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ON SPLIT EQUILIBRIUM, VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS FOR DEMICONTRACTIVE MULTIVALUED MAPPINGS

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Abstract. In this paper, we study the problem of finding the common solutions of split equilibrium problem, variational inequality problem and common fixed point problem for demicontractive multivalued mappings in real Hilbert spaces. We propose a new inertial iterative algorithm with self-adaptive step size for approximating the solution of this problem. Under mild and standard conditions, we prove a strong convergence result for the sequence generated by our proposed algorithm without the knowledge of the operator norm. Moreover, we apply our result to study split minimization problem and provide some numerical experiments to demonstrate the applicability and performance of our algorithm in comparison with some of the related results in the literature.
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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The *Equilibrium Problem* (shortly, (EP)) in the sense of Blum and Oettli [11] is to find $\hat{x} \in C$ such that

$$F(\hat{x}, y) \geq 0, \quad \forall\, y \in C. \tag{1.1}$$

We denote the set of all solutions of EP (1.1) by $EP(F)$. The EP attracts considerable research efforts and serves as a unifying framework for many well-known problems, such as the Optimization Problems (OPs), Nonlinear Complementarity Problems (NCPs), Variational Inequality Problems (VIPs), the Fixed Point Problem (FPP), Saddle Point Problems (SPPs), the Nash equilibria and many others. It finds applications in different fields, such as physics, economics, etc (see, for example [11, 31] and the references therein).
Consider the problem of finding a point $\hat{x} \in C$ satisfying

$$F_1(\hat{x}, x), \text{ for all } x \in C, \tag{1.2}$$

such that $\hat{y} = A\hat{x} \in Q$ satisfies

$$F_2(\hat{y}, y), \text{ for all } y \in Q, \tag{1.3}$$

where C and Q are nonempty closed convex subsets of the real Hilbert spaces H_1 and H_2 respectively, $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are bifunctions, and $A : H_1 \to H_2$ is a bounded linear operator. The problem (1.2)-(1.3) is called the *Split Equilibrium Problem* (SEP). We denote the set of solutions of the SEP by $SEP(F_1, F_2)$.
A lot of research has been carried out on the SEP featuring a great number of applications in physics, structural analysis, management sciences and economics, etc. (see, for example [4, 35, 26]). Several algorithms have been developed for solving the SEP and its related optimization problems (see [2, 3, 13, 17, 18], and the references therein).
Let $\mathcal{C}$ be a nonempty, closed and convex subset of a real Hilbert space H and $A : H \to H$ be an operator. The *Variational Inequality Problem* (VIP) is defined as follows: Find $\hat{x} \in \mathcal{C}$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle \geq 0, \ \forall y \in \mathcal{C}. \tag{1.4}$$

We denote the solution set of the VIP (1.4) by $VI(\mathcal{C}, A)$.
The VIP was first introduced independently by Fichera [15] and Stampacchia [38] for modeling problems arising from mechanics. The VIP can be considered as a central problem in optimization and nonlinear analysis because the theory provides a simple, natural and unified framework for treating many important mathematical problems such as minimization problems, network equilibrium problems, complementary problems, systems of nonlinear equations and others (see [3, 12, 16, 43, 44] and other references therein). This has made this theory an interesting area of research to many authors who have proposed and analyzed several iterative algorithms for solving the VIP and other related optimization problems (see [6, 30, 32], and references therein). It is well known that $u \in VI(C, A)$ if and only if $u = P_C(I - \lambda B)(u)$ for any $\lambda > 0$, where P_C is the metric projection of H onto C.
Given the map $S : C \to C$, the problem of finding a point $x^* \in C$ (called fixed point) such that $Sx^* = x^*$ is called a *Fixed Point Problem* (FPP). We denote the set of all the fixed points of S by $Fix(S)$. That is,

$$Fix(S) = \{x^* \in C : Sx^* = x^*\}. \tag{1.5}$$

If S is a multivalued mapping, i.e. $S : C \to 2^C$, then $\hat{x} \in C$ is called a fixed point of S if

$$\hat{x} \in S\hat{x}. \tag{1.6}$$

The fixed point theory for multivalued mappings has been applied in various areas such as game theory, control theory, mathematical economics, etc. Moreover, several problems arising in sciences and engineering can be formulated as finding solution of FPP of a nonlinear mapping.

Recently, the problem of finding common solutions of the set of fixed points of nonlinear mappings and the set of solutions of optimization problems have been considered by some authors (for instance, see [5, 31, 48, 47, 40, 52] and the references therein). The motivation for studying such a common solution problem is in its potential application to mathematical models whose constraints can be expressed as fixed point problems and optimization problems. This is the case in real world problems such as signal processing, image recovery and network resource allocation (see, for instance, [22, 27] and the references therein). In this paper, we will be studying the problem of approximating the common solutions of split equilibrium problem, variational inequality problem and common fixed point problem for demicontractive multivalued mappings.

To obtain the common solution of the SEP, VIP and FPP of asymptotically nonexpansive mapping, Wang et al [48] introduced the following iterative scheme:

Algorithm 1.1.

$$\begin{cases} x_0 \in C, \\ u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{s_n}^{F_2})A)x_n, \\ y_n = P_C(u_n - \sigma_n G u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \mu_n T^n y_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $T : C \to C$ is an asymptotically nonexpansive mapping with sequence $\{k_n\}$, $\{r_n\} \subset (r, +\infty)$ with $r > 0$, $\{s_n\} \subset (s, +\infty)$ with $s > 0$, $\{\sigma_n\} \subset (0, 2\kappa)$ and $\gamma \subset (0, 1/L^2)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\} \subset (0, 1)$ are three sequences $\alpha_n + \beta_n + \mu_n = 1$ for each $n \in N$. The proved the strong convergence of the Algorithm (1.1) under the following assumptions:

(A1) $\lim\limits_{n\to+\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$, $\limsup\limits_{n\to+\infty} \beta_n < 1$;

(A2) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < +\infty$, $\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < +\infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < +\infty$, $\sum_{n=1}^{\infty} |s_n - s_{n-1}| < +\infty$;

(A3) $\lim\limits_{n\to+\infty} \frac{k_n - 1}{\alpha_n} = 0$, $\lim\limits_{n\to+\infty} \sigma_n = \sigma > 0$;

(A4) T is asymptotically regular, that is, $\lim\limits_{n\to+\infty} \|T^{n+1}z - T^n z\| = 0$ for all $z \in C$.

Under the same conditions above with the regularity assumption replaced with $\lim_{n\to+\infty} \sup_{x\in K} \|T_i^{n+1}x - T_i^n x\| = 0$, Zhang and Gui [52] introduced and proved strong convergence result for the following algorithm for approximating the common solution of split equilibrium problem and common fixed point of finite family of asymptotically nonexpansive mappings in Hilbert spaces:

Algorithm 1.2.

$$\begin{cases} x_0 \in C, \\ u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ x_{n+1} = \alpha_n x_n + \frac{(1-\alpha_n)}{m}\sum_{i=1}^{m} T_i^n u_n, \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $\{T_i\} : C \to C$ is a finite family of asymptotically nonexpansive mappings in Hilbert space.

The condition (A2) has been applied in establishing the convergence of several algorithms in the literature despite its restrictions. These restrictions motivated the research question: *Is it possible to develop an algorithm that generates a strong convergence sequence for finding the common solution of the SEP (1.2)-(1.3), VIP* (1.4) *and the FPP of more general mappings (e.g., the demicontractive mapping) under mild conditions?*
Zhao *et al* [53] proposed the following iterative scheme for finding a common element of the solution sets of SEP, VIP for an inverse strongly monotone mapping and FPP of nonspreading multivalued mapping S in Hilbert spaces

Algorithm 1.3.

$$\begin{cases} x_1 \in C, \ arbitrarily \\ u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = P_C(I - \sigma_n G)u_n, \\ x_{n+1} \in a_n x_n + (1 - a_n)Sy_n, \ \text{for all } n \geq 1. \end{cases} \tag{1.9}$$

They obtained weak convergence result for the sequence generated by the Algorithm 1.3 under certain mild conditions.
Bauschke and Combettes [8] remarked that in solving optimization problems, strong convergence of iterative methods are more preferable and more applicable than their weak convergence counterparts. It is therefore necessary to develop iterative methods that generate sequence which converges strongly.
At this point, it is worth noting that the step size γ in the aforementioned algorithms plays a vital role in the convergence properties of iterative methods. The algorithms above and many other related results in the literature involve step size that is dependent on the operator norm $\|A\|$. Algorithms designed this way are usually difficult to execute because they require computation of the operator norm, which is very difficult to calculate or even estimate. Hence, this further gives rise to the following research

questions: *Can the step size be constructed such that it is independent on the operator norm?* In this work, we provide affirmative answers to the above questions.

In order to accelerate the rate of convergence of algorithms, authors often incorporate the inertial technique. The inertial algorithm involves a two-step iteration where the next iterate is determined by making use of the previous two iterates. Polyak [34] first introduced an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. Several researchers have recently constructed some efficient iterative schemes by employing inertial technique (e.g. see [3, 45, 7, 10, 29, 42, 49]).

Motivated and inspired by the above results and related literature, in this paper, we propose an inertial iterative scheme which does not require prior knowledge of the operator norm and obtain strong convergence result for approximating the common solution of SEP, VIP and FPP for finite family of demicontractive multivalued mappings. Our results complement and generalise several existing results in this direction in the literature.

2. Preliminaries

We first recall some definitions and known results which are needed to prove our main results.

Let C be a nonempty, closed and convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. P_C is called the metric projection of H onto C. It is known that P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \tag{2.1}$$

for all $x, y \in H$. Moreover $\langle x - P_C x, y - P_C x \rangle \leq 0$ holds for all $x \in H$ and $y \in C$, see [41].

