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SOLUTIONS OF p-LAPLACIAN FRACTIONAL DIFFERENTIAL EQUATIONS ON TIME SCALES

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Abstract. We discuss a class of fractional p-Laplacian boundary value problem with a parameter on time scales. The novelty of this paper is that the existence of infinitely many solutions of the equation is obtained by using the genus property of critical point theory under the condition of weaker than the super p times Ambrosetti-Rabinowitz condition. What's more, the existence of infinitely many solutions is proved via the symmetric mountain pass theorem with the Ambrosetti-Rabinowitz condition. At the end, an example is presented to illustrate our main results.
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1. Introduction

In this paper, we are interested in the following class of fractional p-Laplacian boundary value problem with a parameter on time scales

$$\begin{cases} {}_t^{\mathbb{T}}D_b^{\alpha}\Phi_p({}_a^{\mathbb{T}}D_t^{\alpha}u(t)) = \lambda f(t,u(t)),\ \Delta - a.e.\ t \in J, \\ u(a) = u(b) = 0, \end{cases} \tag{1}$$

where $J = [a,b] \cap \mathbb{T}$, $\mathbb{T}$ is a time scale, $a,\ b \in \mathbb{T}$. $1 < p < \infty$, $\frac{1}{p} < \alpha \leq 1$ and $\lambda > 0$. $\Phi_p(s) = |s|^{p-2}s\ (s \neq 0)$, $\Phi_p(0) = 0$, $1 < p < \infty$. ${}_a^{\mathbb{T}}D_t^{\alpha}$ and ${}_t^{\mathbb{T}}D_b^{\alpha}$ denote the left and the right Riemann-Liouville fractional derivative operators of order α defined on $\mathbb{T}$, respectively. $f \in C(J \times \mathbb{R}, \mathbb{R})$ and $F(t,u) = \int_0^u f(t,s)\mathrm{d}s$.

The time scale theory proposed by Hilger in 1988 was a unified theory of continuous and discrete calculus, which revealed a profound connection between continuous calculus and difference equations (see [13]). Most notably, time scale theory can not only unify the study of continuous and discrete systems, but also quantum systems simultaneously. To be specific: If the time scale satisfies $\mathbb{T} = \mathbb{R}$, then the fractional time scale calculus is consistent with the classical fractional calculus, fractional calculus for $\mathbb{T} = \mathbb{R}$ has been extensively studied and there are many works on the subject (see [9], [10], [18], [19]). If the time scale is taken as $\mathbb{T} = \mathbb{Z}$, the fractional time scale calculus becomes the calculus of discrete systems with step size $\mu' = 1$. The pioneering work on fractional calculus on the discrete setting $\mathbb{T} = \mathbb{Z}$ appeared in 1957, related to Kuttner's work on difference sequences (see [17]). The definitions of discrete fractional derivatives and finite fractional differences were proposed by Atici and Eloe in 2007 (see [2]). If the time scale is $\mathbb{T} = q^{\mathbb{N}_0}$, the fractional time scale calculus becomes the discrete fractional q-calculus (see [3]). Therefore, continuous, discrete and quantum fractional systems can be systematically processed with different time scale $\mathbb{T}$ values. There is no doubt that the time scale theory brings many conveniences to the establishment of models. Different time scales can produce different applications, including engineering, physics, economics, biology, population models, control systems and other scientific and engineering fields (see [4], [11], [15], [23]).

Recently, some work has been done to consider the existence of solutions to boundary value problems on time scales (see [1], [8], [22], [24]). In particular, in [24], the authors considered a class of second-order Hamiltonian systems on time scales with three-order boundary conditions as shown below

$$\begin{cases} u^{\Delta\Delta}(t) = f(\sigma(t), u^{\sigma}(t)), \ \Delta - a.e. \ t \in J, \\ u(a) = 0, u(b) = \zeta u(\eta), \end{cases}$$

where $J = [a, b] \cap \mathbb{T}$, $\zeta \in \mathbb{R}$, $\eta \in (a, b) \cap \mathbb{T}$ and a, $b \in \mathbb{T}$. When f met the Ambrosetti-Rabinowitz condition, they used the symmetrical mountain pass theorem to prove that there are infinitely many solutions for this problem.

