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POSITIVE SOLUTION FOR A SYSTEM OF INTEGRAL EQUATIONS VIA KRASNOSEL'SKIĬ-GUO THEOREM

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Abstract. The paper deals with the existence of positive solution to the system of nonlinear Hammerstein integral equations. The main tool used in the proof is Krasnosel'skiĭ-Guo fixed point theorem of compression-expansion cones. An application of specific cone enabled to achieve the results. Some examples are provided in order to illustrate the applicability of presented theorems.
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1. Introduction

In this paper we deal with the system of integral equations of Hammerstein type:

$$\begin{cases} u_1(t) = \int_0^1 k_1(t,s) f_1(s, u_1(s), u_2(s)) ds, \\ \\ u_2(t) = \int_0^1 k_2(t,s) f_2(s, u_1(s), u_2(s)) ds, \end{cases} \tag{1.1}$$

where $k_i : [0,1]^2 \to [0,+\infty)$ and $f_i : [0,1] \times [0,+\infty)^2 \to [0,+\infty)$ are continuous functions for $i = 1, 2$. This type of equations are often used in applied mathematics, we refer the reader for more information to monographs [2] and [9]. Citing just a few examples from the literature, we mention [1], where authors illustrate theoretical results by models of chemical reactors, beams and thermostats. Interesting, from the view of applications, are results contained in [8], where Minhós and Sousa considered a system of integral equations of Hammerstein type that both equations and both variables can have a different regularity. This allows for the specific applications to phenomena modeled by coupled systems composed of differential equations of different orders and distinct boundary conditions.

Existence and localization of positive solutions for a system of Hammerstein integral equations is one of the most studied topics in this family of equations. It will be also the aim of this paper.

In [5] authors establish a localization of the solution of the following operator system

$$\begin{cases} u_1 = T_1(u_1, u_2), \\ u_2 = T_2(u_1, u_2), \end{cases}$$

assuming that for each fixed u_2, the operator $T_1(\cdot, u_2)$ satisfies compression-expansion type conditions in the line of Krasnosel'skiĭ result and for each fixed u_1, the operator $T_2(u_1, \cdot)$ fulfills the hypotheses of the Schauder fixed point theorem. This hybrid approach was applied to a Hammerstein system of the same form as (1.1) and yields a solution with the first component localized in a conical shell and the second one in a certain closed convex set.

A similar hybrid technique has been applied in paper [12]. The author considered the following system of second-order equations with Dirichlet boundary conditions

$$\begin{cases} -u''(t) = f(t, u, v), \\ -v''(t) = g(t, u, v), \\ u(0) = u(1) = 0 = v(0) = v(1), \end{cases}$$

where $f : [0, 1] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ and $g : [0, 1] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions. To obtain solution (u, v), the above system was equivalently rewritten in the form of

$$\begin{cases} u(t) = \int_0^1 G(t, s) f(s, u(s), v(s)) ds, \\ \\ v(t) = \int_0^1 G(t, s) g(s, u(s), v(s)) ds, \end{cases}$$

where $G : [0, 1] \times [0, 1] \to \mathbb{R}$ denotes the corresponding Green's function. The localization of u is obtained as an application of Krasnosel'skiĭ-Guo compression-expansion fixed point theorem in cones, while v is located between a couple of well-ordered lower and upper solutions.

Krasnosel'skiĭ-Guo fixed point theorem is also applied to obtain positive solutions of iterative systems with nonlinear boundary value problems. It is worth mentioning recently published paper [10], which is devoted to determining the eigenvalue intervals of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ for which there exist positive solutions of the iterative systems of second-order equations with impulses and m-point boundary conditions.

Yang and O'Regan in [15] study the existence of positive continuous solutions to specific system of nonlinear Hammerstein integral equations

$$\begin{cases} u(t) = \int_D k(t, s) f(s, u(s), v(s)) ds, \\ \\ v(t) = \int_D k(t, s) g(s, u(s), v(s)) ds, \end{cases}$$

where $D \subset \mathbb{R}^n$ is a bounded closed domain, $k \in C(D \times D, \mathbb{R}_+)$ and $f, g \in C(D \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$. Under the very conditions which characterize above system and via fixed point index theory in a cone authors establish existence and multiplicity results.