We denote the strong convergence and the weak convergence of the sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. For a given sequence $\{x_n\} \subset H, w_\omega(x_n)$ denotes the set of weak limits of $\{x_n\}$, that is,

$$w_\omega(x_n) := \{x \in H : x_{n_j} \rightharpoonup x, \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

Definition 2.1. Let H be a Hilbert space. A function $f : H \to H$ is called a *contraction* if there exists $\mu \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \mu \|x - y\|, \quad \text{for all } x, y \in H.$$

A mapping $G : C \to H$ is called κ-inverse strongly monotone if there exists $\kappa > 0$ such that

$$\langle x - y, Gx - Gy \rangle \geq \kappa \|Gx - Gy\|^2, \quad \forall x, y \in C.$$

A bounded linear operator $D : C \to H$ is called *strongly positive* if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Dx, x \rangle \geq \bar{\gamma} ||x||^2, \quad \text{for all } x \in C.$$

Definition 2.2. A subset K of H is called *proximal* if for each $x \in H$, there exists $y \in K$ such that

$$||x-y|| = d(x,K) = \inf\{\|x-z\| : z \in K\}.$$

We denote by $CB(C), CC(C), K(C)$ and $P(C)$ the families of all nonempty closed bounded subsets of C, nonempty closed convex subsets of C, nonempty compact subsets of C, and nonempty proximal bounded subsets of C, respectively. The Pompeiu-Hausdorff metric on $CB(C)$ is defined by

$$H(A,B) := \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\},$$

for all $A, B \in CB(C)$, where $d(x,B) = \inf_{b\in B}\|x-b\|$. Let $S : C \to 2^C$ be a multivalued mapping. We say that S satisfies the endpoint condition if $Sp = \{p\}$ for all $p \in Fix(S)$. For multivalued mappings $S_i : C \to 2^C (i \in \mathbb{N})$ with $\cap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$, we say that S_i satisfies the common endpoint condition if $S_i(p) = \{p\}$ for all $i \in \mathbb{N}, p \in \cap_{i=1}^{\infty} Fix(S_i)$.
We recall some basic definitions on multivalued mappings.

Definition 2.3. A multivalued mapping $S : C \to CB(C)$ is said to be

(i) *nonexpansive* if

$$H(Sx,Sy) \le ||x-y||, \text{ for all } x,y \in C;$$

(ii) *quasi-nonexpansive* if $F(S) \neq \emptyset$ and

$$H(Sx,Sp) \le ||x-p||, \text{ for all } x \in C, p \in Fix(S);$$

(iii) *nonspreading* if

$$2H(Sx,Sy)^2 \le d(y,Sx)^2 + d(x,Sy)^2, \text{ for all } x,y \in C;$$

(iv) $\lambda-$*hybrid* if there exists $\lambda \in \mathbb{R}$ such that

$$(1+\lambda)H(Sx,Sy)^2 \le (1-\lambda)||x-y||^2 + \lambda d(y,Sx)^2 + \lambda d(x,Sy)^2, \text{ for all } x,y \in C;$$

(v) $k-$*demicontractive* for $0 \le k < 1$ if $F(S) \neq \emptyset$ and

$$H(Sx,Sp)^2 \le ||x-p||^2 + kd(x,Sx)^2, \text{ for all } x \in C, p \in Fix(S).$$

Remark 2.1. From the definitions above, one can easily see that the class of $k-$demicontractive mappings is more general and includes all the other types of mappings defined above.

The *best approximation operator* P_S for a multivalued mapping $S : C \to P(C)$ is defined by

$$P_S(x) := \{y \in Sx : ||x-y|| = d(x,Sx)\}.$$

It is known that $Fix(S) = Fix(P_S)$ and P_S satisfies the endpoint condition. Song and Cho [37] gave an example of a best approximation operator P_S which is nonexpansive, but where S is not necessarily nonexpansive.

Definition 2.4. Let $S : C \to CB(C)$ be a multivalued mapping. The multivalued mapping $I-S$ is said to be demiclosed at zero if for any sequence $\{x_n\} \subset C$ which converges weakly to q and the sequence $\{||x_n - u_n||\}$ converges strongly to 0, where $u_n \in Sx_n$, then $q \in Fix(S)$.

The following results will be needed in the sequel:

Lemma 2.1. *For all $x, y \in H$, we have the following statements* [46]*:*

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
(ii) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$.

Lemma 2.2. [51] *For each $x_1, \dots, x_m \in H$ and $\alpha_1, \dots, \alpha_m \in [0,1]$ with $\sum_{i=1}^m \alpha_i = 1$, the following holds:*

$$\|\alpha_1 x_1 + \ldots + \alpha_m x_m\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{1\leq i<j\leq m} \alpha_i\alpha_j \|x_i - x_j\|^2.$$

Lemma 2.3. [36] *Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^\infty \alpha_n = +\infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n, \text{ for all } n \geq 1,$$

if $\limsup_{k\to+\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to+\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0$, then $\lim_{n\to+\infty} a_n = 0$.

Lemma 2.4. [28] *Let $\{a_n\}, \{c_n\} \subset \mathbb{R}_+, \{\sigma_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that*

$$a_{n+1} \leq (1-\sigma_n)a_n + b_n + c_n \text{ for all } n \geq 0.$$

Assume $\sum_{n=0}^\infty |c_n| < +\infty$. Then the following results hold:
(1) If $b_n \leq \beta\sigma_n$ for some $\beta \geq 0$, then $\{a_n\}$ is a bounded sequence.
(2) If we have

$$\sum_{n=0}^\infty \sigma_n = +\infty \text{ and } \limsup_{n\to+\infty} \frac{b_n}{\sigma_n} \leq 0,$$

then $\lim_{n\to+\infty} a_n = 0$.

Assumption 2.1. In solving the equilibrium problem (EP), we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) F is upper hemicontinuous, that is, for all $x, y, z \in C$, $\lim_{t\downarrow 0} F\big(tz + (1-t)x, y\big) \leq F(x,y)$;
(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.5. *Let C be a nonempty closed convex subset of a Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to C$ as follows:*

$$T_r^F(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall\, y \in C\}. \tag{2.2}$$

Then T_r^F is well defined and the following hold:

(1) *for each $x \in H, T_r^F(x) \neq \emptyset$;*
(2) *T_r^F is single-valued;*

(3) T_r^F *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\|T_r^F x - T_r^F y\|^2 \le \langle T_r^F x - T_r^F y, x - y\rangle;$$

(4) $Fix(T_r^F) = EP(F)$;

(5) $EP(F)$ *is closed and convex.*

3. Main results

In this section, we present our algorithm and prove some strong convergence theorems of the proposed algorithm for solving the SEP, VIP and FPP.

Let H_1 and H_2 be two real Hilbert spaces, $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex subsets, and $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying Assumption 2.1, with F_2 being upper semicontinuous in the first argument. Let $D : H_1 \to H_1$ be a strongly positive bounded linear operator with coefficient $\bar{\xi}$, and $f : H_1 \to H_1$ be a contraction with coefficient $\rho \in (0,1)$ such that $0 < \xi < \frac{\bar{\xi}}{\rho}$. Suppose that $G : C \to H_1$ is a $\kappa-$inverse strongly monotone operator. Let $S_i : C \to CB(C)$ be a finite family of multivalued demicontractive mappings with constant λ_i such that each $I - S_i$ is demiclosed at zero, $S_i(p) = \{p\}$ for all $p \in \cap_{i=1}^m Fix(S_i)$, and $\lambda = \max\{\lambda_i\}$. Suppose that the solution set $\Omega := SEP(F_1, F_2) \cap VI(C, G) \cap \bigcap_{i=1}^m Fix(S_i) \neq \emptyset$. We establish the convergence of the algorithm under the following assumptions on the control parameters:

(C1) Let $\{\alpha_n\} \subset (0,1)$ such that $\lim\limits_{n\to+\infty} \alpha_n = 0$ and $\sum_{n=0}^\infty \alpha_n = +\infty$;

(C2) $\{\delta_{n,i}\} \subset (0,1), \sum_{i=0}^m \delta_{n,i} = 1, \liminf_n (\delta_{n,0} - \lambda)\delta_{n,i} > 0,$ for each $1 \le i \le m$;

(C3) $0 < a \le \tau_n \le b < 1, \{\sigma_n\} \subset (0, 2\kappa)$ such that $\lim\limits_{n\to+\infty} \sigma_n = \sigma > 0$ and $\sigma < 2\kappa$

(C4) $\{r_n\}, \{s_n\}$ are positive real sequences such that $\liminf\limits_{n\to+\infty} r_n > 0$ and $\liminf\limits_{n\to+\infty} s_n > 0$;

(C5) Let $\theta > 0$ and $\{\epsilon_n\}$ be a positive sequence such that $\epsilon_n = o(\alpha_n)$, i.e., $\lim\limits_{n\to+\infty} \frac{\epsilon_n}{\alpha_n} = 0$.

Now, the algorithm is presented as follows:

Algorithm 3.1.

Step 0. Select initial data $x_0, x_1 \in H_1$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \le \theta_n \le \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \ \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}). \tag{3.2}$$

Step 3. Compute

$$t_n = T_{r_n}^{F_1}(w_n + \gamma_n A^*(T_{s_n}^{F_2} - I)Aw_n), \tag{3.3}$$

where

$$\gamma_n := \begin{cases} \tau_n \frac{\|(T_{s_n}^{F_2}-I)Aw_n\|^2}{\|A^*(T_{s_n}^{F_2}-I)Aw_n\|^2}, & \text{if } Aw_n \neq T_{s_n}^{F_2}Aw_n, \\ \gamma, & \text{otherwise } (\gamma \text{ being any nonnegative real number}). \end{cases} \tag{3.4}$$

Step 4. Compute

$$\begin{cases} u_n = P_C(I-\sigma_n G)t_n \\ y_n = \delta_{n,0}u_n + \sum_{i=1}^m \delta_{n,i}v_{n,i}, \quad v_{n,i} \in S_i u_n. \\ x_{n+1} = \alpha_n \xi f(x_n) + (I-\alpha_n D)y_n. \end{cases} \tag{3.5}$$

Set $n := n+1$ and return to **Step 1**.

Remark 3.1. Observe that from (3.1) and conditions (C1) and (C5), it follows that

$$\lim_{n\to+\infty} \theta_n\|x_n - x_{n-1}\| = 0 \text{ and } \lim_{n\to+\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0. \tag{3.6}$$

The following lemmas are crucial in establishing the strong convergence of the proposed algorithm.

