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A NEW INERTIAL ITERATION FOR A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS

Y. SHEHU*, O.S. IYIOLA**, X. QIN*** AND J.C. YAO****

*School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China

**Department of Mathematics, Morgan State University, Baltimore, MD 21251, USA

***Department of Mathematics, Hangzhou Normal University, Hangzhou, China

****Research Center for Interneural Computing, China Medical University Hospital,
China Medical University, Taichung 40447, Taiwan
E-mail: yaojc@mail.cmu.edu.tw (Corresponding author)

Abstract. In this paper, a new inertial iteration to find a common fixed point of a countable family of nonexpansive mappings is presented. Some weak convergence results of common fixed points are obtain in Hilbert spaces. Some applications to monotone inclusion problems are considered. Finally, some numerical experiments are presented to support our new iteration.
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1. Introduction

The theory of fixed points of nonlinear operators play an important role in numerous fields, such as differential equations, optimization theory, economics, medical imaging, and engineering. Recently, various solution methods were devised for fixed points of nonlinear operators in infinite dimensional spaces; see, e.g., [1, 8, 20, 23, 25] and the references therein. Recall that a common problem in diverse fields of mathematics and physical sciences consists of finding a point (solution) with certain constraints (in a finite family of convex sets). Recently, common fixed points of a family of nonlinear operators are under the spotlight of academic research.

Le H as a real Hilbert space with scalar product $\langle .,. \rangle$ and induced norm $\| \cdot \|$. Let T be a mapping on H. Recall that T is nonexpansive iff, for all $x, y \in H$, $\|Tx - Ty\| \leq \|x - y\|$. As an important and popular method, Krasnosel'skiĭ-Mann iteration [16, 19] is powerful to find a fixed point of nonexpansive mappings. It generates an iterative sequence as: $x_1 \in H$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \quad \forall n = 1, 2, \dots \tag{1.1}$$

where $\alpha_n \in (0,1)$. It is known that Krasnosel'skiĭ-Mann iteration (1.1) converges weakly to a fixed point of T under some appropriate conditions on $\{\alpha_n\}$ in infinite dimensional spaces; see, e.g., [5, 15, 21] and the references therein.

Inertial extrapolation, which is viewed as an important acceleration technique, has been extensively studied. Many authors investigated the Krasnosel'skiĭ-Mann iteration with inertial extrapolation steps (inertial Krasnosel'skiĭ-Mann iteration) and it is believe that inertial extrapolation steps could increase the convergence speed of original iterations; see, e.g., [2, 18, 22]). Recall that the following inertial Krasnosel'skiĭ-Mann iteration: $x_0, x_1 \in H$,

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (1-\alpha_n)w_n + \alpha_n T w_n, \end{cases} \tag{1.2}$$

where $\theta_n \in [0,1)$ is the inertial factor and $\alpha_n \in (0,1)$. Weak convergence of the iteration was established in [6, 9, 13, 11, 12, 18, 24]. Indeed, several conditions were imposed on $\{\theta_n\}$ in order to obtain the weak convergence of $\{x_n\}$ to a fixed point of T. In [6], $\{\theta_n\}$ is a non-decreasing sequence with $0 \le \theta_n \le \theta < 1$ for all $n \ge 1$ and $\theta, \sigma, \delta > 0$ such that

(a) $\delta > \frac{\theta^2(1+\theta)+\theta\sigma}{1-\theta^2}$;

(b) $0 < \alpha \le \alpha_n \le \mu := \frac{\delta - \theta[\theta(1+\theta)+\theta\delta+\sigma]}{\delta[1+\theta(1+\theta)+\theta\delta+\sigma]}$.

Similar conditions can be found in [6, 9, 13, 11, 12, 18, 24]. Note that the weak convergence in [6, 9, 13, 11, 12, 18, 24] were obtained under relatively strong assumptions on $\{\theta_n\}$, $0 \le \theta_n \le \theta < 1$ for all $n \ge 1$ and for a single nonexpansive mapping T. One of the major contributions of these results is how to weaken and relax the conditions imposed on the inertial factor. For more recent results, we refer to [9].

In the results mentioned above, the inertial factor was assumed to be bounded away from 1. Furthermore, as far as we know, no results on inertial type Krasnosel'skiĭ-Mann iteration have been proposed and studied for a countable family of nonexpansive mappings for which choice of the inertial factor being 1 is possible. This leads us to ask the following question.

Can we propose a new inertial Krasnosel'skiĭ-Mann iteration for a countable family of nonexpansive mappings in Hilbert spaces in which the choice of inertial factor being 1 is possible? Our aim in this paper to give a positive answer to this question. In details, We obtain a weak convergence result under some weak conditions on the iterative parameters. We give applications of our result to monotone inclusion problems and Douglas-Rachford splitting method. We also give some numerical illustrations of our proposed method and compare our method with some recent inertial Krasnosel'skiĭ-Mann iteration in the literature. Our numerical experiment show that our method has faster convergence speed than these recent inertial Krasnosel'skiĭ-Mann iterations.

This paper is organized paper as follows: We give some basic lemmas and result needed for our convergence results in Section 2. We present our main result, weak convergence analysis of our inertial Krasnosel'skiĭ-Mann iteration, in Section 3. Some applications to monotone inclusion are also given in Section 3. Section 4 gives the numerical implementations of our results. We give final concluding remarks in Section 5, the last section.