In addition, fractional calculus theory is an extended version of integer theory. The first work considered the problem of developing fractional calculus on generic time scales were [5]-[7] in 2011. Since the fractional differential equation is more accurately defined, fractional calculus plays an important role in the qualitative theory of differential equations (see [12], [14], [16]). For example, in [14], the authors studied the following fractional boundary value problem on time scales

$$\begin{cases} {}_a^{\mathbb{T}}D_t^{\alpha}({}_t^{\mathbb{T}}D_b^{\alpha}u(t)) = \nabla F(t, u(t)), \ \Delta - a.e. \ t \in J, \\ u(a) = u(b) = 0, \end{cases} \tag{2}$$

where $0 < \alpha \leq 1$, ${}_a^{\mathbb{T}}D_t^{\alpha}$ and ${}_t^{\mathbb{T}}D_b^{\alpha}$ are the left and right Riemann-Liouville fractional derivatives of order α defined on $\mathbb{T}$, respectively. The authors got the existence of at least one solution of the above problem by the mountain pass theorem provided F satisfied the Ambrosetti-Rabinowitz condition.

Inspired by the above research work and research background, we are interested in the boundary value problem (BVP for short) (1). And it is proved that there are infinitely many solutions to the BVP (1) by using the genus property of critical point theory with f satisfying the sub p times growth. Additionally, when f satisfies the Ambrosetti-Rabinowitz condition, the existence of infinitely many solutions is given through the symmetric mountain pass theorem.

As far as we know, there are very few literatures about the existence of solutions to fractional boundary value problems on time scales through variational methods, let alone the equation with p-Laplacian. Moreover, sufficient condition is given for the existence of infinitely many solutions to the BVP (1) without the super p times Ambrosetti-Rabinowitz condition. On the whole, the research of this paper has certain theoretical significance, and has some innovative points.

2. Preliminaries

In this part, we will recall some definitions and lemmas. For the rest related definitions and lemmas, please refer to reference [1], [8], [14]. Let's first give the definitions of the following notations. Suppose $\mathbb{T}$ is a time scale, $a < b$. Define $J^0_{\mathbb{R}} = [a, b)$, $J_{\mathbb{R}} = [a, b]$, $J^0 = J^0_{\mathbb{R}} \cap \mathbb{T}$ and $J = J_{\mathbb{R}} \cap \mathbb{T}$.

Definition 2.1 [14] For $t \in \mathbb{T}$, we define the forward jump operator $\bar{\sigma} : \mathbb{T} \to \mathbb{T}$ by $\bar{\sigma}(t) = \inf\{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\bar{\rho} : \mathbb{T} \to \mathbb{T}$ is defined by $\bar{\rho}(t) = \sup\{s \in \mathbb{T} : s < t\}$. Put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\bar{\sigma}(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\bar{\rho}(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ denotes the empty set. If $\bar{\sigma}(t) > t$, we say that t is right-scattered, while if $\bar{\rho}(t) < t$, we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $t < \sup \mathbb{T}$ and $\bar{\sigma}(t) = t$, we say that t is right-dense, while if $t > \inf \mathbb{T}$ and $\bar{\rho}(t) = t$, we say that t is left-dense. Points that are right-dense and left-dense at the same time are called dense. The graininess function $\bar{\mu} : \mathbb{T} \to [0, \infty)$ is defined by $\bar{\mu}(t) = \bar{\sigma}(t) - t$. The derivative makes use of the set $\mathbb{T}^{\bar{k}}$, which is derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum $\bar{M}$, then $\mathbb{T}^{\bar{k}} = \mathbb{T} \setminus \{\bar{M}\}$, otherwise, $\mathbb{T}^{\bar{k}} = \mathbb{T}$.

Definition 2.2 [14] Let u be an integrable function defined on J. The left and right fractional integrals of order $0 < \alpha \leq 1$ for the function u denoted by ${}^{\mathbb{T}}_{a}I^{\alpha}_{t}u(t)$ and ${}^{\mathbb{T}}_{t}I^{\alpha}_{b}u(t)$, respectively, are defined by

$$
\begin{aligned}
{}^{\mathbb{T}}_{a}I^{\alpha}_{t}u(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t - \sigma(s))^{\alpha-1} u(s) \Delta s, \\
{}^{\mathbb{T}}_{t}I^{\alpha}_{b}u(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s - \sigma(t))^{\alpha-1} u(s) \Delta s,
\end{aligned}
$$

where $\Gamma > 0$ is the gamma function.