Lemma 3.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then, $\{x_n\}$ is bounded.*

Proof. First, we show that $P_\Omega(I-D+\xi f)$ is a contraction on H_1. For all $x, y \in H_1$, we have that

$$\begin{aligned} &||P_\Omega(I-D+\xi f)(x) - P_\Omega(I-D+\xi f)(y)|| \\ &\quad\le ||(I-D+\xi f)(x) - (I-D+\xi f)(y)|| \\ &\quad\le ||(I-D)x-(I-D)y|| + \xi||fx-fy|| \\ &\quad\le (1-\bar{\xi})||x-y|| + \xi\rho||x-y|| \\ &\quad= (1-(\bar{\xi}-\xi\rho))||x-y||. \end{aligned}$$

Hence, $P_\Omega(I-D+\xi f)$ is a contraction. Thus, by the Banach Contraction Principle there exists an element $p \in \Omega$ such that $p = P_\Omega(I-D+\xi f)(p)$. Since $p \in \Omega$, then we have $p = T_{r_n}^{F_1}p, Ap = T_{s_n}^{F_2}(Ap)$, and $S_i(p) = p$ for all $i = 1, 2, \ldots, m$. Now, by the nonexpansivity of $T_{r_n}^{F_1}$ and by applying Lemma 2.1(ii), we obtain

$$\begin{aligned} \|t_n - p\|^2 &= \|T_{r_n}^{F_1}(w_n + \gamma_n A^*(T_{s_n}^{F_2}Aw_n - Aw_n)) - p\|^2 \\ &\le \|w_n + \gamma_n A^*(T_{s_n}^{F_2}Aw_n - Aw_n) - p\|^2 \\ &= \|w_n - p\|^2 + 2\gamma_n\langle w_n - p, A^*(T_{s_n}^{F_2}Aw_n - Aw_n)\rangle \\ &+ \gamma_n^2\|A^*(T_{s_n}^{F_2}Aw_n - Aw_n)\|^2 \\ &= \|w_n - p\|^2 + 2\gamma_n\langle A(w_n - p), T_{s_n}^{F_2}Aw_n - Aw_n\rangle \\ &+ \gamma_n^2\|A^*(T_{s_n}^{F_2}Aw_n - Aw_n)\|^2. \end{aligned} \tag{3.7}$$

Again, by Lemma 2.1(ii), we have

$$\begin{aligned}
&2\gamma_n\langle A(w_n-p),(T^{F_2}_{s_n}-I)Aw_n\rangle\\
&\quad= 2\gamma_n\langle T^{F_2}_{s_n}Aw_n-Ap-(T^{F_2}_{s_n}-I)Aw_n,\ (T^{F_2}_{s_n}-I)Aw_n\rangle\\
&\quad= 2\gamma_n\big[\langle T^{F_2}_{s_n}Aw_n-Ap,\ (T^{F_2}_{s_n}-I)Aw_n\rangle\\
&\quad- \langle (T^{F_2}_{s_n}-I)Aw_n,\ (T^{F_2}_{s_n}-I)Aw_n\rangle\big]\\
&\quad= 2\gamma_n\big[\langle T^{F_2}_{s_n}Aw_n-Ap,\ (T^{F_2}_{s_n}-I)Aw_n\rangle-\|(T^{F_2}_{s_n}-I)Aw_n\|^2\big]\\
&\quad= \gamma_n\big[\|T^{F_2}_{s_n}Aw_n-Ap\|^2+\|(T^{F_2}_{s_n}-I)Aw_n\|^2\\
&\quad- \|T^{F_2}_{s_n}Aw_n-Ap-(T^{F_2}_{s_n}-I)Aw_n\|^2-2\|(T^{F_2}_{s_n}-I)Aw_n\|^2\big]\\
&\quad= \gamma_n\big[\|T^{F_2}_{s_n}Aw_n-Ap\|^2+\|(T^{F_2}_{s_n}-I)Aw_n\|^2\\
&\quad- \|Aw_n-Ap\|^2-2\|(T^{F_2}_{s_n}-I)Aw_n\|^2\big]\\
&\quad= \gamma_n\big[\|T^{F_2}_{s_n}Aw_n-Ap\|^2-\|Aw_n-Ap\|^2-\|(T^{F_2}_{s_n}-I)Aw_n\|^2\big]\\
&\quad\le \gamma_n\big[\|Aw_n-Ap\|^2-\|Aw_n-Ap\|^2-\|(T^{F_2}_{s_n}-I)Aw_n\|^2\big]\\
&\quad= -\gamma_n\|(T^{F_2}_{s_n}-I)Aw_n\|^2. \qquad (3.8)
\end{aligned}$$

Hence, by applying (3.8) to (3.7), and using the definition of γ_n we get

$$\begin{aligned}
\|t_n-p\|^2 &\le \|w_n-p\|^2-\gamma_n\|(T^{F_2}_{s_n}-I)Aw_n\|^2+\gamma_n^2\|A^*(T^{F_2}_{s_n}Aw_n-Aw_n)\|^2\\
&= \|w_n-p\|^2-\gamma_n(1-\tau_n)\|(T^{F_2}_{s_n}-I)Aw_n\|^2. \qquad (3.9)
\end{aligned}$$

Since $0<\tau_n<1$, then we obtain

$$\|t_n-p\|^2\le\|w_n-p\|^2. \qquad (3.10)$$

Since $p\in VI(C,G)$, then $p=P_C(I-\sigma_nG)p$. Also, since $0<\sigma_n<2\kappa$, then $I-\sigma_nG$ is nonexpansive. Hence, by applying (3.10), we get

$$\begin{aligned}
\|u_n-p\| &= \|P_C(I-\sigma_nG)t_n-P_C(I-\sigma_nG)p\|\\
&\le \|(I-\sigma_nG)t_n-(I-\sigma_nG)p\|\\
&\le \|t_n-p\|\\
&\le \|w_n-p\|. \qquad (3.11)
\end{aligned}$$

Next, by invoking Lemma 2.2 and applying (3.11) together with the definition of demicontractive mapping, we have

$$\begin{aligned}
\|y_n-p\|^2 &= \|\delta_{n,0}u_n+\sum_{i=1}^m\delta_{n,i}v_{n,i}-p\|^2\\
&= \delta_{n,0}\|u_n-p\|^2+\sum_{i=1}^m\delta_{n,i}\|v_{n,i}-p\|^2-\delta_{n,0}\sum_{i=1}^m\delta_{n,i}\|v_{n,i}-u_n\|^2\\
&\le \delta_{n,0}\|u_n-p\|^2+\sum_{i=1}^m\delta_{n,i}H(S_iu_n,S_ip)-\delta_{n,0}\sum_{i=1}^m\delta_{n,i}\|v_{n,i}-u_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \delta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{m} \delta_{n,i}(\|u_n - p\|^2 \\
&+ \lambda_i d(u_n, S_i u_n)) - \delta_{n,0} \sum_{i=1}^{m} \delta_{n,i}\|v_{n,i} - u_n\|^2 \\
&\leq \delta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{m} \delta_{n,i}(\|u_n - p\|^2 \\
&+ \lambda_i \|u_n - v_{n,i}\|^2) - \delta_{n,0} \sum_{i=1}^{m} \delta_{n,i}\|v_{n,i} - u_n\|^2 \\
&= \|u_n - p\|^2 - \sum_{i=1}^{m} \delta_{n,i}(\delta_{n,0} - \lambda_i)\|u_n - v_{n,i}\|^2 \qquad (3.12)\\
&\leq \|u_n - p\|^2. \qquad (3.13)\\
&\leq \|w_n - p\|^2. \qquad (3.14)
\end{aligned}
$$

Applying the triangle inequality, we get

$$
\begin{aligned}
\|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\
&\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\
&= \|x_n - p\| + \alpha_n \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|. \qquad (3.15)
\end{aligned}
$$

Since by Remark 3.1, $\lim_{n\to+\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, it follows that there exists a constant $M_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1$, for all $n \geq 1$. Hence, from (3.15) we obtain

$$
\|w_n - p\| \leq \|x_n - p\| + \alpha_n M_1. \qquad (3.16)
$$

By applying (3.14) and (3.16), we get

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n \xi f(x_n) + (I - \alpha_n D)y_n - p\| \\
&= \|\alpha_n(\xi f(x_n) - Dp) + (I - \alpha_n D)(y_n - p)\| \\
&\leq \alpha_n\|\xi f(x_n) - Dp\| + (1 - \alpha_n\bar{\xi})\|y_n - p\| \\
&= \alpha_n\|\xi(f(x_n) - f(p)) + (\xi f(p) - Dp)\| + (1 - \alpha_n\bar{\xi})\|y_n - p\| \\
&\leq \alpha_n\xi\rho\|x_n - p\| + \alpha_n\|\xi f(p) - Dp\| + (1 - \alpha_n\bar{\xi})(\|x_n - p\| + \alpha_n M_1) \\
&= (1 - \alpha_n(\bar{\xi} - \xi\rho))\|x_n - p\| + \alpha_n\|\xi f(p) - Dp\| + (1 - \alpha_n\bar{\xi})\alpha_n M_1 \\
&= (1 - \alpha_n(\bar{\xi} - \xi\rho))\|x_n - p\| \\
&+ \alpha_n(\bar{\xi} - \xi\rho)\Big\{\frac{\|\xi f(p) - Dp\|}{\bar{\xi} - \xi\rho} + \frac{(1 - \alpha_n\bar{\xi})}{\bar{\xi} - \xi\rho}M_1\Big\} \\
&\leq (1 - \alpha_n(\bar{\xi} - \xi\rho))\|x_n - p\| + \alpha_n(\bar{\xi} - \xi\rho)M^*,
\end{aligned}
$$

where

$$
M^* := \sup_{n\in\mathbb{N}}\Big\{\frac{\|\xi f(p) - Dp\|}{\bar{\xi} - \xi\rho} + \frac{(1 - \alpha_n\bar{\xi})}{\bar{\xi} - \xi\rho}M_1\Big\}.
$$