2. Preliminaries

In this section, we now present some properties and results that we needed for the proof of convergence of our proposed method.

Note that the following identities hold in a Hilbert space H

(i) $\|u+v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2$ for all $u, v \in H$;

(ii) $2\langle u-v, u-w\rangle = \|u-v\|^2 + \|u-w\|^2 - \|v-w\|^2$ for all $u, v, w \in H$;

(iii) $\|\varrho u + \upsilon v\|^2 = \varrho(\varrho+\upsilon)\|u\|^2 + \upsilon(\varrho+\upsilon)\|v\|^2 - \varrho\upsilon\|u-v\|^2$ for all $u, v \in H$, $\varrho, \upsilon \in \mathbb{R}$.

Lemma 2.1. [2] *Let $\{\eta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ be sequences in $[0,+\infty)$ such that $\eta_{n+1} \leq \eta_n + \lambda_n(\eta_n - \eta_{n-1}) + \gamma_n$, where $\sum_{n=1}^{+\infty} \gamma_n < +\infty$. If there exists a real number λ, where $0 \leq \lambda_n \leq \lambda < 1$ for all $n \in \mathbb{N}$, then*

(i) $\sum_{n=1}^{+\infty} [\eta_n - \eta_{n-1}]_+ < +\infty$, where $[t]_+ := \max\{0, t\}$;

(ii) there exists $\eta^* \in [0,+\infty)$ such that $\lim_{n\to\infty} \eta_n = \eta^*$.

Lemma 2.2. [21] *Let C be a subset of H with $C \neq \emptyset$ and let $\{y_n\}$ be a sequence in H that satisfies*

(i) $\lim_{n\to\infty} \|y_n - y\|$ exists for any $y \in C$;

(ii) every sequential weak limit point of $\{y_n\}$ belongs to C.

Then $\{y_n\}$ converges weakly to a point in C.

Recall that a mapping T is said to be demiclosed if any sequence $\{y_n\}$ weakly converging to y and $\{Ty_n\}$ strongly converges to z, then $Ty = z$.

Lemma 2.3. [15] *Let $C \subset H$ be nonempty, convex, and closed and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0.*

We adapt the results in [7] (proved in strictly convex Banach space) in Hilbert spaces as follow.

Lemma 2.4. *Let C be a nonempty, convex, and convex subset of H. Let $\{S_k\}$ be a sequence of nonexpansive mappings of C into H and $\{\beta_k\}$ a sequence of positive real numbers such that $\sum_{k=1}^{\infty} \beta_k = 1$. If $\sum_{k=1}^{\infty} F(S_k)$ is nonempty, then $T = \sum_{k=1}^{\infty} \beta_k S_k$ is well defined and $F(T) = \sum_{k=1}^{\infty} F(S_k)$.*

3. Main Results

We first give the following assumptions for our convergence analysis.

Assumption 3.1. (a) $\{\beta_n^k\}$ is a family of non-negative numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ and the following conditions are satisfied:

(i) $\sum_{k=1}^{n} \beta_n^k = 1, \ \forall n \geq 1$;

(ii) $\lim\limits_{n\to\infty} \beta_n^k > 0, \ \forall k \geq 1$;

(iii) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$;

(b) $\{S_k\}$ is a countable family of nonexpansive mappings with $\cap_{k=1}^{\infty} F(S_k) \neq \emptyset$;

(c) $\{\theta_n\}$ is chosen such that $0 \leq \theta_n \leq \theta_{n+1} \leq 1$;

(d) $\{\alpha_n\}$ is chosen such that $0 < \alpha \leq \alpha_n \leq \alpha_{n+1} \leq \frac{1}{2+\varepsilon} := \delta, \varepsilon > 0$.

Remark 3.2. The condition (a) in Assumption 3.1 is satisfied if (see [3])

$$\beta_n^k = \begin{cases} 2^{-k}, & k < n \\ 2^{1-k}, & k = n \end{cases}$$

for $n, k \geq 1$ with $k \leq n$.

Remark 3.3. Our condition (c) in Assumption 3.1 is weaker than the conditions imposed on $\{\theta_n\}$ in [6, 9, 13, 11, 12, 18, 24] and the choice $\theta_n = 1$ is not possible [6, 9, 13, 11, 12, 18, 24].

We now introduce our iteration and give the weak convergence under Assumption 3.1 in the following.

Theorem 3.4. *Suppose Assumption 3.1 are satisfied. Let $\{x_n\}$ in H be generated by: $x_0, x_1 \in H$,*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \sum_{k=1}^n \beta_n^k S_k w_n. \end{cases} \tag{3.1}$$

Then $\{x_n\}$ converges weakly to a common fixed point of $\{S_k\}$.

