Fixed Point Theory, 27(2026), No. 2, 629-638
DOI: 10.24193/fpt-ro.2026.2.10
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SYSTEM OF ADDITIVE FUNCTIONAL EQUATIONS IN TERNARY BANACH ALGEBRAS

CHOONKIL PARK

Research Institute for Convergence of Basic Science, Hanyang University,
Seoul 04763, Korea
E-mail: baak@hanyang.ac.kr

Abstract. In this paper, we solve the system of additive functional equations

$$\begin{cases} 3f(x+y+z) = g(x)+g(y)+g(z) \\ g\left(\frac{x+y+z}{3}\right) = f(x)+f(y)+f(z) \end{cases}$$

and we prove the Hyers-Ulam stability of ternary f-hom-ders in ternary Banach algebras.
Key Words and Phrases: Ternary Banach algebra, ternary f-hom-der, fixed point method, Hyers-Ulam stability, system of additive functional equations.
2020 Mathematics Subject Classification: 47B47, 17B40, 39B72, 47H10, 39B52, 11E20.

1. Introduction

Let $\mathcal{B}$ be a complex Banach algebra and $g : \mathcal{B} \to \mathcal{B}$ be a $\mathbb{C}$-linear mapping. Mirzavaziri and Moslehian [22] introduced the concept of g-derivation $f : \mathcal{B} \to \mathcal{B}$ as follows:

$$f(xy) = f(x)g(y) + g(x)f(y) \tag{1.1}$$

for all $x, y \in \mathcal{B}$. Park *et al.* [29] introduced the concept of hom-derivation on $\mathcal{B}$, i.e., $g : \mathcal{B} \to \mathcal{B}$ is a homomorphism and f satisfies (1.1) for all $x, y \in \mathcal{B}$. Dehghanian *et al.* [6] introduced the concept of hom-der $g : \mathcal{B} \to \mathcal{B}$ as follows:

$$g(x)g(y) = xg(y) + g(x)y$$

for all $x, y \in \mathcal{B}$. Kheawborisuk *et al.* [18] defined and studied hom-ders in fuzzy Banach algebras.

A ternary Banach algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [36]).

Let A and B be ternary Banach algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *ternary homomorphism* if

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *ternary derivation* if

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in A$ (see [1, 23]).

Definition 1.1 Let $\mathcal{B}$ be a ternary Banach algebra and $f : \mathcal{B} \to \mathcal{B}$ be a ternary homomorphism. A $\mathbb{C}$-linear mapping $g : \mathcal{B} \to \mathcal{B}$ is called a ternary f-hom-der if it satisfies

$$[g(x), g(y), g(z)] = [f(x), g(y), g(z)] + [g(x), f(y), g(z)] + [g(x), g(y), f(z)]$$

for all $x, y, z \in \mathcal{B}$.

Example 1.2 Let $\mathbf{M}_n$ be the collection of all $n \times n$ complex matrices with ternary product $[X, Y, Z] = XY^*Z$ and $g : \mathbf{M}_n \to \mathbf{M}_n$ defined by $g(X) = 3X$ and $f : \mathbf{M}_n \to \mathbf{M}_n$ defined by $f(X) = X$. Then f is a ternary homomorphism and g is a ternary f-hom-der.

The stability problem of functional equations originated from a question of Ulam [33] concerning the stability of group homomorphisms. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [31] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. Park [25, 26] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [3, 4, 5, 7, 9, 10, 12, 13, 14, 19, 30, 35]).

The method provided by Hyers [17] which produces the additive function will be called a direct method. This method is the most significant and strong tool to concerning the stability of different functional equations. That is, the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution [7, 32]. The other significant method is fixed point theorem, that is, the exact solution of the functional equation is explicitly created as a fixed point of some certain map [11, 16, 20, 27, 28, 34].

We remember a fixed point alternative theorem.

