq a *_L b$ for all $a, b \in [0,1]$ and $T\colon X \to X$ be an H'-edge preserving (H,H')-fuzzy graphical ξ-contraction such that $\xi \in \Xi_k$ for some $k \in (0,1)$. Suppose, the following conditions hold:*

(i) there exists $x_0 \in X$ such that $Tx_0 \in [x_0]_{H'}^q$ for some $q \in \mathbb{N}$;
(ii) the graph H' is transitive;
(iii) if an H'-termwise connected T-Picard sequence M_H-convergent to a limit in X, then it is $E(H_I')$-convergent to some $x \in X$.

Then, there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$ with initial value $x_0 \in X$ is H'-termwise connected and M_H-convergent to both $x^\star$ and $Tx^\star$. In addition, if the five-tuple $(X, M_H, *, H', T)$ has the property (S), then T has a fixed point in X. Furthermore, if X_T is connected (as a subgraph of H'), then the fixed point of T is unique.

Proof. Consider the graphical fuzzy metric space $(X, M^{\mathrm{p}}_{\mathrm{H}}, *_p)$ (as we have defined in Proposition 3.19). Then $(X, M^{\mathrm{p}}_{\mathrm{H}}, *_p)$ is an H'-M-complete graphical fuzzy metric space and $\inf_{x,y\in X} M^{\mathrm{p}}_{\mathrm{H}}(x,y,t) > 0$ for all $t > 0$.

Since $T\colon X \to X$ is an (H, H')-fuzzy graphical ξ-contraction on $(X, M_{\mathrm{H}}, *)$, there exists $\xi \in \Xi$ such that

$$M_{\mathrm{H}}(Tx, Ty, t) \geq \xi(M_{\mathrm{H}}(x, y, t)) \text{ for all } x, y \in X, t > 0 \text{ with } (x, y) \in E(H'). \tag{3.3}$$

Define $\chi\colon (0,1] \to (0,1]$ by $\chi(a) = a^k$ for all $a \in (0,1]$, then $\chi \in \Xi$. Finally, we show that T is an (H, H')-fuzzy graphical χ-contraction on $(X, M^{\mathrm{p}}_{\mathrm{H}}, *_p)$. Suppose, $(x, y) \in E(H')$, then using (3.3) we obtain:

$$\begin{aligned} M^{\mathrm{p}}_{\mathrm{H}}(Tx, Ty, t) &= e^{M_{\mathrm{H}}(Tx,Ty,t)-1} \geq e^{\xi(M_{\mathrm{H}}(x,y,t))-1} \geq e^{kM_{\mathrm{H}}(x,y,t)-k} \\ &= \chi\left(M^{\mathrm{p}}_{\mathrm{H}}(x, y, t)\right). \end{aligned}$$

We note that all the conditions of Theorem 3.12 are satisfied by mapping T in $(X, M^{\mathrm{p}}_{\mathrm{H}}, *_p)$. Hence, by Theorem 3.12 the mapping T has a unique fixed point. □

We next use the concept of H'-G-completeness of graphical fuzzy metric spaces and prove the existence and uniqueness of a fixed point of an (H, H')-fuzzy graphical ξ-contraction. Note that, the concept of H'-G-completeness is stronger than H'-M-completeness, nevertheless, the following result differs from the previous theorems in the sense that the transitivity of graph is dropped; also, we observe that the following result does not contain the condition "$x_0 \in X$, $\inf_{k\geq 1} M(x_0, T^k x_0, t) > 0$" in the list of assumptions.

Theorem 3.21. *Let $(X, M_H, *)$ be an H'-G-complete graphical fuzzy metric space and and $T\colon X \to X$ be an H'-edge preserving (H, H')-fuzzy graphical ξ-contraction. Suppose that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $Tx_0 \in [x_0]^q_{H'}$ for some $q \in \mathbb{N}$;

(iii) if an H'-termwise connected T-Picard sequence M_H-convergent to a limit in X, then it is $E(H'_I)$-convergent to some $x \in X$.

Then, there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$, with initial value $x_0 \in X$, is H'-termwise connected and $E(H'_I)$-convergent to both $x^\star$ and $Tx^\star$. In addition, if the five-tuple $(X, M_H, *, H', T)$ has the property (S), then T has a fixed point in X. Furthermore, if X_T is connected (as a subgraph of H'), then the fixed point of T is unique.

Proof. Let $x_0 \in X$ be the given point such that $Tx_0 \in [x_0]^q_{H'}$ and $q \in \mathbb{N}$. We construct a T-Picard sequence $\{x_n\}$ with initial value x_0 as follows: since $Tx_0 \in [x_0]^q_{H'}$, by definition, there must be a path $\{y_i\}_{i=0}^q$ such that $x_0 = y_0, Tx_0 = y_q$ and $(y_{i-1}, y_i) \in E(H')$ for $i = 1, 2, \ldots, q$. As, T is H'-edge preserving, we have $(Ty_{i-1}, Ty_i) \in E(H')$ for $i = 1, 2, \ldots, q$. Therefore, $\{Ty_i\}_{i=0}^q$ is a path from $Ty_0 = Tx_0 = x_1$ to $Ty_q =$