Definition 2.3 [14] Let $t \in \mathbb{T}$ and $u : \mathbb{T} \to \mathbb{R}$. The left and right Riemann-Liouville fractional derivatives of order $0 < \alpha \leq 1$ for the function u denoted by ${}^{\mathbb{T}}_{a}D^{\alpha}_{t}u(t)$ and ${}^{\mathbb{T}}_{t}D^{\alpha}_{b}u(t)$, respectively, are defined by

$$
\begin{aligned}
{}^{\mathbb{T}}_{a}D^{\alpha}_{t}u(t) &= \left({}^{\mathbb{T}}_{a}I^{1-\alpha}_{t}u(t) \right)^{\Delta} = \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t - \sigma(s))^{-\alpha} u(s) \Delta s \right)^{\Delta}, \\
{}^{\mathbb{T}}_{t}D^{\alpha}_{b}u(t) &= -\left({}^{\mathbb{T}}_{t}I^{1-\alpha}_{b}u(t) \right)^{\Delta} = -\frac{1}{\Gamma(1-\alpha)} \left(\int_t^b (s - \sigma(t))^{-\alpha} u(s) \Delta s \right)^{\Delta}.
\end{aligned}
$$

Definition 2.4 [14] Let $t \in \mathbb{T}$ and $u : \mathbb{T} \to \mathbb{R}$. Then the left and right Caputo fractional derivatives of order $0<\alpha\le 1$ for the function u denoted by ${}_a^{\mathbb{T}c}D_t^{\gamma}u(t)$ and ${}_t^{\mathbb{T}c}D_b^{\gamma}u(t)$, respectively, are defined by

$$
\begin{aligned}
{}_a^{\mathbb{T}c}D_t^{\alpha}u(t) &= {}_a^{\mathbb{T}}I_t^{1-\alpha}u^{\Delta}(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-\sigma(s))^{-\alpha}u^{\Delta}(s)\Delta s,\\
{}_t^{\mathbb{T}c}D_b^{\alpha}u(t) &= -{}_t^{\mathbb{T}}I_b^{1-\alpha}u^{\Delta}(t)=-\frac{1}{\Gamma(1-\alpha)}\int_t^b (s-\sigma(t))^{-\alpha}u^{\Delta}(s)\Delta s.
\end{aligned}
$$

Lemma 2.1 [1] *Suppose $\bar{p}\ge 1$, and the set $L_{\Delta}^{\bar{p}}(J^0,\mathbb{R})$ is a Banach space together with the norm defined for every $u\in L_{\Delta}^{\bar{p}}(J^0,\mathbb{R})$ as*

$$
\|u\|_{L_{\Delta}^{\bar{p}}}=\begin{cases} \left(\int_{J^0}|u(s)|^{\bar{p}}\Delta s\right)^{\frac{1}{\bar{p}}}, & \bar{p}\in\mathbb{R},\\ \inf\{M\in\mathbb{R}:|u|\le M\ \Delta-a.e.\ on\ J^0\}, & \bar{p}=\infty. \end{cases}
$$

Proposition 2.1 [1] *Suppose $\bar{p}\in\bar{\mathbb{R}}$ and $\bar{p}\ge 1$, let $\bar{p}'\in\bar{\mathbb{R}}$ be such that $\frac{1}{\bar{p}}+\frac{1}{\bar{p}'}=1$, then, if $u\in L_{\Delta}^{\bar{p}}(J^0,\mathbb{R})$ and $v\in L_{\Delta}^{\bar{p}'}(J^0,\mathbb{R})$, then $u\cdot v\in L_{\Delta}^{1}(J^0,\mathbb{R})$ and $\|u\cdot v\|_{L_{\Delta}^1}\le\|u\|_{L_{\Delta}^{\bar{p}}}\cdot\|v\|_{L_{\Delta}^{\bar{p}'}}$.*

For $\bar{p}\in\mathbb{R}$ and $\bar{p}\ge 1$, we set the space

$$
L_{\Delta}^{\bar{p}}(J^0,\mathbb{R})=\{u:J^0\to\mathbb{R}:\int_{J^0}|u(s)|^{\bar{p}}\Delta s<\infty\}
$$

with the norm $\|u\|_{L_{\Delta}^{\bar{p}}}=\left(\int_{J^0}|u(s)|^{\bar{p}}\Delta s\right)^{\frac{1}{\bar{p}}}$. For BVP (1), we consider the space

$$
W=W_{\Delta}^{\alpha,p}=\left\{u\in L_{\Delta}^{p}(J^0,\mathbb{R}),\ {}_a^{\mathbb{T}}D_t^{\alpha}u\in L_{\Delta}^{p}(J^0,\mathbb{R}):u(a)=u(b)=0\right\}
$$

with the norm

$$
\|u\|_{\alpha,p}=\left(\int_{J^0}|u(t)|^p\Delta t+\int_{J^0}|{}_0^{\mathbb{T}}D_t^{\alpha}u(t)|^p\Delta t\right)^{\frac{1}{p}}.
$$

According to [14], [24], the space W with $1<p<\infty$ is a reflexive Banach space.

Lemma 2.2 *The norm $\|u\|_{\alpha,p}$ in W is equivalent to*

$$
\|u\|=\left(\int_{J^0}|{}_0^{\mathbb{T}}D_t^{\alpha}u(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}.
$$

By analogy with Proposition 4.1 and Proposition 4.2 in [14], the following two lemmas can be obtained.