Whereas in [7], Lan established new coexistence fixed point theorems in product Banach spaces via the classical fixed point index theory for compact maps defined on cones. These results are applied to obtain at least one nonnegative coexistence solution of system of Hammerstein integral equations of the form

$$u_i(t) = \int_0^1 k_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s)) ds,$$

where $i \in \{1, \dots, n\}$, for each $t \in [0,1]$ the kernel $k_i(t,\cdot) : [0,1] \to \mathbb{R}_+$ is measurable and $f_i : [0,1] \times \mathbb{R}_+^n \to \mathbb{R}_+$ satisfies the Carathéodory conditions.

Also in [16] authors are concerned with the existence and multiplicity of positive solutions for the following system

$$u_i(t) = \int_a^b k_i(t,s) g_i(s) f_i(s, u_1(s), u_2(s), \dots, u_n(s)) ds,$$

where $i \in \{1, \dots, n\}$, $f_i \in C([a,b] \times [0,+\infty)^n, [0,+\infty))$, $g_i \in L[a,b]$ is almost everywhere positive on $[a,b]$, and $k_i \in C([a,b] \times [a,b], [0,+\infty))$.

The paper's main structure is as follows. Section 2 presents necessary theoretical background and some preliminary assumptions. Section 3 contains the main results: the existence and localization theorems. In the last section some examples are provided in order to illustrate the application of results.

2. Preliminaries

Our approach is based on the well-known Krasnosel'skiĭ-Guo theorem of compression-expansion cones. Let us remind that a non-empty subset K of a Banach space X is called a cone if K is closed, convex and $\lambda u \in K$ for $u \in K$ and $\lambda \geq 0$ and $K \cap (-K) = \{0\}$.

Theorem 2.1 [3] *Let K be a cone in a Banach space X and suppose that Ω_1 and Ω_2 are bounded open sets in X such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $F : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that one of the following conditions holds:*

(i) $\|Fx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_1$ and $\|Fx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_2$,

(ii) $\|Fx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1$ and $\|Fx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_2$.

Then F has a fixed point in the set $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We will work in the Banach space $X = C[0,1] \times C[0,1]$ and for $u = (u_1, u_2) \in X$ we make use of the norm

$$\|u\| = \max\{\|u_1\|_\infty, \|u_2\|_\infty\},$$

where $\|\cdot\|_\infty$ is the supremum norm in the space $C[0,1]$.

In the sequel, for $i = 0, 1$ and $I \subset [0,1]$, we make the following assumptions on the terms that occur in (1.1). There exists continuous function $\Phi_i : [0,1] \to [0,+\infty)$ such that:

(A_1) $\underset{t\in[0,1]}{\forall} \ \underset{s\in[0,1]}{\forall} \quad k_i(t,s) \leq \Phi_i(s),$

(A_2) $\underset{c_i>0}{\exists} \ \underset{t\in I}{\forall} \ \underset{s\in[0,1]}{\forall} \quad k_i(t,s) \geq c_i \Phi_i(s),$

(A_3) $\underset{d_1>0}{\exists} \underset{s\in[0,1]}{\forall} \quad \Phi_1(s) \geq d_1\Phi_2(s),$

(A_4) $\underset{d_2>0}{\exists} \underset{s\in[0,1]}{\forall} \quad \Phi_2(s) \geq d_2\Phi_1(s).$

Moreover

(A_5) $\underset{\rho>0}{\exists} \underset{e_1>0}{\exists} \underset{s\in[0,1]}{\forall} \underset{u_i\in[0,\rho]}{\forall} \quad f_1(s,u_1,u_2) \geq e_1 f_2(s,u_1,u_2),$

(A_6) $\underset{\rho>0}{\exists} \underset{e_2>0}{\exists} \underset{s\in[0,1]}{\forall} \underset{u_i\in[0,\rho]}{\forall} \quad f_2(s,u_1,u_2) \geq e_2 f_1(s,u_1,u_2).$