Proof. Fix $x^* \in \cap_{k=1}^\infty F(S_k)$. From (3.1), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 = (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|\sum_{k=1}^n \beta_n^k S_k w_n w_n - x^*\|^2 \\ - \alpha_n(1-\alpha_n)\|x_n - \sum_{k=1}^n \beta_n^k S_k w_n\|^2. \end{aligned} \tag{3.2}$$

From $x_{n+1} = (1-\alpha_n)x_n + \alpha_n \sum_{k=1}^n \beta_n^k S_k w_n$, we have

$$\sum_{k=1}^n \beta_n^k S_k w_n - x_n = \frac{1}{\alpha_n}(x_{n+1} - x_n),$$

which togeher with (3.2) yields that

$$\|x_{n+1} - x^*\|^2 \leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|\sum_{k=1}^n \beta_n^k S_k w_n - x^*\|^2 - \frac{(1-\alpha_n)}{\alpha_n}\|x_{n+1} - x_n\|^2. \tag{3.3}$$

Since S_k is nonexpansive for each k, we have

$$\|\sum_{k=1}^n \beta_n^k S_k w_n - x^*\| = \|\sum_{k=1}^n \beta_n^k (S_k w_n - x^*)\| \leq \sum_{k=1}^n \beta_n^k \|S_k w_n - x^*\| \leq \|w_n - x^*\|.$$

Using (3.3), one obtains

$$\|x_{n+1} - x^*\|^2 \leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|w_n - x^*\|^2 - \frac{(1-\alpha_n)}{\alpha_n}\|x_{n+1} - x_n\|^2. \tag{3.4}$$

It follows from (3.1) that

$$\begin{aligned}\|w_n - x^*\|^2 &= \|(1+\theta_n)(x_n - x^*) - \theta_n(x_{n-1} - x^*)\|^2\\ &= (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2,\end{aligned}$$

which together with (3.4) yields

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le\ & (1+\alpha_n\theta_n)\|x_n - x^*\|^2 - \alpha_n\theta_n\|x_{n-1} - x^*\|^2\\ &+ \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 - \frac{1-\alpha_n}{\alpha_n}\|x_{n+1} - x_n\|^2.\end{aligned} \tag{3.5}$$

Define $a_n = \|x_n - x^*\|^2 - \alpha_n\theta_n\|x_{n-1} - x^*\|^2 + \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2$. Since $\alpha_n \le \alpha_{n+1}$ and $\theta_n \le \theta_{n+1}$, it follows that $\alpha_n\theta_n \le \alpha_{n+1}\theta_{n+1}$. Thus

$$\begin{aligned}a_{n+1} - a_n \le\ & \|x_{n+1} - x^*\|^2 - (1+\alpha_n\theta_n)\|x_n - x^*\|^2 + \alpha_n\theta_n\|x_{n-1} - x^*\|^2\\ &+ \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1})\|x_{n+1} - x_n\|^2 - \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2.\end{aligned} \tag{3.6}$$

Observe that

$$\frac{1-\alpha_n}{\alpha_n} - \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1}) \ge \varepsilon + \frac{\varepsilon}{2+\varepsilon} \ge \varepsilon.$$

Using (3.5) in (3.6) (noting Condition (d) in Assumption 3.1), we arrive at

$$\begin{aligned}a_{n+1} - a_n &\le -\Big(\frac{1-\alpha_n}{\alpha_n} - \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1})\Big)\|x_{n+1} - x_n\|^2\\ &\le -\varepsilon\|x_{n+1} - x_n\|^2.\end{aligned}$$

It then follows that $a_{n+1} \le a_n$ for all $n \ge 1$. Thus $\{a_n\}$ is monotone non-increasing. Consequently,

$$\begin{aligned}a_n &= \|x_n - x^*\|^2 - \alpha_n\theta_n\|x_{n-1} - x^*\|^2 + \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2\\ &\ge \|x_n - x^*\|^2 - \alpha_n\theta_n\|x_{n-1} - x^*\|^2\end{aligned}$$

and then

$$\begin{aligned}\|x_n - x^*\|^2 &\le \alpha_n\theta_n\|x_{n-1} - x^*\|^2 + a_n\\ &\le \delta\|x_{n-1} - x^*\|^2 + a_1\\ &\cdots\\ &\le \delta^n\|x_0 - x^*\|^2 + (1+\delta+\cdots+\delta^{n-1})a_1\\ &\le \delta^n\|x_0 - x^*\|^2 + \frac{a_1}{1-\delta}.\end{aligned} \tag{3.7}$$

It follows from (3.7) that $\{\|x_n - x^*\|\}$ is bounded (also means $\{x_n\}$ is bounded). Observe that

$$\begin{aligned}a_{n+1} &= \|x_{n+1} - x^*\|^2 - \alpha_{n+1}\theta_{n+1}\|x_n - x^*\|^2 + \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1})\|x_{n+1} - x_n\|^2\\ &\ge -\alpha_{n+1}\theta_{n+1}\|x_n - x^*\|^2\end{aligned}$$

and so $-a_{n+1} \le \alpha_{n+1}\theta_{n+1}\|x_n - x^*\|^2 \le \delta\|x_n - x^*\|^2$, which follows by (3.7) that

$$-a_{n+1} \le \delta^{n+1}\|x_0 - x^*\|^2 + \frac{\delta a_1}{1-\delta}.$$

By (3.7), we obtain

$$\varepsilon \sum_{n=1}^{m} \|x_{n+1} - x_n\|^2 \ \leq\ a_1 - a_{m+1} \ \leq\ \delta^{m+1}\|x_0 - x^*\|^2 + \frac{a_1}{1-\delta}.$$