Lemma 2.3 *Let $0<\alpha\le 1$ and $1<p<\infty$. For any $u\in W$, if $1-\alpha\ge\frac{1}{p}$ or $\alpha>\frac{1}{p}$, then*

$$
\|u\|_{L_{\Delta}^{p}}\le\frac{b^{\alpha}}{\Gamma(\alpha+1)}\|{}_a^{\mathbb{T}}D_t^{\alpha}u\|_{L_{\Delta}^{p}}:=\Lambda\|{}_a^{\mathbb{T}}D_t^{\alpha}u\|_{L_{\Delta}^{p}}.
$$

If $\alpha > \frac{1}{p}$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\|u\|_\infty \le \frac{b^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|{}_a^{\mathbb{T}}D_t^\alpha u\|_{L_\Delta^p} := \bar{\Lambda}\|{}_a^{\mathbb{T}}D_t^\alpha u\|_{L_\Delta^p},$$

where $\|u\|_\infty = \max_{t\in J}|u(t)|$.

Lemma 2.4 *Let* $0<\alpha\le 1$ *and* $1<p<\infty$. *Assume that* $\alpha > \frac{1}{p}$ *and the sequence* $\{u_n\}$ *converges weakly to* u *in* W, *that is,* $u_n \rightharpoonup u$ *as* $n\to\infty$. *Then* $u_n \to u$ *in* $C(J,\mathbb{R})$ *as* $n\to\infty$.

For all $u\in W$, we consider the functional $\Psi : W\to\mathbb{R}$ defined by

$$\begin{aligned}\Psi(u) =&\frac{1}{p}\int_{J^0}|{}_0^{\mathbb{T}}D_t^\alpha u(t)|^p\Delta t - \lambda\int_{J^0}F(t,u(t))\Delta t\\ =&\frac{1}{p}\|u\|^p - \lambda\int_{J^0}F(t,u(t))\Delta t,\end{aligned}$$

where $F(t,u)=\int_0^u f(t,s)\mathrm{d}s$. It can be proved that the functional Ψ is continuously differentiable on W and

$$\Psi'(u)v = \int_{J^0}\Phi_p({}_0^{\mathbb{T}}D_t^\alpha u(t)){}_0^{\mathbb{T}}D_t^\alpha v(t)\Delta t - \lambda\int_{J^0}f(t,u(t))v(t)\Delta t,\ \forall v\in W.$$

If $u\in W$ is a critical point of Ψ, then, u is a solution to BVP (1).

3. Main Results

In this section, we prove the existence of infinitely many weak solutions of the BVP (1). A few notations are given at the beginning of this section. Suppose E is a Banach space, $\Psi\in C^1(E,\mathbb{R})$ and $z\in\mathbb{R}$. Define

$$\begin{aligned}&\Sigma = \{A\subset E-\{0\} : A \textit{ is closed in } E \textit{ and symmetric about } 0\},\\ &K_z = \{u\in E : \Psi(u)=z, \Psi'(u)=0\},\\ &\Psi^z = \{u\in E : \Psi(u)\le z\}.\end{aligned}$$

Definition 3.1 [20], [21] For $A\in\Sigma$, if there is an odd mapping $\Psi\in C(A,\mathbb{R}^n\setminus\{0\})$, and n is the smallest integer with this property, then the genus of A is called n, and denoted as $\gamma(A)=n$.

Lemma 3.1 [20] *Let* Ψ *be a* C^1 *even functional on the Banach space* E *and satisfy the Palais-Smale ((PS) for short) condition. For any* $n\in\mathbb{N}$, *let* $\Sigma_n=\{A\in\Sigma : \gamma(A)\ge n\}$, $z_n=\inf_{A\in\Sigma_n}\sup_{u\in A}\Psi(u)$.

(1) If $\Sigma_n\ne\emptyset$ *and* $z_n\in\mathbb{R}$, *then* z_n *is the critical value of* Ψ.

(2) If there is $l\in\mathbb{N}$ *such that* $z_n=z_{n+1}=z_{n+2}=\cdots=z_{n+l}=z\in\mathbb{R}$ *and* $z\ne\Psi(0)$, *then* $\gamma(K_z)\ge l+1$.

Remark 3.1 From Remark 7.3 in [20], it can be known that if $K_z\in\Sigma$ and $\gamma(K_z)>1$, then K_z contains infinitely many different points, that is, Ψ has infinitely many different critical points in E.