$T^2x_0 = x_2$ of length q, and so, $x_2 \in [x_1]^q_{H'}$. In this way, for each $n \in \mathbb{N}$, we construct a path $\{T^n y_i\}_{i=0}^q$ from $T^n y_0 = T^n x_0 = x_n$ to $T^n y_q = T^n T x_0 = x_{n+1}$ of length q, which shows that $x_{n+1} \in [x_n]^q_{H'}$ for all $n \in \mathbb{N}$. Thus, $\{x_n\}$ is an H'-termwise connected sequence. Since, $(T^n y_{i-1}, T^n y_i) \in E(H')$ for $i = 1, 2, \dots, q$ and $n \in \mathbb{N}$, ξ is nondecreasing, by (FGξ) we have

$$M_{\mathrm{H}}(T^n y_{i-1}, T^n y_i, t) \geq \xi\left(M_{\mathrm{H}}(T^{n-1} y_{i-1}, T^{n-1} y_i, t)\right) \geq \cdots \geq \xi^n\left(M_{\mathrm{H}}(y_{i-1}, y_i, t)\right) \tag{3.4}$$

for all $t > 0$, $i = 1, 2, \dots, q$ and $n \in \mathbb{N}$. Since, $(T^n y_{i-1}, T^n y_i) \in E(H')$ for $i = 1, 2, \dots, q$ and for each $n \in \mathbb{N}$ we obtain from (GFM4) and (3.4) that

$$\begin{aligned} M_{\mathrm{H}}(x_n, x_{n+1}, t) &= M_{\mathrm{H}}(T^n y_0, T^n y_q, t) \\ &\geq M_{\mathrm{H}}\left(T^n y_0, T^n y_1, \frac{t}{q}\right) * \cdots * M_{\mathrm{H}}\left(T^n y_{q-1}, T^n y_q, \frac{t}{q}\right) \\ &\geq \xi^n\left(M_{\mathrm{H}}\left(y_0, y_1, \frac{t}{q}\right)\right) * \cdots * \xi^n\left(M_{\mathrm{H}}\left(y_{q-1}, y_q, \frac{t}{q}\right)\right) \\ &= \prod_{i=1}^{q} \xi^n\left(M_{\mathrm{H}}\left(y_{i-1}, y_i, \frac{t}{q}\right)\right). \end{aligned}$$

Again, since the sequence $\{x_n\}$ is H'-termwise connected, therefore for all $t > 0$, $n \in \mathbb{N}$ and each $p \in \mathbb{N}$, using the above inequality we obtain

$$\begin{aligned} M_{\mathrm{H}}(x_n, x_{n+p}, t) &\geq M_{\mathrm{H}}\left(x_n, x_{n+1}, \frac{t}{p}\right) * \cdots * M_{\mathrm{H}}\left(x_{n+p-1}, x_{n+p}, \frac{t}{p}\right) \\ &\geq \prod_{i=1}^{q} \xi^n\left(M_{\mathrm{H}}\left(y_{i-1}, y_i, \frac{t}{pq}\right)\right) * \cdots * \prod_{i=1}^{q} \xi^{n+p-1}\left(M_{\mathrm{H}}\left(y_{i-1}, y_i, \frac{t}{pq}\right)\right) \end{aligned}$$

Letting $n \to +\infty$ and using (ξ2), we obtain

$$\lim_{n \to +\infty} M_{\mathrm{H}}(x_n, x_{n+p}, t) \geq 1 * \cdots * 1 = 1$$

Therefore, $\{x_n\}$ is an H'-termwise connected G-Cauchy sequence in X. Now by the H'-G-completeness of X, the sequence $\{x_n\}$ is M_{H}-convergent to a limit in X.

The remaining part of the proof is similar to the proofs of the previous theorems. □

As corollaries of Theorem 3.21, in similar manner as we have done before, one can prove the existence and uniqueness results for (H, H')-fuzzy graphical contractive mappings, (H, H')-fuzzy graphical Tirado contractions, (H, H')-fuzzy graphical ψ-contractions and (H, H')-fuzzy graphical Wardowski contractions in H'-G-complete graphical fuzzy metric spaces.

4. Conclusions

Graph theory has a reasonable influence on applied mathematics and computer science. This fact is the motivation for introducing the graphical structures compatible with the metric structure of spaces; the same is done for fuzzy metric spaces. On the other hand, the contraction mappings are involved in a vast range of applications in theoretical and practical problems. The contraction mappings can be defined in fuzzy metric spaces in a variety of ways, which have privilege over metric spaces. Here, we have introduced a new class of contractions on graphical fuzzy metric spaces, which generalizes, extends, and unifies several types of contractions defined on fuzzy metric spaces. So, the fixed point results proved for such mappings extend several known results in two senses. On the one hand, the domain of mapping is a generalized space, and on the other hand, the class of contractions considered is wider than some existing classes. Several examples verify and illustrate our claims and results. Note that, while proving the existence of a fixed point, we neither use any special kind of the concerned t-norm nor use any special kind of fuzzy metric (e.g., Miheţ [14, 15] used the positivity of t-norm and non-Archimedeanness of fuzzy metric). However, an additional condition on T-Picard sequence is imposed to obtain a fixed point.

We conclude the discussion with the following open problems:

- Can we prove Theorem 3.9, Theorem 3.11 and Theorem 3.12 without assuming the condition $\inf_{k\geq 1} M_{\mathrm{H}}(x_0, x_k, t) > 0$?
- Can we prove Theorem 3.9, Theorem 3.11 and Theorem 3.12 without assuming the transitivity of graph H'?

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