We are working with the cone

$$K = \{(u_1,u_2) \in X : \quad u_i(t) \geq 0, \quad \min_{t\in I} u_i(t) \geq c\|u\|, \quad i=1,2\}, \tag{2.1}$$

here c is a positive constant defined as a

$$c = \min\{c_i, c_i d_i e_i\}, \quad i=1,2.$$

An application of cone (2.1) is crucial because it enables to obtain, in a simple way, the positive solution of the considered system (1.1). By a term positive solution we mean function $u=(u_1,u_2)$, that solves (1.1), with both components u_1, u_2 nontrivial and positive on the interval I. To the best of our knowledge this specific cone has not been previously used for this purpose. For the first time we exploited it in our recent papers [13] and [14] to establish sufficient conditions for the existence of positive increasing solutions to a third order differential equation subject to nonlinear and nonlocal boundary conditions. The main advantage in using the cone (2.1) is that the minimum of the functions u_1 and u_2 on the interval I is compared with the norm $\|u\|$ instead of $\|u_i\|_\infty$, which in turn, gives us important information about localization of both components u_1 and u_2, i.e.,

$$\min_{t\in I} u_i(t) \geq c\|u\| > 0.$$

In comparison, in a vector version of Krasnosel'skiĭ theorem presented in [11] author considers cones

$$K_i = \{v \in P : v(t) \geq M_i\|v\|_\infty \text{ for all } t \in \mathbb{R}\}, \quad i=1,2,$$

where P is the cone of all nonnegative functions from space $X = \{v \in C(\mathbb{R},\mathbb{R}) : v(t) = v(t+\omega),\ \omega > 0\}$, $M_i \in (0,1)$ is some coefficient, and the corresponding cone is $K = K_1 \times K_2$. As a result, there were obtained positive ω-periodic solution of the first-order differential equation system. Also in the papers [5] and [7] mentioned earlier in the introduction, where a vector approach is employed as well, the common feature is that in every definition of exploited cones appears only standard supremum norm. Thus the results obtained in our paper differ from those mentioned.

Note that assumptions $(A_1), (A_2)$ often occur in the related literature. Further, less typical are (A_3) and (A_4), which require a link between the bounds for the kernels k_1 and k_2, it basically comes down to comparing the functions of one variable. The most restrictive seem to be the last assumptions regarding the relationship between functions f_1 and f_2, but these requirements (A_5) and (A_6) are necessary to exploit our cone (2.1).

3. Main results

Let $0 < r_1 < r_2$. To formulate the existence theorem we will exploit the following conditions:

(C_1) $c_1 \min\limits_{I\times[cr_1,r_1]^2} f_1(s,u_1,u_2)\int_I \Phi_1(s)ds \geq r_1$,

(C_2) $c_2 \min\limits_{I\times[cr_1,r_1]^2} f_2(s,u_1,u_2)\int_I \Phi_2(s)ds \geq r_1$,

(C_3) $\max\limits_{[0,1]\times[0,r_2]^2} f_1(s,u_1,u_2)\int_0^1 \Phi_1(s)ds \leq r_2$,

(C_4) $\max\limits_{[0,1]\times[0,r_2]^2} f_2(s,u_1,u_2)\int_0^1 \Phi_2(s)ds \leq r_2$.

Finally we are ready to present and prove our main results.

Theorem 3.1 *Let assumptions $(A_1)-(A_6)$ be fulfilled. Assume that there exist r_1, r_2, such that conditions (C_3), (C_4) hold and (C_1) or (C_2) are satisfied. Then problem* (1.1) *has a positive solution $u = (u_1, u_2)$, such that $r_1 \leq \|(u_1, u_2)\| \leq r_2$. Moreover* $\min\limits_{t\in I} u_i(t) \geq cr_1,\ i = 1, 2.$

Proof. Let us remind that K is a cone given by (2.1) and for $(u_1, u_2) \in K$ we define operator:

$$T = (T_1, T_2), \quad \text{where} \quad T_i(u_1,u_2)(t) = \int_0^1 k_i(t,s)f_i(s,u_1(s),u_2(s))ds, \quad i = 1,2.$$

Our goal is to prove that the operator T has a fixed point that, in turn, is a solution of the considered system (1.1).