Now, we assume that $f(t,u)$ and $F(t,u)$ satisfy the following conditions.
(H_1) $F(t,0)=0$ for all $t\in J$, there exists $1<\theta<p$, such that for all $(t,u)\in J\times\mathbb{R}$, $F(t,u)\geq g(t)|u|^\theta$ and $|f(t,u)|\leq h(t)|u|^{\theta-1}$, where $g:J\to\mathbb{R}^+$ is a bounded continuous function, and $h:J\to\mathbb{R}^+$ is a continuous function such that $h\in L_\Delta^{\frac{p}{p-\theta}}(J^0,\mathbb{R})$.
(H_2) There exists $1<\vartheta\leq\theta<p$, such that

$$f(t,u)u\leq\vartheta F(t,u),\ for\ t\in J,\ u\in\mathbb{R}\setminus\{0\}.$$

Theorem 3.1 *If (H_1)-(H_2) hold, and $f(t,u)$ is odd about u, then the BVP (1) has infinitely many nontrivial solutions $\{u_n\}_{n=1}^\infty$ with $\frac{1}{p}\|u_n\|^p-\lambda\int_{J^0}F(t,u_n(t))\Delta t\to 0^-$ as $n\to\infty$.*

Proof. According to the definition of Ψ, we know that Ψ is even and $\Psi(0)=0$.

Next, we prove that Ψ satisfies the (PS) condition in W. Suppose $\{u_n\}_{n=1}^\infty\subset W$ be a sequence such that $\Psi(u_n)\leq k$, $\lim_{n\to\infty}\Psi'(u_n)=0$, where k is a positive constant.

Let's prove that $\{u_n\}_{n=1}^\infty$ is bounded in W. Based on (H_2), the definitions of Ψ and Ψ',

$$\begin{aligned}(1-\frac{\vartheta}{p})\|u_n\|^p=&\Psi'(u_n)u_n-\vartheta\Psi(u_n)+\lambda\int_{J^0}\Big(f(t,u_n(t))u_n(t)-\vartheta F(t,u_n(t))\Big)\Delta t\\ \leq& k\|u_n\|+\vartheta k,\end{aligned}$$

which means that $\{u_n\}_{n=1}^\infty$ is bounded in W. On the other hand, according to the fact that W is a Banach space and f is continuous, it follows from Lemma 2.4 that $u_n\to u$ in W as $n\to\infty$. That is, Ψ satisfies the (PS) condition.

Let $\{e_n\}_{n=1}^\infty$ be the standard orthogonal basis of W, that is, $\|e_q\|=1$ and $<e_q,e_{q'}>=0$, $1\leq q\neq q'$. For any $n\in\mathbb{N}$, define

$$E_n=\operatorname{span}\{e_1,\cdots,e_n\},\ S_n=\{u\in E_n:\|u\|=1\},$$

then for any $u\in E_n$, there exist $\omega_i\in\mathbb{R}$ $(i=1,\cdots,n)$, such that

$$u(t)=\sum_{i=1}^n\omega_ie_i(t),\ \forall t\in J.$$

Therefore, for $u\in E_n$,

$$\begin{aligned}\|u\|^p=\int_{J^0}|{}_a^{\mathbb{T}}D_t^\alpha u(t)|^p\Delta t&=\sum_{i=1}^n|\omega_i|^p\int_{J^0}|{}_a^{\mathbb{T}}D_t^\alpha e_i(t)|^p\Delta t\\ &=\sum_{i=1}^n|\omega_i|^p\|e_i\|^p=\sum_{i=1}^n|\omega_i|^p.\end{aligned}$$

In view of (H_1), for all bounded open interval $\mathbb{I}\subset J$, there is $\eta>0$ (dependent on $\mathbb{I}$), such that

$$F(t,u)\geq g(t)|u|^\theta\geq\eta|u|^\theta,\ \forall(t,u)\in\mathbb{I}\times\mathbb{R}.$$

Then, for any $u \in S_n$ and $\tau > 0$,

$$\begin{aligned}\Psi(\tau u) =& \frac{\tau^p}{p}\|u\|^p - \lambda \int_{J^0} F(t, \tau u(t))\Delta t \\ \leq& \frac{\tau^p}{p} - \lambda \tau^\theta \eta \int_{\mathbb{I}} \left| \sum_{i=1}^{n} \omega_i e_i(t) \right|^\theta \Delta t.\end{aligned}$$

It's easy to prove that $\int_{\mathbb{I}} |\sum_{i=1}^{n} \omega_i e_i(t)|^\theta \Delta t > 0$. So, there exist $\zeta > 0$ and $\varpi > 0$, such that for $u \in S_n$, $\Psi(\varpi u) < -\zeta$.