Theorem 1.3 [8] *Assume that $(\mathcal{B}, d)$ is a complete generalized metric space and $\mathfrak{I} : \mathcal{B} \to \mathcal{B}$ is a strictly contractive mapping, that is,*

$$d(\mathfrak{I}u, \mathfrak{I}v) \leq Ld(u, v)$$

for all $u, v \in \mathcal{B}$ and a Lipschitz constant $L < 1$. Then for each given element $u \in \mathcal{B}$, either

$$d(\mathfrak{I}^n u, \mathfrak{I}^{n+1} u) = +\infty, \qquad \forall n \geq 0,$$

or

$$d(\mathfrak{I}^n u, \mathfrak{I}^{n+1} u) < +\infty, \qquad \forall n \geq n_0,$$

for some positive integer n_0. Furthermore, if the second alternative holds, then

(i) the sequence $(\mathfrak{I}^n u)$ is convergent to a fixed point v^* of $\mathfrak{I}$;

(ii) v^* is the unique fixed point of $\mathfrak{I}$ in the set $V := \{v \in \mathcal{B}, d(\mathfrak{I}^{n_0} u, v) < +\infty\}$;

(ii) $d(v, v^*) \leq \frac{1}{1-L} d(v, \mathfrak{I} v)$ for all $u, v \in V$.

In this paper, we consider the following system of additive functional equations

$$\begin{cases} 3f(x+y+z) = g(x) + g(y) + g(z) \\ g\left(\frac{x+y+z}{3}\right) = f(x) + f(y) + f(z) \end{cases} \tag{1.2}$$

for all $x, y, z \in \mathcal{B}$.

The aim of the present paper is to solve the system of additive functional equations (1.2) and prove the Hyers-Ulam stability of ternary f-hom-ders in ternary Banach algebras by using the fixed point method.

Throughout this paper, assume that $\mathcal{B}$ is a ternary Banach algebra.

2. Stability of the system of additive functional equations (1.2)

We solve and investigate the system of additive functional equations (1.2) in ternary Banach algebras.

Lemma 2.1 *Let $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying* (1.2) *for all $x, y \in \mathcal{B}$. Then the mappings $f, g : \mathcal{B} \to \mathcal{B}$ are additive.*

Proof. Letting $x = y = z = 0$ in (1.2), we get

$$f(0) = g(0) = 0.$$

Putting $y = z = x$ in (1.2), we have

$$\begin{aligned} 3f(3x) &= 3g(x), \\ g(x) &= 3f(x) \end{aligned} \tag{2.1}$$

and so $f(3x) = 3f(x)$ and $g\left(\frac{x}{3}\right) = g(x)$ for all $x \in \mathcal{B}$. Thus

$$g(x+y) = 3g\left(\frac{x+y+0}{3}\right) = 3f(x) + 3f(y) + 3f(0) = g(x) + g(y)$$

for all $x, y \in \mathcal{B}$. Hence the mapping $g : \mathcal{B} \to \mathcal{B}$ is additive and thus by (2.1) the mapping $f : \mathcal{B} \to \mathcal{B}$ is additive.

Using the fixed point technique, we prove the Hyers-Ulam stability of the system of the additive functional equations (1.2) in complex Banach algebras.

Theorem 2.2 *Suppose that $\Delta : \mathcal{B}^3 \to [0,\infty)$ is a function such that there exists an $L < 1$ with*

$$\Delta\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) \le \frac{L}{3}\Delta(x,y,z) \tag{2.2}$$

for all $x, y, z \in \mathcal{B}$. Let $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying

$$\begin{cases} \|3f(x+y+z) - g(x) - g(y) - g(z)\| \le \Delta(x,y,z) \\ \|g\left(\frac{x+y+z}{3}\right) - f(x) - f(y) - f(z)\| \le \Delta(x,y,z) \end{cases} \tag{2.3}$$

for all $x, y, z \in \mathcal{B}$. Then there exist unique additive mappings $F, G : \mathcal{B} \to \mathcal{B}$ such that

$$\|F(x) - f(x)\| \le \frac{4L}{9(1-L)}\Delta(x,x,x),$$
$$\|G(x) - g(x)\| \le \frac{L+3}{3(1-L)}\Delta(x,x,x)$$

for all $x \in \mathcal{B}$.