Setting $a_n := \|x_n - p\|$; $b_n := \alpha_n(\bar{\xi} - \xi\rho)M^*$; $c_n := 0$, and $\sigma_n := \alpha_n(\bar{\xi} - \xi\rho)$. By Lemma 2.4(1) and the assumptions on the control sequences, we have that $\{\|x_n - p\|\}$ is bounded and this implies that $\{x_n\}$ is bounded. Consequently, $\{w_n\}, \{t_n\}, \{u_n\}$ and $\{y_n\}$ are also bounded. □

Lemma 3.2. *The following inequality holds for all $p \in \Omega$ and $n \in \mathbb{N}$:*

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq \Big(1 - \frac{2\alpha_n(\bar{\xi} - \xi\rho)}{(1 - \alpha_n\xi\rho)}\Big)||x_n - p||^2 + \frac{2\alpha_n(\bar{\xi} - \xi\rho)}{(1 - \alpha_n\xi\rho)}\Big\{\frac{\alpha_n\bar{\xi}^2}{2(\bar{\xi} - \xi\rho)}M \\
+ \frac{3M_2(1 - \alpha_n\bar{\xi})^2}{2(\bar{\xi} - \xi\rho)}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \\
+ \frac{1}{(\bar{\xi} - \xi\rho)}\langle \xi f(p) - Dp, x_{n+1} - p\rangle\Big\} \\
- \frac{(1 - \alpha_n\bar{\xi})^2}{(1 - \alpha_n\xi\rho)}\Big\{\gamma_n(1 - \tau_n)||(T^{F_2}_{s_n} - I)Aw_n||^2 \\
+ \sigma_n(2\kappa - \sigma_n)\|Gt_n - Gp\|^2 + \sum_{i=1}^{m}\delta_{n,i}(\delta_{n,0} - \lambda_i)\|u_n - v_{n,i}\|^2\Big\}
\end{aligned}$$

Proof. Let $p \in \Omega$. Then by the Cauchy-Schwartz inequality and Lemma 2.1(ii), we get

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle \\
&\leq \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - p\| \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|(\theta_n\|x_n - x_{n-1}\| + 2\|x_n - p\|) \\
&\leq \|x_n - p\|^2 + 3M_2\theta_n\|x_n - x_{n-1}\| \\
&= \|x_n - p\|^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|, \qquad (3.17)
\end{aligned}$$

where $M_2 := \sup_{n\in\mathbb{N}}\{\|x_n - p\|, \theta_n\|x_n - x_{n-1}\|\} > 0$.

Now, by applying Lemma 2.1, (3.9), (3.12), and (3.17) together with the fact that $p = P_C(I - \sigma_n G)(p)$, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n\xi f(x_n) + (I - \alpha_n D)y_n - p\|^2 \\
&= \|\alpha_n(\xi f(x_n) - Dp) + (I - \alpha_n D)(y_n - p)\|^2 \\
&\leq (1 - \alpha_n\bar{\xi})^2\|y_n - p\|^2 + 2\alpha_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \qquad (3.18) \\
&\leq (1 - \alpha_n\bar{\xi})^2\Big\{\|u_n - p\|^2 - \sum_{i=1}^{m}\delta_{n,i}(\delta_{n,0} - \lambda_i)\|u_n - v_{n,i}\|^2\Big\} \\
&\quad + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\
&\leq (1 - \alpha_n\bar{\xi})^2\Big\{\|P_C(I - \sigma_n G)t_n - P_C(I - \sigma_n G)p\|^2
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{m} \delta_{n,i}(\delta_{n,0} - \lambda_i)\|u_n - v_{n,i}\|^2\Big\} \\
&+ 2\alpha_n \xi\rho\|x_n - p\|\|x_{n+1} - p\| + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\
&\leq (1 - \alpha_n\bar{\xi})^2\Big\{\|(I - \sigma_n G)t_n - (I - \sigma_n G)p\|^2 \\
&- \sum_{i=1}^{m} \delta_{n,i}(\delta_{n,0} - \lambda_i)\|u_n - v_{n,i}\|^2\Big\} \\
&\quad + \alpha_n\xi\rho\big(\|x_n - p\|^2 + \|x_{n+1} - p\|^2\big) + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\
&= (1 - \alpha_n\bar{\xi})^2\Big\{\|t_n - p - \sigma_n(Gt_n - Gp)\|^2 \\
&- \sum_{i=1}^{m} \delta_{n,i}(\delta_{n,0} - \lambda_i)\|u_n - v_{n,i}\|^2\Big\} \\
&+ \alpha_n\xi\rho\big(\|x_n - p\|^2 + \|x_{n+1} - p\|^2\big) + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\
&= (1 - \alpha_n\bar{\xi})^2\Big\{\|t_n - p\|^2 - 2\sigma_n\langle t_n - p, Gt_n - Gp\rangle + \sigma_n^2\|Gt_n - Gp\|^2 \\
&- \sum_{i=1}^{m} \delta_{n,i}(\delta_{n,0} - \lambda_i)\|u_n - v_{n,i}\|^2\Big\} + \alpha_n\xi\rho\big(\|x_n - p\|^2 + \|x_{n+1} - p\|^2\big) \\
&+ 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\
&\leq (1 - \alpha_n\bar{\xi})^2\Big\{\|t_n - p\|^2 - 2\sigma_n\kappa\|Gt_n - Gp\|^2 \\
&+ \sigma_n^2\|Gt_n - Gp\|^2 - \sum_{i=1}^{m} \delta_{n,i}(\delta_{n,0} - \lambda_i)\|u_n - v_{n,i}\|^2\Big\} \\
&+ \alpha_n\xi\rho\big(\|x_n - p\|^2 + \|x_{n+1} - p\|^2\big) + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\
&\leq (1 - \alpha_n\bar{\xi})^2\Big\{\|x_n - p\|^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&- \gamma_n(1 - \tau_n)\|(T_{S_n}^{F_2} - I)Aw_n\|^2 \\
&\quad - \sigma_n(2\kappa - \sigma_n)\|Gt_n - Gp\|^2 - \sum_{i=1}^{m} \delta_{n,i}(\delta_{n,0} - \lambda_i)\|u_n - v_{n,i}\|^2\Big\} \\
&+ \alpha_n\xi\rho\big(\|x_n - p\|^2 + \|x_{n+1} - p\|^2\big) \\
&+ 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\
&= \big((1 - \alpha_n\bar{\xi})^2 + \alpha_n\xi\rho\big)\|x_n - p\|^2 + \alpha_n\xi\rho\|x_{n+1} - p\|^2 \\
&+ 3M_2(1 - \alpha_n\bar{\xi})^2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&\quad + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\
&- (1 - \alpha_n\bar{\xi})^2\Big\{\gamma_n(1 - \tau_n)\|(T_{s_n}^{F_2} - I)Aw_n\|^2 + \sigma_n(2\kappa - \sigma_n)\|Gt_n - Gp\|^2
\end{aligned}
$$

$$+\sum_{i=1}^{m}\delta_{n,i}(\delta_{n,0}-\lambda_i)\|u_n-v_{n,i}\|^2\Big\}.$$

By bringing the term $\alpha_n\xi\rho\big\|x_{n+1}-p\|^2$ to the left hand side and dividing through by $1-\alpha_n\xi\rho$, we have

$$\begin{aligned}
||x_{n+1}-p||^2 \leq\ & \frac{(1-2\alpha_n\bar{\xi}+(\alpha_n\bar{\xi})^2+\alpha_n\xi\rho)}{(1-\alpha_n\xi\rho)}||x_n-p||^2\\
&+3M_2\frac{(1-\alpha_n\bar{\xi})^2}{(1-\alpha_n\xi\rho)}\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\\
&+\frac{2\alpha_n}{(1-\alpha_n\xi\rho)}\langle\xi f(p)-Dp,x_{n+1}-p\rangle\\
&-\frac{(1-\alpha_n\bar{\xi})^2}{(1-\alpha_n\xi\rho)}\Big\{\gamma_n(1-\tau_n)||(T_{s_n}^{F_2}-I)Aw_n||^2\\
&+\sigma_n(2\kappa-\sigma_n)\|Gt_n-Gp\|^2+\sum_{i=1}^{m}\delta_{n,i}(\delta_{n,0}-\lambda_i)\|u_n-v_{n,i}\|^2\Big\}\\
=\ & \frac{(1-2\alpha_n\bar{\xi}+\alpha_n\xi\rho)}{(1-\alpha_n\xi\rho)}||x_n-p||^2+\frac{(\alpha_n\bar{\xi})^2}{(1-\alpha_n\xi\rho)}||x_n-p||^2\\
&+3M_2\frac{(1-\alpha_n\bar{\xi})^2}{(1-\alpha_n\xi\rho)}\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\\
&\quad+\frac{2\alpha_n}{(1-\alpha_n\xi\rho)}\langle\xi f(p)-Dp,x_{n+1}-p\rangle\\
&-\frac{(1-\alpha_n\bar{\xi})^2}{(1-\alpha_n\xi\rho)}\Big\{\gamma_n(1-\tau_n)||(T_{s_n}^{F_2}-I)Aw_n||^2\\
&+\sigma_n(2\kappa-\sigma_n)\|Gt_n-Gp\|^2+\sum_{i=1}^{m}\delta_{n,i}(\delta_{n,0}-\lambda_i)\|u_n-v_{n,i}\|^2\Big\}\\
\leq\ & \Big(1-\frac{2\alpha_n(\bar{\xi}-\xi\rho)}{(1-\alpha_n\xi\rho)}\Big)||x_n-p||^2+\frac{2\alpha_n(\bar{\xi}-\xi\rho)}{(1-\alpha_n\xi\rho)}\Big\{\frac{\alpha_n\bar{\xi}^2}{2(\bar{\xi}-\xi\rho)}M\\
&+\frac{3M_2(1-\alpha_n\bar{\xi})^2}{2(\bar{\xi}-\xi\rho)}\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\\
&\quad+\frac{1}{(\bar{\xi}-\xi\rho)}\langle\xi f(p)-Dp,x_{n+1}-p\rangle\Big\}\\
&-\frac{(1-\alpha_n\bar{\xi})^2}{(1-\alpha_n\xi\rho)}\Big\{\gamma_n(1-\tau_n)||(T_{s_n}^{F_2}-I)Aw_n||^2\\
&+\sigma_n(2\kappa-\sigma_n)\|Gt_n-Gp\|^2+\sum_{i=1}^{m}\delta_{n,i}(\delta_{n,0}-\lambda_i)\|u_n-v_{n,i}\|^2\Big\},
\end{aligned}$$

where $M:=\sup\{||x_n-p||^2:n\in\mathbb{N}\}$. This completes the proof. □

Lemma 3.3. *The following inequality holds for all $p \in \Omega$ and $n \in \mathbb{N}$:*