Therefore, $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 \ \leq\ \frac{a_1}{\varepsilon(1-\delta)} < +\infty$. Hence, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Using (3.5) again yields

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq\ & (1 + \alpha_n\theta_n)\|x_n - x^*\|^2 - \alpha_n\theta_n\|x_{n-1} - x^*\|^2 + 2\alpha_n\|x_n - x_{n-1}\|^2 \\ & - \frac{(1-\alpha_n)}{\alpha_n}\|x_{n+1} - x_n\|^2 \\ \leq\ & \|x_n - x^*\|^2 + \alpha_n\theta_n(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2) \\ & + 2\delta\|x_n - x_{n-1}\|^2.\end{aligned} \tag{3.8}$$

By Lemma 2.1, we see that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Thus

$$\|x_{n+1} - w_n\| \leq \|x_{n+1} - x_n\| + \|x_n - w_n\| \leq \ \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\|,$$

which tends to 0 as n tends to the infinity. By (3.1), we conclude that $\|x_{n+1} - x_n\| \geq \alpha\|x_n - \sum_{k=1}^{n} \beta_n^k S_k w_n\|$. Thus

$$\|x_n - \sum_{k=1}^{n} \beta_n^k S_k w_n\| \leq \frac{1}{\alpha}\|x_{n+1} - x_n\| \to 0,\ n \to \infty. \tag{3.9}$$

Furthermore,

$$\|w_n - \sum_{k=1}^{n} \beta_n^k S_k w_n\| \leq \|x_n - x_{n-1}\| + \|x_n - \sum_{k=1}^{n} \beta_n^k S_k w_n\| \ \to 0,\ n \to \infty.$$

Observe that $\|x_n - w_n\| \leq \|x_n - x_{n-1}\|$. It follows that

$$\begin{aligned}\|x_n - \sum_{k=1}^{n} \beta_n^k S_k x_n\| &\leq \|x_n - w_n\| + \|w_n - \sum_{k=1}^{n} \beta_n^k S_k w_n\| + \|\sum_{k=1}^{n} \beta_n^k (S_k w_n - S_k x_n)\| \\ &\leq \|x_n - w_n\| + \|w_n - \sum_{k=1}^{n} \beta_n^k S_k w_n\| + \|\sum_{k=1}^{n} \beta_n^k \|S_k w_n - S_k x_n\| \\ &\leq 2\|x_n - w_n\| + \|w_n - \sum_{k=1}^{n} \beta_n^k S_k w_n\| \to 0, n \to \infty.\end{aligned}$$

Since S_k is nonexpansive for each k, then $\|S_k x - x^*\| \leq \|x - x^*\|$ for all $x \in H$. Therefore, $\sum_{k=1}^{n} \|\beta_n^k S_k x\| \leq \|x - x^*\| + \|x^*\|$. Hence, $\sum_{k=1}^{n} \beta_n^k S_k$ is well-defined on H. Define $T_n x := \sum_{k=1}^{n} \beta_n^k S_k x$ for all $x \in H$. From the Condition A(i) in Assumption 3.1, T_n is nonexpansive for each n. Clearly, $\cap_{k=1}^{\infty} F(S_k) \subset \cap_{n=1}^{\infty} F(T_n)$. Using Condition A (ii), we know that, for every $k \geq 1$, there exists $n_0 \geq 1$ such that $\beta_{n_0}^k > 0$. Hence, $F(T_{n_0}) \subset F(S_k)$, for all $k \geq 1$ by Lemma 2.4 and so $\cap_{n=1}^{\infty} F(T_n) \subset F(S_k)$ for all $k \geq 1$. Therefore, $\cap_{n=1}^{\infty} F(T_n) = \cap_{k=1}^{\infty} F(S_k) \neq \emptyset$. Assume that B is a bounded subset of H. Since $\cap_{k=1}^{\infty} F(S_k) \neq \emptyset$, we have that $\{S_k x : x \in B, k \geq 1\}$ is bounded. Define $M := \sup\{\|S_k x\| : x \in B, k \geq 1\}$. Then, for each $x \in B$ and $n \geq 1$,

$$\begin{aligned}\|T_{n+1}x - T_n x\| &\leq \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| \|S_k x\| + \beta_{n+1}^{n+1} \|S_{n+1}x\| \\ &\leq \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| M + \Big(1 - \sum_{k=1}^{n} |\beta_{n+1}^k\Big) M \\ &\leq 2M \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k|.\end{aligned}$$

Therefore, $\sup\{\|T_{n+1}x - T_n x\| : x \in B\} \leq 2M \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k|$. Using Condition A(iii), we obtain

$$\sum_{k=1}^{\infty} \sup\{\|T_{n+1}x - T_n x\| : x \in B\} \leq 2M \sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty. \tag{3.10}$$

Now, for $l, m \geq 1$ with $l > m$, we have $|\beta_l^k - \beta_m^k| \leq \sum_{n=m}^{l-1} |\beta_{n+1}^k - \beta_n^k|$ for all $k \geq 1$ and $k \leq m$. Hence,

$$\sum_{k=1}^{m} |\beta_l^k - \beta_m^k| \leq \sum_{k=1}^{m} \sum_{n=m}^{l-1} |\beta_{n+1}^k - \beta_n^k| = \sum_{n=m}^{l-1} \sum_{k=1}^{m} |\beta_{n+1}^k - \beta_n^k| \leq \sum_{n=m}^{l-1} \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k|.$$