Proof. Putting $x = y = z = 0$ in (2.3), we get

$$\begin{cases} \|3f(0) - 3g(0)\| \le \Delta(0,0,0) = 0 \\ \|g(0) - 3g(0)\| \le \Delta(0,0,0) = 0 \end{cases}$$

and so $f(0) = g(0) = 0$.

Letting $y = z = x$ in (2.3), we obtain

$$\begin{cases} \|3f(3x) - 3g(x)\| \le \Delta(x,x,x) \\ \|g(x) - 3f(x)\| \le \Delta(x,x,x) \end{cases} \tag{2.4}$$

Let $\Gamma = \{\gamma : \mathcal{B} \to \mathcal{B} : \gamma(0) = 0\}$. We define a generalized metric on Γ as follows: $d : \Gamma \times \Gamma \to [0,\infty]$ by

$$d(\delta,\gamma) = \inf\{\mu \in \mathbb{R}_+ : \|\delta(x) - \gamma(x)\| \le \mu\Delta(x,x,x), \forall x \in \mathcal{B}\}$$

and we consider $\inf \emptyset = +\infty$. Then d is a complete generalized metric on Γ (see [21]).

Now, we define the mapping $\mathcal{J} : (\Gamma, d) \to (\Gamma, d)$ such that

$$\mathcal{J}\delta(x) := 3\delta\left(\frac{x}{3}\right)$$

for all $x \in \mathcal{B}$.

Let $\delta, \gamma \in (\Gamma, d)$ be given such that $d(\delta,\gamma) = \mu$. Then

$$\|\delta(x) - \gamma(x)\| \le \mu\Delta(x,x,x)$$

for all $x \in \mathcal{B}$. Hence

$$\|\mathcal{J}\delta(x) - \mathcal{J}\gamma(x)\| = \left\|3\delta\left(\frac{x}{3}\right) - 3\gamma\left(\frac{x}{3}\right)\right\| \le 3\mu\Delta\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\right) \le L\mu\Delta(x,x,x)$$

for all $x \in \mathcal{B}$. It follows that $d(\mathcal{J}\delta(x), \mathcal{J}\gamma(x)) \le L\mu$. So

$$d(\mathcal{J}\delta(x), \mathcal{J}\gamma(x)) \le Ld(\delta,\gamma)$$

for all $x \in \mathcal{B}$ and all $\delta, \gamma \in \Gamma$.

Using (2.4), we obtain that

$$\begin{cases} \left\|f(x) - 3f\left(\frac{x}{3}\right)\right\| \le \frac{1}{3}\Delta\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\right) + \Delta\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\right) \le \frac{4L}{9}\Delta(x,x,x) \\ \left\|g(x) - 3g\left(\frac{x}{3}\right)\right\| \le \Delta\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\right) + \Delta(x,x,x) \le \frac{L+3}{3}\Delta(x,x,x) \end{cases}$$

for all $x \in \mathcal{B}$, which imply that $d(f, \mathcal{J}f) \le \frac{4L}{9}$ and $d(g, \mathcal{J}g) \le \frac{L+3}{3}$.

Using the fixed point alternative, we deduce the existence of a unique fixed point of $\mathcal{J}$ and a unique fixed point of $\mathcal{J}$, that is, the existence of mappings $F, G : \mathcal{B} \to \mathcal{B}$, respectively, such that

$$F(x) = 3F\left(\frac{x}{3}\right), \qquad G(x) = 3G\left(\frac{x}{3}\right)$$

with the following property: there exist $\mu, \eta \in (0, \infty)$ satisfying

$$\|f(x) - F(x)\| \le \mu\Delta(x,x,x), \qquad \|g(x) - G(x)\| \le \eta\Delta(x,x,x)$$

for all $x \in \mathcal{B}$.