Let

$$S_n^\varpi = \{\varpi u : u \in S_n\},\ \Theta = \{(\omega_1, \cdots, \omega_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |\omega_i|^p < \varpi^p\}.$$

Then, for $u \in S_n^\varpi$, $\Psi(u) < -\zeta$, which together with the fact that even functional $\Psi \in C^1(W, \mathbb{R})$, yields that $S_n^\varpi \subset \Psi^{-\zeta} \in \Sigma$.

In addition, there exists an odd homeomorphism mapping $\Upsilon \in C(\partial\Theta, S_n^\varpi)$. By some properties of the genus, we obtain $\gamma(\Psi^{-\zeta}) \geq \gamma(S_n^\varpi) = n$, so $\Psi^{-\zeta} \in \Sigma_n$, that is, $\Sigma_n \neq \emptyset$. Let

$$z_n = \inf_{A \in \Sigma_n} \sup_{u \in A} \Psi(u),$$

then, from the fact that Ψ is bounded from below on W, we have $-\infty \leq z_n < -\zeta < 0$, that is, for any $n \in \mathbb{N}$, z_n is a negative real number. Thus, by Lemma 3.1 and Remark 3.1, Ψ has infinitely many nontrivial critical points, in other words, BVP (1) admits infinitely many nontrivial solutions.

Finally, define

$$\bar{E}_n = \operatorname{span}\{e_n\},\ Z_n = \overline{\bigoplus_{i=n}^{\infty} \bar{E}_i}.$$

Set $\mu_n = \sup_{u \in Z_n, \|u\|=1} \|u\|_{L^p_\Delta}$. Since $0 < \mu_{n+1} \leq \mu_n$, $\lim_{n\to\infty} \mu_n = \mu \geq 0$. For every $n \geq 1$, there exists $u_n \in Z_n$ such that $\|u_n\| = 1$ and $\|u_n\|_{L^p_\Delta} \geq \frac{\mu_n}{p}$. Furthermore, according to $u_n \rightharpoonup 0$ in W, we have $u_n \to 0$ in $L^p_\Delta(J^0, \mathbb{R})$. Thus, $\lim_{n\to\infty} \mu_n = \mu = 0$.

Combining (H_1), Lemma 2.3 and Hölder's inequality,

$$\begin{aligned}\int_{J^0} F(t, u(t))\Delta t \leq& \frac{1}{\theta} \int_{J^0} h(t)|u(t)|^\theta \Delta t \leq \frac{1}{\theta} \left(\int_{J^0} |h(t)|^{\frac{p-\theta}{p}} \Delta t \right)^{\frac{p}{p-\theta}} \left(\int_{J^0} |u(t)|^p \Delta t \right)^{\frac{\theta}{p}} \\ \leq& \frac{\Lambda^\theta}{\theta} \|h\|_{L_\Delta^{\frac{p}{p-\theta}}} \|u\|^\theta,\end{aligned}$$

then, we can get

$$\Psi(u) \geq \frac{1}{p}\|u\|^p - \frac{\Lambda^\theta}{\theta} \|h\|_{L_\Delta^{\frac{p}{p-\theta}}} \|u\|^\theta,$$

which indicates that $\Psi(u)$ is coercive. Therefore, there exists $\xi > 0$, such that $\Psi(u) > 0$ for $\|u\| \geq \xi$. What's more, for any $A \in \Sigma_n$, owing to $\gamma(A) \geq n$, one can get

$A \cap Z_n \neq \emptyset$. Combining the above, we can get

$$\begin{aligned}\sup_{u\in A}\Psi(u) \geq \inf_{u\in Z_n,\|u\|\leq\xi}\Psi(u) \geq \inf_{u\in Z_n,\|u\|\leq\xi}\left(\frac{1}{p}\|u\|^p - \frac{\mu_n^\theta}{\theta}\|h\|_{L_\Delta^{\frac{p}{p-\theta}}}\|u\|^\theta\right)\\ \geq -\frac{\mu_n^\theta}{\theta}\|h\|_{L_\Delta^{\frac{p}{p-\theta}}}\xi^\theta,\end{aligned}$$

which means that

$$z_n = \inf_{A\in\Sigma_n}\sup_{u\in A}\Psi(u) \geq -\frac{\mu_n^\theta}{\theta}\|h\|_{L_\Delta^{\frac{p}{p-\theta}}}\xi^\theta.$$

Notice $z_n < 0$ and $\lim_{n\to\infty}\mu_n = 0$, so, $\lim_{n\to\infty} z_n = 0^-$.