Taking the limit, we arrive at

$$\sum_{k=1}^{m} |\beta_{n+1}^k - \beta_n^k| \leq \sum_{n=m}^{\infty} \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty,$$

where $\beta^k := \lim\limits_{n\to\infty} \beta_n^k,\ k \geq 1$. Thus $\lim\limits_{m\to\infty} \sum_{k=1}^{m} |\beta^k - \beta_m^k| = 0$. Therefore,

$$|\sum_{k=1}^{\infty} \beta^k - 1| = \lim_{m\to\infty} |\sum_{k=1}^{m} \beta^k - \sum_{k=1}^{m} \beta_m^k| \leq \lim_{m\to\infty} \sum_{k=1}^{m} |\beta^k - \beta_m^k| = 0$$

and so $\sum_{k=1}^{\infty} |\beta^k = 1$. Define mapping T by $Tx := \sum_{k=1}^{m} \beta^k S_k x$ for all $x \in H$. Using Lemma 2.4, we obtain that T is a nonexpansive mapping on H such that $F(T) = \cap_{k=1}^{\infty} F(S_k) = \cap_{n=1}^{\infty} F(T_n)$, since $\beta^k > 0$ for all $k \geq 1$. Furthermore,

$$\begin{aligned}\|T_m x - Tx\| &\leq \|\sum_{k=1}^{m} (\beta_m^k - \beta^k) S_k x\| + \sum_{k=m+1}^{\infty} \|\beta^k S_k x\| \\ &\leq \sum_{k=1}^{m} |\beta_m^k - \beta^k| M + \sum_{k=m+1}^{\infty} \beta^k M.\end{aligned}$$

Passing to the limit, we see $\lim_{m\to\infty} T_m x = Tx$, $x \in H$. By (3.10), we have that $\{T_n x\}$ is Cauchy for each $x \in H$. Let $k, l \geq 1$ such that $k > l$. Then

$$\begin{aligned}
\|T_k x - T_l x\| &\leq \sup\{\|T_k z - T_l z\| : z \in B\} \\
&\leq \sup\{\|T_k z - T_{k-1} z\| : z \in B\} + \sup\{\|T_{k-1} z - T_l z\| : z \in B\} \\
&\cdots \\
&\leq \sum_{n=l}^{k-1} \sup\{\|T_{n+1} z - T_n z\| : z \in B\} \\
&\leq \sum_{n=l}^{\infty} \sup\{\|T_{n+1} z - T_n z\| : z \in B\}.
\end{aligned}$$

Hence, $\{T_n x\}$ is Cauchy and converges strongly to a point in H. Furthermore,

$$\|Tx - T_l x\| = \lim_{n\to\infty} \|T_k x - T_l x\| \leq \sum_{n=l}^{\infty} \sup\{\|T_{n+1} z - T_n z\| : z \in B\}$$

for all $x \in H$ and $l \geq 1$. It follows that

$$\sup\{\|Tx - T_l x\| : x \in B\} \leq \sum_{n=l}^{\infty} \sup\{\|T_{n+1} z - T_n z\| : z \in B\}.$$

Thus

$$\lim_{l\to\infty} \sup\{\|Tx - T_l x\| : x \in B\} = 0. \tag{3.11}$$

Observe that

$$\begin{aligned}
&\|Tw_n - w_n\| \\
&\leq \|Tw_n - T_n w_n\| + \|T_n w_n - x_{n+1}\| + \|x_{n+1} - w_n\| \\
&\leq \|Tw_n - T_n w_n\| + 2\|x_{n+1} - x_n\| + \|x_n - w_n\| + \|x_n - T_n w_n\| \\
&\leq \sup\{\|Tx - T_n x\| : x \in B\} + 2\|x_{n+1} - x_n\| + \|x_n - w_n\| + \|x_n - T_n w_n\|.
\end{aligned}$$

By (3.9) and (3.11), we obtain $\lim_{n\to\infty} \|Tw_n - w_n\| = 0$. Since $\{x_n\}$ is bounded, there exists $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup p \in H$. Since $x_n - w_n \to 0, n \to \infty$, we obtain $w_{n_j} \rightharpoonup p \in H$. By Lemma 2.3, we find $p \in F(T) = \cap_{k=1}^{\infty} F(S_k)$. Observe that

(i) $\lim_{n\to\infty} \|x_n - x^*\|$ exists for every $x^* \in \cap_{k=1}^{\infty} F(S_k)$;

(ii) the weak sequential limit point of $\{x_n\}$ is in $\cap_{k=1}^{\infty} F(S_k)$.

By Opial's lemma Lemma 2.2, we obtain that $\{x_n\}$ converges weakly to a point in $\cap_{k=1}^{\infty} F(S_k)$. □

If $\theta_n = 1$, then we have the following result for a single nonexpansive mapping.

Corollary 3.5. *Let $S : H \to H$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be generated by: $x_0, x_1 \in H$,*

$$\begin{cases} w_n = 2x_n - x_{n-1} \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S w_n \end{cases}$$

with $0 < \alpha \leq \alpha_n \leq \alpha_{n+1} \leq \frac{1}{2+\varepsilon} := \delta, \varepsilon > 0$. Then $\{x_n\}$ converges weakly to a fixed point of S.