Since $\lim_{n\to\infty} d(\mathcal{J}^n f, F) = 0$ and $\lim_{n\to\infty} d(\mathcal{J}^n g, G) = 0$,

$$\lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right) = F(x), \qquad \lim_{n\to\infty} 3^n g\left(\frac{x}{3^n}\right) = G(x)$$

for all $x \in \mathcal{B}$.

Next, $d(f, F) \le \frac{1}{1-L} d(f, \mathcal{J}f)$ and $d(g, G) \le \frac{1}{1-L} d(g, \mathcal{J}g)$ which imply

$$\|f(x) - F(x)\| \le \frac{4L}{9(1-L)}\Delta(x,x,x), \qquad \|g(x) - G(x)\| \le \frac{L+3}{3(1-L)}\Delta(x,x,x) \tag{2.5}$$

for all $x \in \mathcal{B}$.

Using (2.2) and (2.3), we conclude that

$$\begin{aligned} &\|3F(x+y+z) - G(x) - G(y) - G(z)\| \\ &= \lim_{n\to\infty} 3^n \left\|3f\left(\frac{x+y+z}{3^n}\right) - g\left(\frac{x}{3^n}\right) - g\left(\frac{y}{3^n}\right) - g\left(\frac{z}{3^n}\right)\right\| \\ &\le \lim_{n\to\infty} 3^n \Delta\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) \le \lim_{n\to\infty} L^n \Delta(x,y,z) = 0 \end{aligned}$$

and

$$\begin{aligned} &\left\|G\left(\frac{x+y+z}{3}\right) - F(x) - F(y) - F(z)\right\| \\ &= \lim_{n\to\infty} 3^n \left\|g\left(\frac{x+y+z}{3^n}\right) - f\left(\frac{x}{3^n}\right) - f\left(\frac{y}{3^n}\right) - f\left(\frac{z}{3^n}\right)\right\| \\ &\le \lim_{n\to\infty} 3^n \Delta\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) \le \lim_{n\to\infty} L^n \Delta(x,y,z) = 0 \end{aligned}$$

for all $x, y, z \in \mathcal{B}$, since $L < 1$. Hence

$$\begin{cases} 3F(x+y+z) = G(x) + G(y) + G(z) \\ G\left(\frac{x+y+z}{3}\right) = F(x) + F(y) + F(z) \end{cases}$$

for all $x, y, z \in \mathcal{B}$. Therefore by Lemma 2.1, the mappings $F, G : \mathcal{B} \to \mathcal{B}$ are additive.

Corollary 2.3 *Let η, p be nonnegative real numbers with $p > 1$ and $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying*

$$\begin{cases} \|3f(x+y+z) - g(x) - g(y) - g(z)\| \le \eta(\|x\|^p + \|y\|^p + \|z\|^p) \\ \|g\left(\frac{x+y+z}{3}\right) - f(x) - f(y) - f(z)\| \le \eta(\|x\|^p + \|y\|^p + \|z\|^p) \end{cases}$$

for all $x, y, z \in \mathcal{B}$. Then there exist unique additive mappings $F, G : \mathcal{B} \to \mathcal{B}$ such that

$$\|F(x) - f(x)\| \le \frac{4\eta}{3^p - 3}\|x\|^p,$$
$$\|G(x) - g(x)\| \le \frac{3(3^p+1)\eta}{3^p - 3}\|x\|^p$$

for all $x \in \mathcal{B}$.

Proof. The proof follows from Theorem 2.2 by taking

$$\Delta(x, y, z) = \eta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in \mathcal{B}$. Choosing $L = 3^{1-p}$, we obtain the desired result.

3. Stability of ternary F-hom-ders in ternary Banach algebras

In this section, by using the fixed point technique, we prove the Hyers-Ulam stability of ternary F-hom-ders in ternary Banach algebras.