Let $0 < r_1 < r_2 \leq \rho$, $\Omega_\rho = \{u \in X : \|u\| < \rho\}$, $\Omega_1 = \{u \in X : \|u\| < r_1\}$, and $\Omega_2 = \{u \in X : \|u\| < r_2\}$. Note that $(K \cap \overline{\Omega}_2 \setminus \Omega_1) \subset (K \cap \overline{\Omega}_\rho)$. We will show that $T(K \cap \overline{\Omega}_\rho) \subset K$. By (A_1), for $i = 1, 2$, we have

$$\|T_i(u_1,u_2)\|_\infty \leq \int_0^1 \Phi_i(s)f_i(s,u_1(s),u_2(s))ds,$$

then

$$\|T(u_1,u_2)\| \leq \max\{\int_0^1 \Phi_i(s)f_i(s,u_1(s),u_2(s))ds,\ i = 1,2\}.$$

On the other hand, from (A_2) we obtain for $u = (u_1, u_2) \in K \cap \overline{\Omega}_\rho$, $t \in I$ and $s \in [0, 1]$

$$\begin{aligned} T_i(u_1,u_2)(t) &= \int_0^1 k_i(t,s)f_i(s,u_1(s),u_2(s))ds \geq \int_0^1 c_i\Phi_i(s)f_i(s,u_1(s),u_2(s))ds \\ &\geq c\int_0^1 \Phi_i(s)f_i(s,u_1(s),u_2(s))ds \geq c\|T_i(u_1,u_2)\|_\infty, \end{aligned}$$

hence

$$\min_{t\in I} T_i(u_1,u_2)(t) \geq c\|T_i(u_1,u_2)\|_\infty.$$

Moreover, by $(A_2), (A_3)$ and (A_5), we get

$$\begin{aligned}T_1(u_1,u_2)(t) &= \int_0^1 k_1(t,s)f_1(s,u_1(s),u_2(s))ds \geq \int_0^1 c_1\Phi_1(s)e_1f_2(s,u_1(s),u_2(s))ds \\ &\geq \int_0^1 c_1d_1\Phi_2(s)e_1f_2(s,u_1(s),u_2(s))ds \geq c\int_0^1 \Phi_2(s)f_2(s,u_1(s),u_2(s))ds \\ &\geq c\|T_2(u_1,u_2)\|_\infty.\end{aligned}$$

Thus

$$\min_{t\in I} T_1(u_1,u_2)(t) \geq c\|T_2(u_1,u_2)\|_\infty.$$

On the other hand, from $(A_2), (A_4)$ and (A_6), we obtain

$$\begin{aligned}T_2(u_1,u_2)(t) &= \int_0^1 k_2(t,s)f_2(s,u_1(s),u_2(s))ds \geq \int_0^1 c_2\Phi_2(s)e_2f_1(s,u_1(s),u_2(s))ds \\ &\geq \int_0^1 c_2d_2\Phi_1(s)e_2f_1(s,u_1(s),u_2(s))ds \geq c\int_0^1 \Phi_1(s)f_1(s,u_1(s),u_2(s))ds \\ &\geq c\|T_1(u_1,u_2)\|_\infty.\end{aligned}$$

Therefore

$$\min_{t\in I} T_2(u_1,u_2)(t) \geq c\|T_1(u_1,u_2)\|_\infty.$$

Finally, we can conclude that

$$\min_{t\in I} T(u_1,u_2)(t) \geq c\max\{\|T_1(u_1,u_2)\|_\infty, \|T_2(u_1,u_2)\|_\infty\} = c\|T(u_1,u_2)\|,$$

and so $T(K\cap\overline{\Omega}_\rho)\subset K$.

Obviously $(0,0)\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$. Applying the Ascoli-Arzelà theorem, we can prove that T is a completely continuous operator on $K\cap(\overline{\Omega}_2\setminus\Omega_1)$.

If $u\in K\cap\partial\Omega_2$, then $\|u\| = r_2$ and $u_i\in[0,r_2]$ for $i=1,2$. From (A_1), (C_3) and (C_4)

$$\begin{aligned}T_i(u_1,u_2)(t) &\leq \int_0^1 \Phi_i(s)f_i(s,u_1(s),u_2(s))ds \\ &\leq \max_{[0,1]\times[0,r_2]^2} f_i(s,u_1,u_2)\int_0^1 \Phi_i(s)ds \leq r_2.\end{aligned}$$