If we replace the "$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \sum_{k=1}^{n} \beta_n^k S_k w_n$" in proposed method (3.1) with "$x_{n+1} = (1-\alpha_n)w_n + \alpha_n \sum_{k=1}^{n} \beta_n^k S_k w_n$", then Theorem 3.4 may no hold, which can be seen via the he following example.

Example 3.6. Take $Tx = -x + 2,\ x \in \mathbb{R}$. Then T is nonexpansive and $F(T) = \{1\}$. We take $\alpha_n = \dfrac{1}{4},\ n \geq 1$ and $\theta_n = 1$. Then

$$x_{n+1} = (1-\alpha_n)w_n + \alpha_n T w_n = \frac{3}{4}w_n + \frac{1}{4}Tw_n = \frac{1}{2}x_n - \frac{1}{2}x_{n-1} + \frac{1}{2}.$$

It can be shown that $\{x_n\}$ converges to $\frac{1}{2} \notin F(T)$.

Next, we give some applications of Theorem 3.4 to a monotone inclusion problem. Recall that an operator $A : D(A) \subset H \to 2^H$ is said to be monotone iff $\langle u-v, x-y\rangle \geq 0$ for all $x, y \in D(A)$, $u \in Ax$, and $v \in Ay$; A is maximal monotone iff A is monotone and its graph $G(A) := \{(x,y) : x \in D(A), y \in Ax\}$ is not properly contained in the graph of any other monotone operator. The resolvent operator $J_{\gamma A}$ associated with A and λ is the mapping $J_{\gamma A} : H \to H$ defined by $J_{\gamma A}(x) := (I + \gamma A)^{-1}x$, for all $x \in H$ and $\lambda > 0$. Note that $J_{\gamma A}$ is single-valued and nonexpansive (see, e.g., [4]). Moreover $x \in A^{-1}(0)$ $(A^{-1}(0) := \{x \in H : 0 \in A(x)\})$ if and only if $x = J_{\gamma A}(x)$ for all $\lambda > 0$.

Let us now consider the following inclusion problem involving three operators, Find x in H such that $0 \in Ax$, where A is a maximal monotone operator defined on H.

Corollary 3.7. *Let $A : H \to 2^H$ be a maximal monotone operator such that $A^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be generated by: $x_0, x_1 \in H$,*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n(1-\rho)w_n + \alpha_n \rho J_{\gamma A} w_n \end{cases}$$

with $\rho \in (0,2)$, $0 \leq \theta_n \leq \theta_{n+1} \leq 1$, and $0 < \alpha \leq \alpha_n \leq \alpha_{n+1} \leq \frac{1}{2+\varepsilon} := \delta, \varepsilon > 0$. Then $\{x_n\}$ converges weakly to a point in $A^{-1}(0)$.

Proof. Define $u_n := (1-\rho)w_n + \rho J_{\gamma A} w_n,\ \forall n \geq 1$. Picking $x^* \in A^{-1}(0)$, one has

$$\begin{aligned} \|u_n -^* \| &= \|w_n - x^*\|^2 + 2\rho\langle w_n - x^*, J_{\gamma A} - w_n\rangle + \rho^2 \|J_{\gamma A} - w_n\|^2 \\ &\leq \|w_n - x^*\|^2 - \rho(2-\rho)\|J_{\gamma A} - w_n\|^2. \end{aligned}$$

Note that $x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n$. Hence,

$$\|x_{n+1} - x^*\|^2 \leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|w_n - x^*\|^2 - \frac{(1-\alpha_n)}{\alpha_n}\|x_{n+1} - x_n\|^2.$$

Following the similar proof in Theorem 3.4 and noting that $A^{-1}(0) = F(J_{\gamma A})$, we obtain the desired conclusion immediately. □

Let us apply our results to the Douglas-Rachford splitting method to find zeros of $A + B$ with $A, B : H \to 2^H$ being maximal monotone operators on a Hilbert space H (see [14, 17]). Define

$$R_{\gamma A} := 2J_{\gamma A} - I \quad \text{and} \quad R_{\gamma B} := 2J_{\gamma B} - I$$

for the corresponding *reflections* (also called *Cayley operators*), and observe that these reflections are nonexpansive mappings.

One can show that $0 \in Ax + Bx$ if and only if $x = J_{\gamma B}(y)$, where y is a fixed point of the nonexpansive mapping $R_{\gamma A} R_{\gamma B}$. The Douglas-Rachford splitting method to find a zero of $A + B$

$$x_{n+1} := (1 - \alpha_n)x_n + \alpha_n R_{\gamma A} R_{\gamma B} x_n, \quad n \geq 1. \tag{3.12}$$

One can rewrite (3.12) as

$$\begin{aligned} x_{n+1} &:= (1 - \alpha_n)x_n + \alpha_n \big(2J_{\gamma A}(2J_{\gamma B}x_n - x_n) - 2J_{\gamma B}x_n + x_n\big) \\ &= x_n + 2\alpha_n \big(J_{\gamma A}(2J_{\gamma B}x_n - x_n) - J_{\gamma B}x_n\big). \end{aligned}$$

We now have the following weak convergence result.