Lemma 3.1 [24] *Let $\mathcal{B}$ be a ternary Banach algebra and $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ be an additive mapping such that $\mathcal{F}(\alpha x) = \alpha\mathcal{F}(x)$ for all $\alpha \in \mathbb{T}^1 := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ and all $x \in \mathcal{B}$. Then $\mathcal{F}$ is $\mathbb{C}$-linear.*

Lemma 3.2 *Let $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying*

$$\begin{cases} 3f(\lambda(x+y+z)) = \lambda(g(x) + g(y) + g(z)) \\ g\left(\lambda\frac{x+y+z}{3}\right) = \lambda(f(x) + f(y) + f(z)) \end{cases} \tag{3.1}$$

for all $x, y, z \in \mathcal{B}$ and all $\lambda \in \mathbb{T}^1$. Then the mappings $f, g : \mathcal{B} \to \mathcal{B}$ are $\mathbb{C}$-linear.

Proof. If we put $\lambda = 1$ in (3.1), then f and g are additive by Lemma 2.1.

Letting $y = z = x$ in (3.1), we have

$$\begin{cases} 3f(3\lambda x) = 3\lambda g(x) \\ g(\lambda x) = 3\lambda f(x) \end{cases}$$

for all $x \in \mathcal{B}$ and all $\lambda \in \mathbb{T}^1$. Since the mappings f and g are additive,

$$\begin{cases} f(\lambda x) = \lambda f(x) \\ g(\lambda x) = \lambda g(x) \end{cases}$$

for all $x \in \mathcal{B}$ and all $\lambda \in \mathbb{T}^1$. So by Lemma 3.1, the mappings f and g are $\mathbb{C}$-linear.

Theorem 3.3 *Suppose that $\Delta : \mathcal{B}^3 \to [0, \infty)$ is a function such that there exists an $L < 1$ with*

$$\Delta(x, y, z) \le \frac{L}{27}\Delta(3x, 3y, 3z) \tag{3.2}$$

for all $x, y, z \in \mathcal{B}$. *Let* $f, g : \mathcal{B} \to \mathcal{B}$ *be mappings satisfying*

$$\begin{cases} \|3f(\lambda(x+y+z)) - \lambda(g(x)+g(y)+g(z))\| \leq \Delta(x,y,z) \\ \|g\left(\lambda\frac{x+y+z}{3}\right) - \lambda(f(x)+f(y)+f(z))\| \leq \Delta(x,y,z) \end{cases}, \tag{3.3}$$

$$\|f([x,y,z]) - [f(x), f(y), f(z)]\| \leq \Delta(x,y,z), \tag{3.4}$$

$$\begin{aligned} &\|[g(x),g(y),g(z)] - [f(x),g(y),g(z)] - [g(x),f(y),g(z)] - [g(x),g(y),f(z)]\| \\ &\leq \Delta(x,y,z) \end{aligned} \tag{3.5}$$

for all $x, y, z \in \mathcal{B}$ *and all* $\lambda \in \mathbb{T}^1$. *Then there exist unique mappings* $F, G : \mathcal{B} \to \mathcal{B}$ *such that* F *is a ternary homomorphism and* G *is a ternary* F*-hom-der and*

$$\|F(x) - f(x)\| \leq \frac{4L}{9(1-L)}\Delta(x,x,x), \tag{3.6}$$

$$\|G(x) - g(x)\| \leq \frac{L+3}{3(1-L)}\Delta(x,x,x) \tag{3.7}$$

for all $x \in \mathcal{B}$.

Proof. Let $\lambda = 1$ in (3.3). By Theorem 2.2, there are unique additive mappings $F, G : \mathcal{B} \to \mathcal{B}$ satisfying (3.6) and (3.7) given by

$$\lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right) = F(x), \qquad \lim_{n\to\infty} 3^n g\left(\frac{x}{3^n}\right) = G(x)$$

for all $x \in \mathcal{B}$.