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq\ & (1-\alpha_n\bar{\xi})^2\|x_n-p\|^2 + 3M_2(1-\alpha_n\bar{\xi})^2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| \\
& + 2\alpha_n\langle \xi f(x_n) - Dp, x_{n+1}-p\rangle \\
& + 2M_3(1-\alpha_n\bar{\xi})^2\|Gt_n - Gp\| + 2M_4(1-\alpha_n\bar{\xi})^2\|A^*(T^{F_2}_{S_n}-I)Aw_n\| \\
& - (1-\alpha_n\bar{\xi})^2\Big\{\|t_n-w_n\|^2 + \|u_n-t_n\|^2\Big\},
\end{aligned}
$$

where $M_3 := \sup_{n\in\mathbb{N}}\{\sigma_n\|u_n-t_n\|\}$, $M_4 := \sup_{n\in\mathbb{N}}\{\gamma_n\|t_n-w_n\|\}$.

Proof. Since P_C is firmly nonexpansive and $(I-\sigma_n G)$ is nonexpansive, then by applying Lemma 2.1, we get

$$
\begin{aligned}
\|u_n-p\|^2 &= \|P_C(1-\sigma_nG)t_n - P_C(1-\sigma_nG)p\|^2 \\
&\leq \langle u_n-p, (I-\sigma_nG)t_n - (I-\sigma_nG)p\rangle \\
&= \frac{1}{2}\Big\{\|u_n-p\|^2 + \|(I-\sigma_nG)t_n - (I-\sigma_nG)p\|^2 \\
&- \|u_n-t_n+\sigma_n(Gt_n-Gp)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|u_n-p\|^2 + \|t_n-p\|^2 - \|u_n-t_n+\sigma_n(Gt_n-Gp)\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|u_n-p\|^2 + \|t_n-p\|^2 - \|u_n-t_n\|^2 - \sigma_n^2\|Gt_n-Gp\|^2 \\
&+ 2\sigma_n\langle t_n-u_n, Gt_n-Gp\rangle\Big\} \\
&\leq \frac{1}{2}\Big\{\|u_n-p\|^2 + \|t_n-p\|^2 - \|u_n-t_n\|^2 - \sigma_n^2\|Gt_n-Gp\|^2 \\
&+ 2\sigma_n\|t_n-u_n\|\|Gt_n-Gp\|\rangle\Big\}.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\|u_n-p\|^2 &\leq \|t_n-p\|^2 - \|u_n-t_n\|^2 - \sigma_n^2\|Gt_n-Gp\|^2 + 2\sigma_n\|t_n-u_n\|\|Gt_n-Gp\|\rangle \\
&\leq \|t_n-p\|^2 - \|u_n-t_n\|^2 + 2M_3\|Gt_n-Gp\|,
\end{aligned}
\tag{3.19}
$$

where $M_3 := \sup\limits_{n\in\mathbb{N}}\{\sigma_n\|t_n-u_n\|\}$.

By applying Lemma 2.1 and the firmly nonexpansivity of $T^{F_1}_{r_n}$, we have

$$
\begin{aligned}
||t_n-p||^2 &= ||T^{F_1}_{r_n}(w_n+\gamma_nA^*(T^{F_2}_{s_n}-I)Aw_n) - p||^2 \\
&\leq \langle t_n-p, w_n+\gamma_nA^*(T^{F_2}_{s_n}-I)Aw_n - p\rangle \\
&= \frac{1}{2}\big\{||t_n-p||^2 + ||w_n+\gamma_nA^*(T^{F_2}_{s_n}-I)Aw_n-p||^2 \\
&- ||(t_n-p)-(w_n+\gamma_nA^*(T^{F_2}_{s_n}-I)Aw_n-p)||^2\big\} \\
&\leq \frac{1}{2}\big\{||t_n-p||^2 + ||w_n-p||^2 - ||t_n-w_n-\gamma_nA^*(T^{F_2}_{s_n}-I)Aw_n||^2\big\}
\end{aligned}
$$

$$= \frac{1}{2}\{||t_n - p||^2 + ||w_n - p||^2 - \big(||t_n - w_n||^2 + \gamma_n^2||A^*(T_{s_n}^{F_2} - I)Aw_n||^2$$
$$- 2\gamma_n\langle t_n - w_n, A^*(T_{s_n}^{F_2} - I)Aw_n\rangle\big)\}$$
$$\leq \frac{1}{2}\{||t_n - p||^2 + ||w_n - p||^2 - ||t_n - w_n||^2 - \gamma_n^2||A^*(T_{s_n}^{F_2} - I)Aw_n||^2$$
$$+ 2\gamma_n||t_n - w_n||||A^*(T_{s_n}^{F_2} - I)Aw_n||\}$$
$$\leq \frac{1}{2}\{||t_n - p||^2 + ||w_n - p||^2 - ||t_n - w_n||^2$$
$$+ 2\gamma_n||t_n - w_n||||A^*(T_{s_n}^{F_2} - I)Aw_n||\},$$

From this we obtain

$$\begin{aligned} ||t_n - p||^2 &\leq ||w_n - p||^2 - ||t_n - w_n||^2 + 2\gamma_n||t_n - w_n||||A^*(T_{s_n}^{F_2} - I)Aw_n|| \\ &\leq \|w_n - p\|^2 - \|t_n - w_n\|^2 + 2M_4\|A^*(T_{s_n}^{F_2} - I)Aw_n\|, \end{aligned} \tag{3.20}$$

where $M_4 := \sup_{n\in\mathbb{N}}\{\gamma_n\|t_n - w_n\|\}$.

Substituting (3.20) in (3.19) and applying (3.17), we get

$$\begin{aligned} \|u_n - p\|^2 \leq \|x_n - p\|^2 &+ 3M_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &- \|t_n - w_n\|^2 + 2M_4\|A^*(T_{s_n}^{F_2} - I)Aw_n\| \\ &- \|u_n - t_n\|^2 + 2M_3\|Gt_n - Gp\|. \end{aligned}$$

Now, by applying (3.13) and (3.21) in (3.18), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 \leq (1 - \alpha_n\bar{\xi})^2\Big\{&\|x_n - p\|^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &- \|t_n - w_n\|^2 + 2M_4\|A^*(T_{S_n}^{F_2} - I)Aw_n\| \\ &- \|u_n - t_n\|^2 + 2M_3\|Gt_n - Gp\|\Big\} + 2\alpha_n\langle\xi f(x_n) - Dp, x_{n+1} - p\rangle \\ &= (1 - \alpha_n\bar{\xi})^2\|x_n - p\|^2 + 3M_2(1 - \alpha_n\bar{\xi})^2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &+ 2\alpha_n\langle\xi f(x_n) - Dp, x_{n+1} - p\rangle \\ &\quad + 2M_3(1 - \alpha_n\bar{\xi})^2\|Gt_n - Gp\| + 2M_4(1 - \alpha_n\bar{\xi})^2\|A^*(T_{S_n}^{F_2} - I)Aw_n\| \\ &\quad - (1 - \alpha_n\bar{\xi})^2\Big\{\|t_n - w_n\|^2 + \|u_n - t_n\|^2\Big\}, \end{aligned}$$

which is the desired inequality. □

Now, we prove the strong convergence theorem for the proposed algorithm.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces, and $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex subsets. Let the linear operator $A : H_1 \to H_2$ with adjoint A^* be bounded, and suppose that $S_i : C \to CB(C)$ is a finite family of multivalued demicontractive mappings with constant λ_i such that each $I - S_i$ is demiclosed at zero, $S_i(p) = \{p\}$ for all $p \in \cap_{i=1}^m Fix(S_i)$, and $\lambda = \max\{\lambda_i\}$. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.1 and F_2 is upper* *semicontinuous in the first argument. Suppose that the sequence $\{x_n\}$ defined by the iterative scheme (3.1) satisfies the assumptions (C1)-(C5), then the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \Omega$.*

Proof. Let $\hat{x} = P_\Omega(I - D + \xi f)(\hat{x})$. Then, it follows from Lemma 3.3 that

$$\begin{aligned}\|x_{n+1} - \hat{x}\|^2 \le& \Big(1 - \frac{2\alpha_n(\bar{\xi} - \xi\rho)}{(1 - \alpha_n\xi\rho)}\Big)||x_n - \hat{x}||^2 \\ &+ \frac{2\alpha_n(\bar{\xi} - \xi\rho)}{(1 - \alpha_n\xi\rho)}\Big\{\frac{\alpha_n\bar{\xi}^2}{2(\bar{\xi} - \xi\rho)}M + \frac{3M_2(1 - \alpha_n\bar{\xi})^2}{2(\bar{\xi} - \xi\rho)}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \\ &+ \frac{1}{(\bar{\xi} - \xi\rho)}\langle \xi f(\hat{x}) - D\hat{x}, x_{n+1} - \hat{x}\rangle\Big\}. \end{aligned} \tag{3.21}$$

Next, we claim $\lim\limits_{n\to+\infty} ||x_n - \hat{x}|| = 0$. In view of Lemma 2.3, it suffices to show that $\limsup\limits_{k\to+\infty}\langle \xi f(\hat{x}) - D\hat{x}, x_{n_k+1} - \hat{x}\rangle \le 0$ for every subsequence $\{||x_{n_k} - \hat{x}||\}$ of $\{||x_n - \hat{x}||\}$ satisfying

$$\liminf_{k\to+\infty}(||x_{n_k+1} - \hat{x}|| - ||x_{n_k} - \hat{x}||) \ge 0.$$