Theorem 3.8. *Let $\gamma \in (0, \infty)$ and assume that $0 \in \operatorname{ran}(A+B)$. Let $\{x_n\}$ be generated by: $x_0, x_1 \in H$,*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(2J_{\gamma A} - I)(2J_{\gamma B} - I)w_n \end{cases}$$

with $0 \leq \theta_n \leq \theta_{n+1} \leq 1$ and $0 < \alpha \leq \alpha_n \leq \alpha_{n+1} \leq \frac{1}{2+\varepsilon} := \delta, \varepsilon > 0$. Then $\{x_n\}$ converges weakly to some point $y \in H$ such that $J_{\gamma B}y \in (A+B)^{-1}(0)$, i.e., $x := J_{\gamma B}y$ is a solution of the monotone inclusion problem for $A + B$.

Proof. Define $T := R_{\gamma A} R_{\gamma B}$. Then T is nonexpansive. Applying Theorem 3.4 and [4, Theorem 25.6], we obtain the desired conclusion. □

4. Numerical Experiments

This section is focused on the numerical implementation of our proposed algorithm. The example here is related to the applications provided above. Furthermore, we compare our algorithm with other algorithms in the literature to test the efficiency and accuracy of the new scheme. All codes are written in MATLAB R2019a and performed on a Intel(R) Core(TM) i7-6600U CPU @ 2.60GHz 2.81 GHz, RAM 16.00 GB computer.

Example 4.1. [9] Consider the following complementarity problem. Let $Q := \{x : x_i \geq 0, \, i = 1, \cdots, N\}$ and

$$A_1 = \begin{pmatrix} D & -I & & & & \\ -I & D & -I & & & \\ & \cdots & \cdots & & & \\ & & \cdots & \cdots & & \\ & & -I & D & -I \\ & & & -I & D \end{pmatrix}, \qquad A_2 = \frac{hc}{2} \begin{pmatrix} O & I & & & & \\ -I & O & I & & & \\ & \cdots & \cdots & & & \\ & & \cdots & \cdots & & \\ & & -I & O & I \\ & & & -I & O \end{pmatrix},$$

are two $N \times N$ block tridiagonal matrices, where $N = m^2$, $h = \dfrac{1}{m+1}$, $c = 100$, I is an $m \times m$ identity matrix and D is an $m \times m$ matrix taking the form $4I + M + M^T$ where $M_{i,j} = 1$ if $j = i + 1, \; i = 1, \cdots, m - 1$, otherwise $M_{i,j} = 0$.

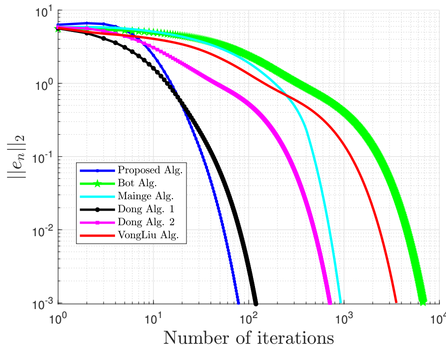

FIGURE 1. Example 4.1: $n = 100$

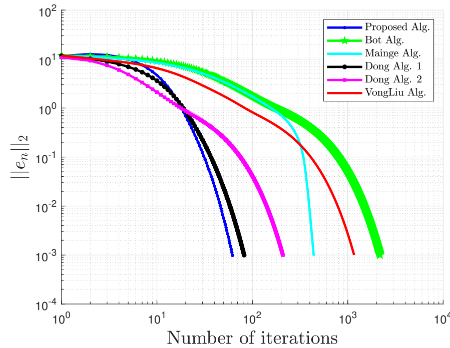

FIGURE 2. Example 4.1: $n = 225$

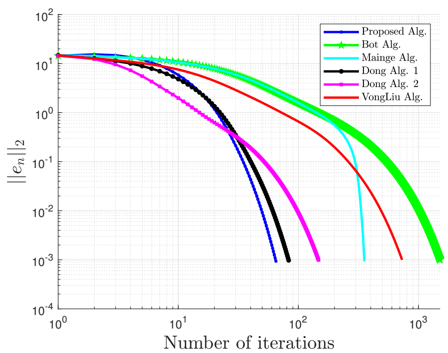

FIGURE 3. Example 4.1: $n = 625$

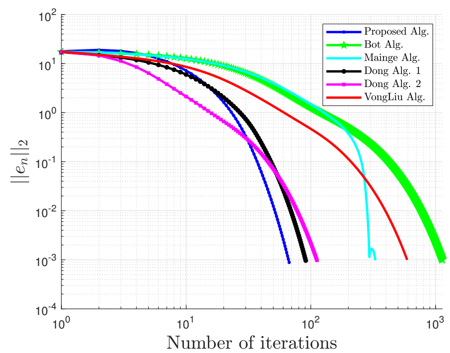

FIGURE 4. Example 4.1: $n = 900$

Define $A = A_1 + A_2$ and consider the following complementarity problem which was also given in [9, Section 5] $0 \in Ax - b + \partial\delta_Q(x)$, where $\partial\delta_Q(x)$ is the sub-differential of the indicator function

$$\delta_Q(x) = \begin{cases} 0, & \text{if } x \in Q, \\ +\infty, & \text{if } x \notin Q. \end{cases}$$