Using (3.2) and (3.3), we conclude that

$$\begin{aligned} &\|3F(\lambda(x+y+z)) - \lambda(G(x)+G(y)+G(z))\| \\ &= \lim_{n\to\infty} 3^n \left\| 3f\left(\frac{\lambda(x+y+z)}{3^n}\right) - \lambda\left(g\left(\frac{x}{3^n}\right) + g\left(\frac{y}{3^n}\right) + g\left(\frac{z}{3^n}\right)\right)\right\| \\ &\leq \lim_{n\to\infty} 3^n \Delta\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) \leq \lim_{n\to\infty} \frac{L^n}{9^n}\Delta(x,y,z) = 0 \end{aligned}$$

and

$$\begin{aligned} &\left\|G\left(\lambda\frac{x+y+z}{3}\right) - \lambda(F(x)+F(y)+F(z))\right\| \\ &= \lim_{n\to\infty} 3^n \left\| g\left(\frac{\lambda(x+y+z)}{3^n}\right) - \lambda\left(f\left(\frac{x}{3^n}\right) + f\left(\frac{y}{3^n}\right) + f\left(\frac{z}{3^n}\right)\right)\right\| \\ &\leq \lim_{n\to\infty} 3^n \Delta\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) \leq \lim_{n\to\infty} \frac{L^n}{9^n}\Delta(x,y,z) = 0 \end{aligned}$$

for all $x, y, z \in \mathcal{B}$, since $L < 1$. Hence

$$\begin{cases} 3F(\lambda(x+y+z)) = \lambda(G(x)+G(y)+G(z)) \\ G\left(\lambda\frac{x+y+z}{3}\right) = \lambda(F(x)+F(y)+F(z)) \end{cases}$$

for all $x, y, z \in \mathcal{B}$ and all $\lambda \in \mathbb{T}^1$. Therefore by Lemma 3.2, the mappings $F, G : \mathcal{B} \to \mathcal{B}$ are $\mathbb{C}$-linear.

It follows from (3.4) that

$$\begin{aligned}
&\|F([x,y,z]) - [F(x),F(y),F(z)]\| \\
&= \lim_{n\to\infty} 27^n \left\| f\left(\frac{[x,y,z]}{27^n}\right) - \left[f\left(\frac{x}{3^n}\right), f\left(\frac{y}{3^n}\right), f\left(\frac{z}{3^n}\right)\right]\right\| \\
&\le \lim_{n\to\infty} 27^n \Delta\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) \\
&\le \lim_{n\to\infty} L^n \Delta(x,y,z) = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{B}$. So

$$F([x,y,z]) = [F(x),F(y),F(z)]$$

for all $x, y, z \in \mathcal{B}$. Thus the $\mathbb{C}$-linear mapping F is a ternary homomorphism.

It follows from (3.5) that

$$\begin{aligned}
&\|[G(x),G(y),G(z)] - [F(x),G(y),G(z)] - [G(x),F(y),G(z)] - [G(x),G(y),F(z)]\| \\
&= \lim_{n\to\infty} 27^n \left(\left\| \left[g\left(\frac{x}{3^n}\right), g\left(\frac{y}{3^n}\right), g\left(\frac{z}{3^n}\right)\right] - \left[f\left(\frac{x}{2^n}\right), g\left(\frac{y}{2^n}\right), g\left(\frac{z}{2^n}\right)\right] \right.\right. \\
&\qquad \left.\left. - \left[g\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), g\left(\frac{z}{2^n}\right)\right] - \left[g\left(\frac{x}{2^n}\right), g\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right] \right\| \right) \\
&\le \lim_{n\to\infty} 27^n \Delta\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) \\
&\le \lim_{n\to\infty} L^n \Delta(x,y,z) = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{B}$. So

$$[G(x),G(y),G(z)] = [F(x),G(y),G(z)] + [G(x),F(y),G(z)] + [G(x),G(y),F(z)]$$

for all $x, y, z \in \mathcal{B}$. Thus the $\mathbb{C}$-linear mapping G is a ternary F-hom-der.