Suppose that $\{||x_{n_k} - \hat{x}||\}$ is a subsequence of $\{||x_n - \hat{x}||\}$ such that

$$\liminf_{k\to+\infty}(||x_{n_k+1} - \hat{x}|| - ||x_{n_k} - \hat{x}||) \ge 0. \tag{3.22}$$

From Lemma 3.2, we obtain

$$\begin{aligned}&\frac{(1 - \alpha_{n_k}\bar{\xi})^2}{(1 - \alpha_{n_k}\xi\rho)}\sigma_{n_k}(2\kappa - \sigma_{n_k})\|Gt_{n_k} - Gp\|^2 \\ &\quad\le \Big(1 - \frac{2\alpha_{n_k}(\bar{\xi} - \xi\rho)}{(1 - \alpha_{n_k}\xi\rho)}\Big)||x_{n_k} - p||^2 - \|x_{n_k+1} - p\|^2 \\ &\quad+ \frac{2\alpha_{n_k}(\bar{\xi} - \xi\rho)}{(1 - \alpha_{n_k}\xi\rho)}\Big\{\frac{\alpha_{n_k}\bar{\xi}^2}{2(\bar{\xi} - \xi\rho)}M \\ &\quad+ \frac{3M_2(1 - \alpha_{n_k}\bar{\xi})^2}{2(\bar{\xi} - \xi\rho)}\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}|| \\ &\quad+ \frac{1}{(\bar{\xi} - \xi\rho)}\langle \xi f(p) - Dp, x_{n_k+1} - p\rangle\Big\}\end{aligned}$$

Applying the fact that $\lim\limits_{k\to+\infty} \alpha_{n_k} = 0$ together with (3.22), we get

$$\frac{(1 - \alpha_{n_k}\bar{\xi})^2}{(1 - \alpha_{n_k}\xi\rho)}\sigma_{n_k}(2\kappa - \sigma_{n_k})\|Gt_{n_k} - Gp\|^2 \to 0,\ k \to +\infty.$$

Consequently,

$$\|Gt_{n_k} - Gp\| \to 0,\ k \to +\infty. \tag{3.23}$$

By following similar argument, from 3.2 we obtain

$$\|u_{n_k} - v_{n_k,i}\| \to 0,\ k \to +\infty, \quad \forall i = 1, 2, \dots, m, \tag{3.24}$$

and

$$\gamma_n(1 - \tau_{n_k})||(T^{F_2}_{s_{n_k}} - I)Aw_{n_k}||^2 \to 0,\ k \to +\infty.$$

From the definition of γ_n, we have

$$\tau_{n_k}(1-\tau_{n_k})\frac{||(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||^4}{||A^*(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||^2}\to 0, \quad k\to+\infty.$$

By the condition on τ_n, it follows that

$$\frac{||(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||^2}{||A^*(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||}\to 0, \quad k\to+\infty.$$

Since $||A^*(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||$ is bounded, then it follows that

$$||(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||\to 0, \quad k\to+\infty. \tag{3.25}$$

From this, we obtain

$$||A^*(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||\leq ||A^*||||(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||=||A||||(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||\to 0, \tag{3.26}$$

$k\to+\infty$. Similarly, from Lemma 3.3 we have

$$\begin{aligned}(1-\alpha_{n_k}\bar{\xi})^2\|t_{n_k}-w_{n_k}\|^2 \leq{}& (1-\alpha_{n_k}\bar{\xi})^2\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\\ &+3M_2(1-\alpha_{n_k}\bar{\xi})^2\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|\\ &+2\alpha_{n_k}\langle \xi f(x_{n_k})-Dp, x_{n_k+1}-p\rangle\\ &+2M_3(1-\alpha_{n_k}\bar{\xi})^2\|Gt_{n_k}-Gp\|\\ &+2M_4(1-\alpha_{n_k}\bar{\xi})^2\|A^*(T^{F_2}_{S_{n_k}}-I)Aw_{n_k}\|.\end{aligned}$$

By applying (3.22), (3.23), (3.26) together with Remark 3.1 we get

$$\|t_{n_k}-w_{n_k}\|\to 0,\ k\to+\infty. \tag{3.27}$$

Following similar argument, from Lemma 3.3 we obtain

$$\|u_{n_k}-t_{n_k}\|\to 0,\ k\to+\infty. \tag{3.28}$$

We now apply (3.24)-(3.28) and Remark (3.1) to obtain the following as $k\to+\infty$:

$$\|w_{n_k}-x_{n_k}\|=\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|\to 0; \tag{3.29}$$

$$\|y_{n_k}-u_{n_k}\|\leq \delta_{n_k,0}\|u_{n_k}-u_{n_k}\|+\sum_{i=1}^{m}\delta_{n_k,i}\|v_{n_k,i}-u_{n_k}\|\to 0. \tag{3.30}$$

Applying (3.24)-(3.30) and the triangular inequality further yields the following as $k\to+\infty$:

$$\|t_{n_k}-x_{n_k}\|\to 0, \|u_{n_k}-x_{n_k}\|\to 0, \|u_{n_k}-w_{n_k}\|\to 0, \|y_{n_k}-w_{n_k}\|\to 0, \tag{3.31}$$

and

$$\|y_{n_k}-x_{n_k}\|\to 0, \|v_{n_k,i}-x_{n_k}\|\to 0 \quad \forall i=1,2,\dots,m. \tag{3.32}$$

Note that

$$\|x_{n_k+1}-x_{n_k}\|\leq \alpha_{n_k}\|\xi f(x_{n_k})-Dx_{n_k}\|+(1-\alpha_{n_k}\bar{\xi})\|y_{n_k}-x_{n_k}\|. \tag{3.33}$$

Thus, from (3.31) and the fact that $\lim_{k\to+\infty} \alpha_{n_k} = 0$, we have

$$\|x_{n_k+1} - x_{n_k}\| \to 0,\ k \to +\infty.$$

To complete the proof, we need to show that $w_\omega(x_n) \subset \Omega$. We first show that $w_\omega(x_n) \subset \bigcap_{i=1}^m Fix(S_i)$. Let $x^* \in w_\omega(x_n)$, then by (3.31) we have $u_{n_k} \rightharpoonup x^*$. Since $I - S_i$ is demiclosed for all $i = 1, \cdots, m$ then it follows from (3.24) that $x^* \in Fix(S_i)$ for all $i = 1, \cdots, m$. This implies that $x^* \in \bigcap_{i=1}^m Fix(S_i)$. Next, we show that $w_\omega(x_n) \subset VI(C, G)$. It suffices to show that $x^* = P_C(I - \sigma G)x^*$. In view of the (3.28) and the definition of u_n, we have

$$\|t_{n_k} - P_C(I - \sigma_{n_k} G)t_{n_k}\| \to 0,\ k \to +\infty. \tag{3.34}$$

Consequently, we get

$$\begin{aligned}\|t_{n_k} - P_C(I - \sigma G)t_{n_k}\| &\le \|t_{n_k} - P_C(I - \sigma_{n_k} G)t_{n_k}\| \\ &\quad + \|P_C(I - \sigma_{n_k} G)t_{n_k} - P_C(I - \sigma G)t_{n_k}\| \\ &\le \|t_{n_k} - P_C(I - \sigma_{n_k} G)t_{n_k}\| \\ &\quad + \|(I - \sigma_{n_k} G)t_{n_k} - (I - \sigma G)t_{n_k}\| \\ &= \|t_{n_k} - P_C(I - \sigma_{n_k} G)t_{n_k}\| + |\sigma - \sigma_{n_k}|\|Gt_{n_k}\|.\end{aligned}$$

By (3.34) and the fact that $\lim_{k\to+\infty} \sigma_{n_k} = \sigma$, we have $\|t_{n_k} - P_C(I - \sigma G)t_{n_k}\| \to 0$. Since $\sigma > 0$ and $\{\sigma_{n_k}\} \subset (0, 2\kappa)$, then $\sigma \in (0, 2\kappa]$. Hence, $I - \sigma G$ is nonexpansive and consequently $P_C(I - \sigma G)$ is nonexpansive. From (3.31), we have that $t_{n_k} \rightharpoonup x^*$ as $k \to +\infty$. By the demiclosedness principle for nonexpansive mappings, we have $x^* = P_C(I - \sigma G)x^*$. Hence, we have $w_\omega(x_n) \subset VI(C, G)$.

Finally, we show that $w_\omega(x_n) \subset SEP(F_1, F_2)$. First, we show that $w_\omega(x_n) \subset EP(F_1)$. Since $t_{n_k} = T^{F_1}_{r_{n_k}}(w_{n_k} + \gamma_{n_k} A^*(T^{F_2}_{s_{n_k}} - I)Aw_{n_k})$, we have

$$F_1(t_{n_k}, y) + \frac{1}{r_{n_k}}\langle y - t_{n_k}, t_{n_k} - w_{n_k} - \gamma_{n_k} A^*(T^{F_2}_{s_{n_k}} - I)Aw_{n_k}\rangle \ge 0, \quad \forall\ y \in C,$$

which implies that

$$F_1(t_{n_k}, y) + \frac{1}{r_{n_k}}\langle y - t_{n_k}, t_{n_k} - w_{n_k}\rangle - \frac{1}{r_{n_k}}\langle y - t_{n_k}, \gamma_{n_k} A^*(T^{F_2}_{s_{n_k}} - I)Aw_{n_k}\rangle \ge 0, \quad \forall\ y \in C.$$

It follows from the monotonicity of F_1 that

$$\frac{1}{r_{n_k}}\langle y - t_{n_k}, t_{n_k} - w_{n_k}\rangle - \frac{1}{r_{n_k}}\langle y - t_{n_k}, \gamma_{n_k} A^*(T^{F_2}_{s_{n_k}} - I)Aw_{n_k}\rangle \ge F_1(y, t_{n_k}), \quad \forall\ y \in C.$$

Since $t_{n_k} \rightharpoonup x^*$, then it follows from (3.26), (3.27), $\liminf_{k\to+\infty} r_{n_k} > 0$, and condition (B4) that

$$F_1(y, x^*) \le 0, \quad \forall\ y \in C. \tag{3.35}$$

Now, for $y \in C$, let $y_t := ty + (1 - t)x^*$ for all $t \in (0, 1]$. This implies that $y_t \in C$, and it follows from (3.35) that $F_1(y_t, x^*) \le 0$. Thus, by applying Assumptions (B1)-(B4), we have

$$\begin{aligned}0 = F_1(y_t, y_t) &\le tF_1(y_t, y) + (1 - t)F_1(y_t, x^*) \\ &\le tF_1(y_t, y).\end{aligned}$$

Hence, we have

$$F_1(y_t, y) \geq 0, \quad \forall\, y \in C. \tag{3.36}$$

Letting $t \to 0$, by condition (B3), we have

$$F_1(x^*, y) \geq 0, \quad \forall\, y \in C. \tag{3.37}$$

This implies that $x^* \in EP(F_1)$.