Lemma 3.2 [20] *Let E be a Banach space and let $\Psi \in C^1(E,\mathbb{R})$ be even, satisfy the (PS) condition, and have $\Psi(0) = 0$. Suppose that $E = Y \bigoplus Z$, where Y is finite dimensional, and Ψ satisfies*
(1) there exist $\sigma > 0$ and $\rho > 0$, such that $\Psi(u) \geq \sigma$ for all $u \in E$ with $\|u\| = \rho$, and
(2) for any finite dimensional subspace $E' \subset E$, there is $\bar{R} = \bar{R}(E')$, such that $\Psi(u) \leq 0$ for all $u \in E'$ with $\|u\| \geq \bar{R}$.
Then Ψ possesses an unbounded sequence of critical values.

To get the second result, we make the following assumption.
(H_1') There exist $\theta' > p$ and $R > 0$, such that

$$0 < \theta' F(t,u) \leq f(t,u)u,\ t \in J,\ |u| \geq R.$$

Theorem 3.2 *If (H_1') holds, and $f(t,u)$ is odd about u, then the BVP (1) has infinitely many nontrivial solutions.*

Proof. It is easy to know that $\Psi(0) = 0$ and Ψ is an even functional. In what follows, we show that Ψ satisfies the (PS) condition. Assume that $\{u_n\}_{n=1}^{\infty} \subset W$ be a sequence such that $\Psi(u_n) \leq k'$, $\lim_{n\to\infty}\Psi'(u_n) = 0$, where k' is a positive constant. It follows from (H_1') that

$$F(t,u) \leq \frac{1}{\theta'}f(t,u)u + m,\ \forall (t,u) \in J \times \mathbb{R},$$

where $m > 0$. Then,

$$\begin{aligned}k' \geq &\Psi(u_n) \geq \frac{1}{p}\|u_n\|^p - \lambda\int_{J^0}F(t,u_n(t))\Delta t\\ \geq &\frac{1}{p}\|u_n\|^p - \frac{\lambda}{\theta'}\int_{J^0}f(t,u_n(t))u_n(t)\Delta t - \lambda\int_{J^0}m\Delta t\\ \geq &(\frac{1}{p} - \frac{1}{\theta'})\|u_n\|^p + \frac{1}{\theta'}\Psi'(u_n)u_n - \lambda\int_{J^0}m\Delta t\\ \geq &(\frac{1}{p} - \frac{1}{\theta'})\|u_n\|^p - \frac{k'}{\theta'}\|u_n\| - \lambda\int_{J^0}m\Delta t,\end{aligned}$$

which means that $\{u_n\}_{n=1}^{\infty}$ is bounded in W. Since W is a reflexive space, there exists $\{u_n\}_{n=1}^{\infty}$, such that $u_n \rightharpoonup u$ in W as $n \to \infty$. Then, it follows from Lemma 2.4 that $u_n \to u$ in $C(J, \mathbb{R})$ as $n \to \infty$. Furthermore,

$$< \Psi'(u_n) - \Psi'(u), u_n - u > \to 0,$$
$$\Big(f(t, u_n(t)) - f(t, u(t))\Big)(u_n(t) - u(t)) \to 0$$

as $n \to \infty$. Because

$$\begin{aligned}\|u_n - u\|^p =& < \Psi'(u_n) - \Psi'(u), u_n - u > \\ &- \lambda \int_{J^0} \Big(f(t, u_n(t)) - f(t, u(t))\Big)(u_n(t) - u(t))\Delta t \to 0\end{aligned}$$

as $n \to \infty$, thus, $u_n \to u$ in W. In conclusion, Ψ satisfies the (PS) condition.

Let B_r denotes the open ball in Ψ with radius r and centered at 0 and let ∂B_r and $\bar{B}_r$ represent the boundary and closure of B_r, respectively. It is easy to get $\bar{B}_{\frac{1}{\bar{\Lambda}}}$ is a bounded weak closed set. It can be seen from (H_1') that there exists $P_1 > 0$ with $\int_{J^0} P_1 \Delta t < \frac{1}{\lambda p \bar{\Lambda}}$, such that

$$F(t, u) \le P_1|u|^{\theta'},\ \forall (t, u) \in J \times [-1, 1].$$

Choose $\rho = \frac{1}{\bar{\Lambda}}$, for any $u \in W$ with $\|u\| = \rho$, then $\|u\|_\infty \le \bar{\Lambda}\|u\| = 1$. Based on the above, for any $u \in \partial B_r$ $(r \le \frac{1}{\bar{\Lambda}})$, we can get

$$\begin{aligned}\Psi(u) \ge& \frac{1}{p}\|u\|^p - \lambda \int_{J^0} P_1|u|^{\theta'}\Delta t \\ \ge& \frac{1}{p}\|u\|^p - \lambda\|u\|_\infty^{\theta'} \int_{J^0} P_1\Delta t \\ \ge& \frac{1}{p}r^p - \lambda\bar{\Lambda}^{\theta'} r^{\theta'} \int_{J^0} P_1\Delta t.\end{aligned}$$

And further, for any $u \in \partial B_{\frac{1}{\bar{\Lambda}}}$,

$$\Psi(u) \ge \frac{1}{p\bar{\Lambda}^p} - \lambda \int_{J^0} P_1\Delta t = \sigma > 0,\ \forall \|u\| = \rho,$$

which indicates that the condition (1) of Lemma 3.2 holds.