Remark 3.4 In Theorem 3.3, $G(x) = 3F(x)$. So F is a ternary homomorphism if and only if G is a ternary F-hom-der.

Corollary 3.5 *Let p, q, r and η be nonnegative real numbers with $p + q + r > 3$ and $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying*

$$\begin{cases} \|f(\lambda(x+y)) + g(\lambda(y-x)) - 2\lambda f(x)\| \le \eta \|x\|^p \|y\|^q \|z\|^r \\ \|g(\lambda(x+y)) - f(\lambda(y-x)) - 2\lambda g(y)\| \le \eta \|x\|^p \|y\|^q \|z\|^r \end{cases}$$

and

$$\begin{aligned}
&\|f([x,y,z]) - [f(x),f(y),f(z)]\| \le \|x\|^p \|y\|^q \|z\|^r, \\
&\|[g(x),g(y),g(z)] - [f(x),g(y),g(z)] - [g(x),f(y),g(z)] - [g(x),g(y),f(z)]\| \\
&\qquad \le \|x\|^p \|y\|^q \|z\|^r
\end{aligned}$$

for all $x, y, z \in \mathcal{B}$ and all $\lambda \in \mathbb{T}^1$. Then there exist unique mappings $F, G : \mathcal{B} \to \mathcal{B}$ such that F is a ternary homomorphism and G is a teranry F-hom-der and

$$\|F(x) - f(x)\| \le \frac{4\eta}{3(3^{p+q+r} - 3)} \|x\|^{p+q+r},$$

$$\|G(x) - g(x)\| \le \frac{(3^{p+q+r} + 1)\eta}{3^{p+q+r} - 3} \|x\|^{p+q+r}$$

for all $x \in \mathcal{B}$.

Proof. The proof follows from Theorem 3.3 by taking $\Delta(x, y, z) = \eta\|x\|^p\|y\|^q\|z\|^r$ for all $x, y, z \in \mathcal{B}$. Choosing $L = 3^{3-p-q-r}$, we obtain the desired result.

References

[1] M. Amyari, M.S. Moslehian, *Approximate homomorphisms of ternary semigroups*, Lett. Math. Phys., **77**(2006), 1-9.

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2**(1950), 64-66.

[3] M. Dehghanian, S.M.S. Modarres, *Ternary γ-homomorphisms and ternary γ-derivations on ternary semigroups*, J. Inequal. Appl., **2012**(2012), Paper No. 34.

[4] M. Dehghanian, S.M.S. Modarres, C. Park, D. Shin, *C^*-Ternary 3-derivations on C^*-ternary algebras*, J. Inequal. Appl., **2013**(2013), Paper No. 124.

[5] M. Dehghanian, C. Park, *C^*-Ternary 3-homomorphisms on C^*-ternary algebras*, Results Math., **66**(2014), no. 3, 385-404.

[6] M. Dehghanian, C. Park, Y. Sayyari, *On the stability of hom-der on Banach algebras* (preprint).

[7] M. Dehghanian, Y. Sayyari, C. Park, *Hadamard homomorphisms and Hadamard derivations on Banach algebras*, Miskolc Math. Notes, **24**(2023), no. 1, 129-137.

[8] J.B. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74**(1968), 305-309.

[9] N. Eghbali, J.M. Rassias, M. Taheri, *On the stability of a k-cubic functional equation in intuitionistic fuzzy n-normed spaces,* Results Math., **70**(2016), 233-248.

[10] I. El-Fassi, J. Brzdęk, A. Chahbi, S. Kabbaj, *On hyperstability of the biadditive functional equation*, Acta Math. Sci. Ser. B (Engl. Ed.), **37**(2017), 1685-1694.