Finally, we show that $Ax^* \in EP(F_2)$. Since A is a bounded linear operator and $w_\omega(x_n) = w_\omega(w_n)$ by (3.29), then we have $Aw_{n_k} \rightharpoonup Ax^*$. It then follows from (3.25) that

$$T^{F_2}_{s_{n_k}} Aw_{n_k} \rightharpoonup Ax^*, \quad k \to +\infty. \tag{3.38}$$

By the definition of $T^{F_2}_{s_{n_k}} Aw_{n_k}$, we have

$$F_2(T^{F_2}_{s_{n_k}} Aw_{n_k}, y) + \frac{1}{s_{n_k}} \langle y - T^{F_2}_{s_{n_k}} Aw_{n_k}, T^{F_2}_{s_{n_k}} Aw_{n_k} - Aw_{n_k}\rangle \geq 0, \quad \forall\, y \in Q.$$

Since F_2 is upper semi-continuous in the first argument, it follows from (3.25), (3.38) and $\liminf\limits_{k\to+\infty} s_{n_k} > 0$ that

$$F_2(Ax^*, y) \geq 0, \quad \forall\, y \in Q. \tag{3.39}$$

This shows that $Ax^* \in EP(F_2)$. Hence, $w_\omega(x_n) \subset \Omega$ as required.
Furthermore, from (3.31) we have that $w_\omega\{t_{n_k}\} = w_\omega\{x_{n_k}\}$. By the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup x^\dagger$ and

$$\begin{aligned}\lim_{j\to+\infty}\langle \xi f(\hat{x}) - D\hat{x}, x_{n_{k_j}} - \hat{x}\rangle &= \limsup_{k\to+\infty}\langle \xi f(\hat{x}) - D\hat{x}, x_{n_k} - \hat{x}\rangle \\ &= \limsup_{k\to+\infty}\langle \xi f(\hat{x}) - D\hat{x}, t_{n_k} - \hat{x}\rangle.\end{aligned} \tag{3.40}$$

Since $\hat{x} = P_\Omega(I - D + \xi f)(\hat{x})$, it follows from (3.40) that

$$\limsup_{k\to+\infty}\langle \xi f(\hat{x}) - D\hat{x}, x_{n_k} - \hat{x}\rangle = \lim_{j\to+\infty}\langle \xi f(\hat{x}) - D\hat{x}, x_{n_{k_j}} - \hat{x}\rangle = \langle \xi f(\hat{x}) - D\hat{x}, x^\dagger - \hat{x}\rangle \leq 0. \tag{3.41}$$

Hence, by (3.33) and (3.41), we have

$$\begin{aligned}\limsup_{k\to+\infty}\langle \xi f(\hat{x}) - D\hat{x}, x_{n_k+1} - \hat{x}\rangle &= \limsup_{k\to+\infty}\langle \xi f(\hat{x}) - D\hat{x}, x_{n_k+1} - x_{n_k}\rangle \\ &\quad + \limsup_{k\to+\infty}\langle \xi f(\hat{x}) - D\hat{x}, x_{n_k} - \hat{x}\rangle \\ &= \langle \xi f(\hat{x}) - D\hat{x}, x^\dagger - \hat{x}\rangle \leq 0.\end{aligned} \tag{3.42}$$

Applying Lemma 2.3 to (3.21), and using (3.42) together with Remark 3.1 and the condition on α_n, we deduce that $\lim\limits_{n\to+\infty} ||x_n - \hat{x}|| = 0$ as required. This completes the proof. □

By the properties of the best approximation operator, we obtain the following result.

Corollary 3.1. *Let H_1 and H_2 be two real Hilbert spaces, and $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex subsets. Let the linear operator $A : H_1 \to H_2$ with adjoint A^* be bounded. Let $S_i : C \to P(C)$ be a finite family of multivalued mappings such that P_{S_i} is λ_i-demicontractive with $\lambda = \max\{\lambda_i\}$ and suppose that $I - P_{S_i}$ is demiclosed at zero for each $i = 1, 2, \ldots, m$. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.1 and F_2 is upper semicontinuous in the first argument. Let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 3.2.

Step 0. Select initial data $x_0, x_1 \in H_1$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \le \theta_n \le \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \ \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.43}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}). \tag{3.44}$$

Step 3. Compute

$$t_n = T_{r_n}^{F_1}(w_n + \gamma_n A^*(T_{s_n}^{F_2} - I)Aw_n), \tag{3.45}$$

where

$$\gamma_n := \begin{cases} \tau_n \frac{\|(T_{s_n}^{F_2} - I)Aw_n\|^2}{\|A^*(T_{s_n}^{F_2} - I)Aw_n\|^2}, & \text{if } Aw_n \neq T_{s_n}^{F_2} Aw_n, \\ \gamma, & \text{otherwise } (\gamma \text{ being any nonnegative real number}). \end{cases} \tag{3.46}$$

Step 4. Compute

$$\begin{cases} u_n = P_C(I - \sigma_n G)t_n \\ y_n = \delta_{n,0} u_n + \sum_{i=1}^m \delta_{n,i} v_{n,i}, \quad v_{n,i} \in P_{S_i}(u_n). \\ x_{n+1} = \alpha_n \xi f(x_n) + (I - \alpha_n D)y_n. \end{cases} \tag{3.47}$$

Set $n := n + 1$ and return to **Step 1**.

Suppose that conditions (C1)-(C5) are satisfied. Then, the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to a point $\hat{x} \in \Gamma$, where

$$\Gamma := SEP(F_1, F_2) \cap VI(C, G) \cap \bigcap_{i=1}^m Fix(S_i) \neq \emptyset.$$

Proof. Since P_{S_i} satisfies the common endpoint condition, $F(S_i) = F(P_{S_i})$ and $I - S_i$ is demiclosed at zero for each $i = 1, 2, \ldots, m$, hence the result is obtained immediately from Theorem 3.1. $\square$

4. Application and numerical examples

4.1. **Application.** In this subsection, we apply our result to study split minimization problem.
Let H_1 and H_2 be two real Hilbert spaces, $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex subsets, and $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. Let $f : C \to \mathbb{R}, g : Q \to \mathbb{R}$ be two convex and lower semi-continuous mappings. The *Split Minimization Problem* (SMP) is formulated as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*) \leq f(x), \quad \forall x \in C, \tag{4.1}$$

$$\text{and } y^* = Ax^* \text{ such that } g(y^*) \leq g(y), \quad \forall y \in Q. \tag{4.2}$$

We denote the solution set of the SMP (4.1)-(4.2) by $\mathcal{F}$ and assume that $\mathcal{F} \neq \emptyset$. Let $F_1(x, y) := f(y) - f(x)$ for all $x, y \in C$ and $F_2(u, v) := g(v) - g(u)$ for all $u, v \in Q$. Then, $F_1(x, y)$ and $F_2(x, y)$ satisfy conditions (B1)-(B4). Therefore, Theorem 3.1 provides a strong convergence theorem for approximating the common solution of split minimization problem, variational inequality problem and common fixed point problem for a finite family of demicontractive multivalued mappings in Hilbert spaces.

4.2. **Numerical Examples.** In this subsection, we present some numerical experiments to illustrate the performance of our proposed methods Algorithm 3.1 in comparison with Algorithm 1.1, Algorithm 1.2 and Algorithm 1.3. All numerical computations were carried out using Matlab version R2019(b).

Example 4.1. Let $H_1 = \mathbb{R} = H_2, C = [-3, 0]$, and $Q = (-\infty, 0]$. Define $G : C \to H_1$ and $A : H_1 \to H_2$ by $Gx = 2x$ and $Ax = \frac{x}{2}$ for each $x \in H_1$. It then follows that $A^*y = \frac{y}{2}$ for each $y \in H_2$. For $i = 1, 2, \dots, 5$, define multivalued mappings $S_i : C \to CB(C)$ by

$$S_i(x) = \begin{cases} \left[-\frac{i|x|}{i|x|+1}, 0\right], & x \in [-3, -2); \\ \{0\}, & x \in [-2, 0]; \end{cases}$$

It can easily be verified that S_i is a hybrid multivalued mapping with $Fix(S_i) = \{0\}$ for each i. Hence, S_i is 0-demicontractive. Moreover, for each $x, y \in C$, define the bifunction $F_1 : C \times C \to \mathbb{R}$ by $F_1(x, y) = xy + y - x - x^2$, and for each $u, v \in Q$, define the bifunction $F_2 : Q \times Q \to \mathbb{R}$ by $F_2(u, v) = uv + 10v - 10u - u^2$. It can easily be verified that F_1, F_2 both satisfy Assumption 2.1. After simple calculation and using Lemma 2.5, we obtain

$$T_r^{(F_1,\phi_1)}(u) = \frac{u - r}{1 + r}, \quad \forall\ u \in C,$$

and

$$T_s^{(F_2,\phi_2)}(v) = \frac{v - 10s}{1 + s}, \quad \forall\ v \in Q.$$

Choose

$$f(x) = \frac{x}{2},\ D(x) = \frac{x}{3},\ \xi = \frac{1}{2},\ \alpha_n = \frac{1}{n+1},\ \epsilon_n = \frac{1}{(n+1)^2},$$

$$\delta_{n,0} = \frac{n}{2n+1}, \delta_{n,i} = \frac{n+1}{5(2n+1)},\ \sigma_n = \frac{n}{2n+1},\ \tau_n = 0.7,\ \theta = 0.8,$$

$$r_n = \frac{n}{n+1},\ s_n = \frac{n+1}{2n+1}.$$

It can easily be checked that all the conditions of Theorem 3.1 are satisfied. We take $Tx = \frac{x}{2}, \beta_n = \mu_n = \frac{n}{2(n+1)}$ in Algorithm 1.1 and $T_i x = \frac{x}{2i}$ in Algorithm 1.2 for $i = 1, 2, \dots, 5$.