Based on the fact that all norms are equivalent in finite dimensional space, for any finite dimensional subspace $\widetilde{W}$ of space W, for any $u \in \widetilde{W}$, there is a constant $G' > 0$, so that

$$\|u\|_{L_\Delta^q} = (\int_{J^0} |u(t)|^q \Delta t)^{\frac{1}{q}} \ge G'\|u\|,\ \forall q \ge 1.$$

According to (H_1'), there exist $P_2 > 0$ and $P_3 > 0$, such that

$$F(t, u) \ge P_2|u|^{\theta'} - P_3,\ \forall (t, u) \in J \times \mathbb{R}.$$

Then, for $u \in \widetilde{W}$, there exists $G > 0$, such that

$$\int_{J^0} F(t, u(t))\Delta t \ge P_2\|u\|_{L_\Delta^{\theta'}}^{\theta'} - \int_{J^0} P_3\Delta t \ge P_2 G^{\theta'}\|u\|^{\theta'} - \int_{J^0} P_3\Delta t,$$

which means that

$$\Psi(u) \leq \frac{1}{p}\|u\|^p - \lambda P_2 G^{\theta'}\|u\|^{\theta'} + \lambda \int_{J^0} P_3 \Delta t.$$

Thus, for $u \in \widetilde{W}$, $\Psi(u) \to -\infty$ as $\|u\| \to \infty$. Further, there exists $\bar{R} = \bar{R}(\widetilde{W})$, such that for any $u \in \widetilde{W}$ with $\|u\| \geq \bar{R}$, $\Psi(u) \leq 0$.

In general, it can be seen from Lemma 3.2 that Ψ admits an unbounded sequence $\{c_n\}_{n=1}^{\infty}$ of critical values with $c_n = \Psi(u_n)$, where u_n guarantees that $\Psi'(u_n) = 0$ $(n = 1, 2, ...)$.

Proof by contradiction. Suppose $\|u_n\| \leq d$ for $n = 1, 2, ...$, and d is a constant. Looking at the above discussion, we know that $u_n \to u$ in W as $n \to \infty$. Hence,

$$\begin{aligned}\frac{1}{p}\|u_n\|^p =& c_n + \lambda \int_{J^0} F(t, u_n(t))\Delta t \\ \geq& c_n + \lambda P_2 \|u_n\|^{\theta'}_{L^{\theta'}_{\Delta}} - \lambda \int_{J^0} P_3 \Delta t,\end{aligned}$$

that is,

$$c_n \leq \frac{1}{p}\|u_n\|^p - \lambda P_2 \|u_n\|^{\theta'}_{L^{\theta'}_{\Delta}} + \lambda \int_{J^0} P_3 \Delta t < \infty.$$

It contradicts the fact that $\{c_n\}_{n=1}^{\infty}$ is unbounded, thus, $\{u_n\}_{n=1}^{\infty}$ is unbounded.

4. Examples

We provide an example below to illustrate the derived conclusions.

Example 4.1 Let $\mathbb{T} = \frac{1}{2}\mathbb{Z}$, $\alpha = \lambda = \frac{1}{2}$, $p = 2$, $a = 0$, $b = 1$ and $J = [0, 1] \cap \frac{1}{2}\mathbb{Z}$. Consider the following fractional boundary value problem

$$\begin{cases} {}_t^{\mathbb{T}}D_1^{\frac{1}{2}} \Phi_2({}_0^{\mathbb{T}}D_t^{\frac{1}{2}} u(t)) = \frac{1}{2} f(t, u(t)), \ \Delta - a.e.\ t \in J, \\ u(0) = u(1) = 0, \end{cases} \tag{3}$$

where

$$f(t, u) = u^3\bigg(2 + 3\sin(|t| + 1)\bigg),$$

then

$$F(t, u) = \int_0^u f(t, s)\mathrm{d}s = \frac{1}{4}u^4\bigg(2 + 3\sin(|t| + 1)\bigg).$$

Therefore, (H_1') holds with $R = 1$ and $\theta' = 3$, that is, from Theorem 3.2, the BVP (3) has infinitely many nontrivial solutions.

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