[11] M. Eshaghi Gordji, H. Khodaei, Th. M. Rassias, R. Khodabakhsh, *J^*-homomorphisms and J^*-derivations on J^*-algebras for a generalized Jensen type functional equation*, Fixed Point Theory, **13**(2012), no. 2, 481-494.

[12] W. Fechner, *Stability of a functional inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math., **71**(2006), 149-161.

[13] M. Fošner, B. Marcen, J. Vukman, *On functional equation related to a class of generalized inner derivations in prime rings*, Bull. Malays. Math. Sci. Soc., **41**(2018), 687-707.

[14] M. Fošner, J. Vukman, *On some functional equations arising from (m, n)-Jordan derivations and commutativity of prime rings*, Rocky Mountain J. Math., **42**(2012), 1153-1168.

[15] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184**(1994), 431-436.

[16] I. Hwang, C. Park, *Bihom derivations in Banach algebras*, J. Fixed Point Theory Appl., **21**(2019), Paper No. 81.

[17] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27**(1941), 222-224.

[18] A. Kheawborisuk, S. Paokanta, J. Senasukh, C. Park, *Ulam stability of hom-ders in fuzzy Banach algebars*, AIMS Math. **7**(2022), no. 9, 16556-16568.

[19] H. Khodaei, *On the stability of additive, quadratic, cubic and quartic set-valued functional equations,* Results Math., **68**(2015), 1-10.

[20] G. Lu, C. Park, *Hyers-Ulam stability of general Jensen-type mappings in Banach algebras*, Results Math., **66**(2014), 87-98.

[21] D. Mihet, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl., **343**(2008), 567-572.

[22] M. Mirzavaziri, M.S. Moslehian, *Automatic continuity of σ-derivations on C^*-algebras*, Proc. Amer. Math. Soc., **134**(2006), 3319-3327.

[23] M.S. Moslehian, *Almost derivations on C^*-ternary rings*, Bull. Belg. Math. Soc.-Simon Stevin, **14**(2006), 135-142.

[24] C. Park, *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc., **36**(2005), 79-97.

[25] C. Park, *Additive ρ-functional inequalities and equations*, J. Math. Inequal., **9** (2015), 17-26.

[26] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal., **9**(2015), 397-407.

[27] C. Park, *An additive (α, β)-functional equation and linear mappings in Banach spaces*, J. Fixed Point Theory Appl., **18**(2016), 495-504.

[28] C. Park, *The stability of an additive (ρ_1, ρ_2)-functional inequality in Banach spaces*, J. Math. Inequal., **13**(2019), 95-104.

[29] C. Park, J. Lee, X. Zhang, *Additive s-functional inequality and hom-derivations in Banach algebras*, J. Fixed Point Theory Appl., **21**(2019), Paper No. 18.

[30] C. Park, J.M. Rassias, A. Bodaghi, S. Kim, *Approximate homomorphisms from ternary semigroups to modular spaces*, Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **113**(2019), 2175-2188.

[31] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.

[32] Y. Sayyari, M. Dehghanian, C. Park, J. Lee, *Stability of hyper homomorphisms and hyper derivations in complex Banach algebras*, AIMS Math., **7**(2022), no. 6, 10700-10710.

[33] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ., New York, 1960.

[34] M.R. Velayati, N. Salehi, *Stability of maximum preserving functional equation on multi-Banach lattice by fixed point method*, Fixed Point Theory, **21**(2020), no. 1, 363-373.

[35] C. Wang, T.Z. Xu, *Solution and Hyers-Ulam-Rassias stability of a mixed type quadratic-additive functional equation with a parameter in quasi-Banach spaces*, Acta Math. Sci. Ser. B (Chin. Ed.), **37**(2017), 846-859.

[36] H. Zettl, *A characterization of ternary rings of operators*, Adv. Math., **48**(1983), 117-143.

Received: November 14, 2022; Accepted: April 7, 2026.