We choose four different initial values as follows with $\gamma = 0.01$ in each case:

Case I: $x_0 = 870$ and $x_1 = -317$;

Case II: $x_0 = -915$ and $x_1 = -317$;

Case III: $x_0 = 977$ and $x_1 = -858$;

Case IV: $x_0 = 678$ and $x_1 = -446$.

We compare the performance of our Algorithm 3.1 with Algorithm 1.1, Algorithm 1.2 and Algorithm 1.3. The stopping criterion used for our computation is $|x_{n+1} - x_n| < 10^{-2}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 1 and Table 1.

TABLE 1. **Numerical results for Example 4.1**

		Alg. 1.1	Alg. 1.2	Alg. 1.3	Alg. 3.1
Case I	CPU time (sec)	0.0469	0.0079	0.0096	0.0390
	No of Iter.	18	9	9	5
Case II	CPU time (sec)	0.0102	0.0069	0.0079	0.0146
	No. of Iter.	16	9	9	5
Case III	CPU time (sec)	0.0102	0.0051	0.0068	0.0105
	No of Iter.	18	9	9	5
Case IV	CPU time (sec)	0.0101	0.0050	0.0085	0.0126
	No of Iter.	17	9	9	5

Example 4.2. We consider the second example in infinite dimensional Hilbert spaces. Let $H_1 = H_2 = L^2([0,1])$ with the inner product defined by $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$ and induced norm $\|x\| = (\int_0^1 |x(t)|^2)^{\frac{1}{2}}$ for all $x, y \in L_2([0,1])$. Let

$$C = \Big\{x \in L_2([0,1]) : \int_0^1 x(t)dt \le 1\Big\} \quad \text{and} \quad Q = \Big\{u \in L_2([0,1]) : \int_0^1 tu(t)dt \le 2\Big\}.$$

Then, C and Q are nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Define $G : C \to L_2([0,1])$ and $A : L_2([0,1]) \to L_2([0,1])$ by $(Gx)(t) = \frac{1}{2}x(t)$ and $(Ax)(t) = \frac{1}{2}x(t)$. Then, G is an inverse strongly monotone mapping and A is a bounded linear operator. We define $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ by

$$F_1(x, y) = \langle P_1(x), y - x\rangle, \quad \text{where } P_1(x(t)) = \frac{x(t)}{2}$$

and

$$F_2(x, y) = \langle P_2(x), y - x\rangle, \quad \text{where } P_2(x(t)) = \frac{x(t)}{3}.$$

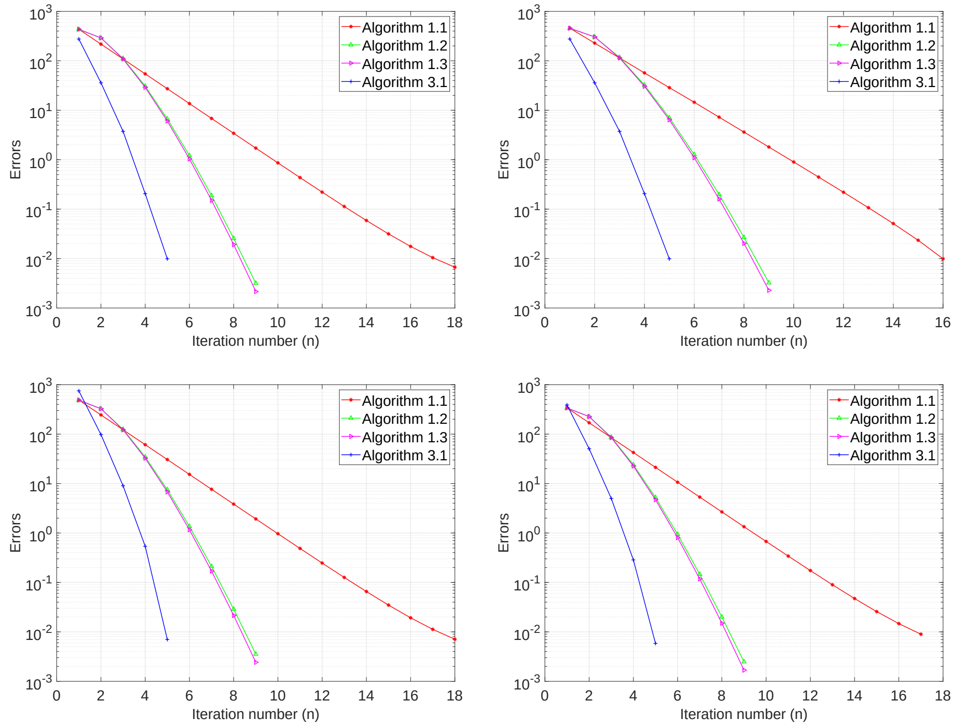

FIGURE 1. Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

Then, F_1 and F_2 satisfy Assumption 2.1. After simple calculation and applying Lemma 2.5, we obtain

$$T_r^{F_1}(u) = \frac{2u}{r+2}, \quad \forall\ u \in C,$$

and

$$T_s^{F_2}(v) = \frac{3v}{s+3}, \quad \forall\ v \in Q.$$

We define $S_i : C \to CB(C)$ by

$$S_i(x) = \Big\{x \in C : \int_0^1 x(t)dt \le \frac{1}{2i}\Big\}.$$

Then, S_i is demicontractive for each i.
Choose

$$(fx)(t) = \frac{x(t)}{3},\ (Dx)(t) = x(t),\ \xi = 1, \alpha_n = \frac{1}{n+1},\ \epsilon_n = \frac{1}{(n+1)^2},$$

$$\delta_{n,0} = \frac{n}{3n+1},\ \delta_{n,i} = \frac{n+1}{5(3n+1)},\ \sigma_n = \frac{n}{2n+1},\ \tau_n = 0.8,\ \theta = 0.9,$$

$$r_n = \frac{n}{2n+1}, \ s_n = \frac{n+1}{3n+1}.$$

It can easily be checked that all the conditions of Theorem 3.1 are satisfied. In Algorithm 1.1, we take $\beta_n = \mu_n = \frac{n}{2(n+1)}$ and we define $T : L^2([0,1]) \to L^2([0,1])$ by

$$(Tx)(t) = \int_0^1 tx(s)ds \quad \text{for all } t \in [0,1].$$

while in Algorithm 1.2, we define the mappings $T_i : L^2([0,1]) \to L^2([0,1])$ for each $i = 1, 2, \dots, 5$, by

$$(T_ix)(t) = \int_0^1 t^i x(s)ds \quad \text{for all } t \in [0,1].$$

It can easily be verified that T and T_i are nonexpansive and thus asymptotically nonexpansive.

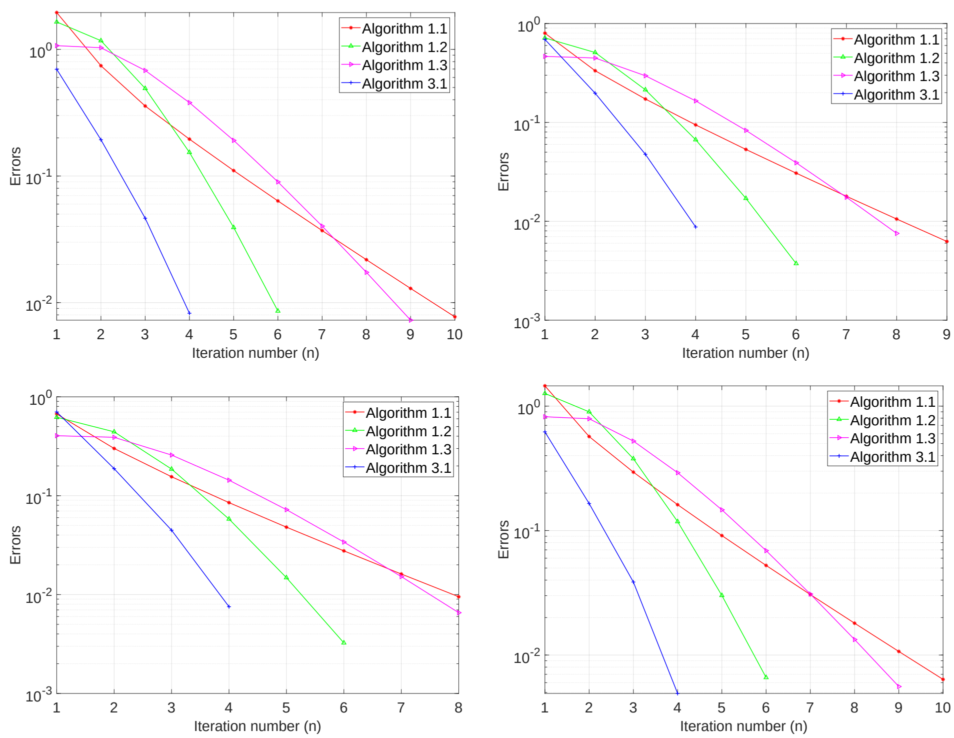

FIGURE 2. Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

We choose four different initial values as follows with $\gamma = 0.05$ in each case:
Case I: $x_0 = t + 3$ and $x_1 = t^3 + t$;
Case II: $x_0 = t + 1$ and $x_1 = t^3 + t$;

Case III: $x_0 = t^4 + t^2 + t$ and $x_1 = t^3 + t$;
Case IV: $x_0 = 2t^5 + t^2 + 3t$ and $x_1 = t^3 + t^2$.
We compare the performance of our Algorithm 3.1 with Algorithm 1.1, Algorithm 1.2 and Algorithm 1.3. The stopping criterion used for our computation is $\|x_{n+1} - x_n\| < 10^{-2}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 2.

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