Setting $b = Ae_1$, where e_1 is the first column of the identity matrix with order $N \times 1$ corresponding to the matrix A, as seen in [9, Section 5], the unique solution of the complementarity problem is given as $x^* = e_1$. Therefore, we set $b := Ae_1$, $A(x) := Ax - b$, and $B(x) := \partial\delta_Q(x)$. (3.1) is reduced to $x_0, x_1 \in \mathbb{R}^n$

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n(2J_{\gamma_n A} - I)(2J_{\gamma_n B} - I)w_n, \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where $0 \leq \theta_n \leq \theta_{n+1} \leq 1$, $0 < \alpha \leq \alpha_n \leq \alpha_{n+1} \leq \frac{1}{2+\varepsilon}$, and $\varepsilon > 0$.

For the computation of γ_n, the method in [10, 9] is adopted for an update. In addition, we assume that A is continuous. Then at n-th iterate, for a known γ_n, first calculate

$$\rho_n := \frac{\gamma_n \|A(x_n) - A(x_{n-1})\|}{\|x_n - x_{n-1}\|}.$$

Then we update γ_n by

$$\gamma_{n+1} = \begin{cases} 1.5\gamma_n, & \text{if } \rho_n \leq 0.5, \\ \gamma_n, & \text{if } 0.5 < \rho_n < 2, \\ 0.5\gamma_n, & \text{if } \rho_n \geq 2. \end{cases} \tag{4.2}$$

TABLE 1. Example 4.1: Proposed Alg. vs Bot Alg. vs Mainge Alg. vs Dong Alg. 1 vs Dong Alg. 2 vs VongLiu Alg. for different values of N

		100	400	625	900
Proposed Alg.	No. of Iteration	78	62	65	67
	CPU (Time)	0.25836	1.2856	3.0014	6.1123
Bot Alg.	No. of Iteration	6766	2174	1506	1130
	CPU (Time)	10.3016	25.9696	43.0128	66.4849
Mainge Alg.	No. of Iteration	924	440	355	329
	CPU (Time)	1.3916	5.1897	10.2117	19.2608
Dong Alg. 1	No. of Iteration	119	82	83	91
	CPU (Time)	0.33693	1.5737	3.952	8.3006
Dong Alg. 2	No. of Iteration	712	209	147	112
	CPU (Time)	1.1316	2.4655	4.1002	6.4836
VongLiu Alg.	No. of Iteration	3542	1165	730	592
	CPU (Time)	5.3151	13.9433	20.7978	34.8906

For the implementation of method (4.1), $J_{\gamma_n B}(w_n) = P_Q(w_n) = \max\{0, w_n\}$. All algorithms are terminated when $\|x_n - x^*\| < 10^{-3}$. We demonstrate the efficiency and accuracy of our proposed method (3.1) over several inertial Krasnosel'skiĭ-Mann iteration methods for solving a monotone inclusion problem $0 \in A(x) + B(x)$ in the literature. We denote Bot et al. algorithm in [6, (5)] as Bot Alg., Mainge algorithm in [18, (1.2), Theorem 3.1 (C2)] as Mainge Alg., Dong algorithm in [9, Algorithm 1 in Section 4] as Dong Alg. 1, Dong et al. algorithm in [13, Algorithm 3.1] as Dong Alg. 2, Vong & Liu algorithm in [24, (rmiMann)] as VongLiu Alg. We give the choice of parameters for our numerical experiments as follows

Proposed Alg.: $\theta_n = 1$, $\alpha_n = 0.499$, and $\gamma_1 = 0.02$.

Bot Alg.: $\alpha_n = 0.1$, $\theta_1 = 0$, $\theta_n = 0.11$, and $\gamma_n = 0.2$.

Mainge Alg.: $\alpha_n = 0.1$, $\theta = 0.9$, and $\gamma_n = 0.2$.

Dong Alg. 1: $\epsilon = 0.01$, $\alpha_n = 0.5 + \epsilon$, $\theta_1 = 0$ $\theta_n = 0.2350$, and $\gamma_1 = 0.02$.

Dong Alg. 2: $\mu = 1$, $\alpha_n = 0.1$, $\lambda = 0.9$, $\beta_n = \frac{1}{(n+1)^2}$, $\gamma_n = 0.2$, and $\gamma_n^* = \frac{1}{1+n}$.
VongLiu Alg.: $\mu = 0.9$, $\alpha_n = 0.1$, $\lambda = 0.9$, $\gamma_n = 0.2$, and $\gamma_n^* = 0.9$.

5. Final Remarks

In this paper, we introduced a new inertial Krasnosel'skiĭ-Mann iteration for which the choice $0 \leq \theta_n \leq 1, \forall n \geq 1$ is allowed for a countable family of nonexpansive mappings and we gave the weak convergence of the iteration in Hilbert spaces. The choice $0 \leq \theta_n \leq 1$ for all $n \geq 1$ is possible, which is the major highlight of this paper. We also considered the zero problem of two maximal monotone operators. Finally, we presented some numerical results to discuss the benefit of the choice $\theta_n = 1$ over some inertial Krasnosel'skiĭ-Mann iterations where $0 \leq \theta_n \leq \theta < 1$ for all $n \geq 1$.

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