Therefore

$$\max\{\|T_i(u_1,u_2)\|_\infty,\ i=1,2\} \leq r_2,$$

$$\|T(u)\| \leq \|u\|.$$

If $u\in K\cap\partial\Omega_1$, then $\|u\| = r_1$ and $u_i(t)\in[cr_1,r_1]$ for all $t\in I$, $i=1,2$. To obtain reverse inequality $\|T(u)\|\geq\|u\|$, it is enough to prove

$$\max\{\|T_i(u_1,u_2)\|_\infty,\ i=1,2\} \geq r_1.$$

If (C_1) and (A_2) hold, we get for $t \in I$

$$\begin{aligned} T_1(u_1,u_2)(t) &\geq \int_I c_1\Phi_1(s) f_1(s,u_1(s),u_2(s))ds \\ &\geq c_1 \min_{I\times[cr_1,r_1]^2} f_1(s,u_1,u_2)\int_I \Phi_1(s)ds \geq r_1. \end{aligned}$$

If (C_2) is satisfied, together with (A_2), we obtain

$$\begin{aligned} T_2(u_1,u_2)(t) &\geq \int_I c_2\Phi_2(s) f_2(s,u_1(s),u_2(s))ds \\ &\geq c_2 \min_{I\times[cr_1,r_1]^2} f_2(s,u_1,u_2)\int_I \Phi_2(s)ds \geq r_1. \end{aligned}$$

Thus $\|T_1(u_1,u_2)\|_\infty \geq r_1$ or $\|T_2(u_1,u_2)\|_\infty \geq r_1$, which gives

$$\|T(u)\| \geq r_1 = \|u\|.$$

An application of Theorem 2.1 (i) completes the proof.

Taking into account the conditions below we can prove, in a similar way, the second existence theorem.

(D_1) $c_1 \min\limits_{I\times[cr_2,r_2]^2} f_1(s,u_1(s),u_2(s)) \int_I \Phi_1(s)ds \geq r_2,$

(D_2) $c_2 \min\limits_{I\times[cr_2,r_2]^2} f_2(s,u_1(s),u_2(s)) \int_I \Phi_2(s)ds \geq r_2,$

(D_3) $\max\limits_{[0,1]\times[0,r_1]^2} f_1(s,u_1(s),u_2(s)) \int_0^1 \Phi_1(s)ds \leq r_1,$

(D_4) $\max\limits_{[0,1]\times[0,r_1]^2} f_2(s,u_1(s),u_2(s)) \int_0^1 \Phi_2(s)ds \leq r_1.$

Theorem 3.2 *Let assumptions $(A_1)-(A_6)$ be fulfilled. Assume that there exist r_1, r_2, such that conditions (D_3), (D_4) hold and (D_1) or (D_2) are satisfied. Then problem* (1.1) *has a positive solution (u_1,u_2), such that $r_1 \leq \|(u_1,u_2)\| \leq r_2$. Moreover* $\min\limits_{t\in I} u_i(t) \geq cr_1,\ i=1,2.$

4. Examples

As a first example we will consider the system of second order differential equations with Dirichlet boundary conditions.

Example 4.1

$$\begin{cases} -u_1'' + m^2u_1 = 2 - t\sin(u_1+u_2), \\ -u_2'' = \sqrt{t^2+1}(e^{-u_1} + \sqrt{u_2}), \\ u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0, \end{cases} \tag{4.1}$$

where $m > 0$. The kernel functions corresponding to above system are

$$k_1(t,s) = \frac{1}{m\sinh m}\begin{cases} \sinh(ms)\sinh[m(1-t)], & s \leq t, \\ \sinh(mt)\sinh[m(1-s)], & s > t \end{cases}$$

for the first equation and

$$k_2(t,s) = \begin{cases} s(1-t), & s \le t, \\ t(1-s), & s > t \end{cases}$$

for the second one.

Assumptions $(A_1), (A_2)$ are satisfied for

$$\Phi_1(s) = \frac{\sinh(ms)\sinh[m(1-s)]}{m\sinh m} \quad \text{and} \quad c_1(t) = \min_{t\in I}\left\{\frac{\sinh(mt)}{\sinh m}, \frac{\sinh[m(1-t)]}{\sinh m}\right\},$$

$$\Phi_2(s) = s(1-s) \quad \text{and} \quad c_2(t) = \min_{t\in I}\{t, 1-t\}.$$

(A_3) will be satisfied if

$$\sinh(ms)\sinh[m(1-s)] - d_1 s(1-s)m\sinh m \ge 0.$$

Hence, for $s \in [0,1]$, consider the auxiliary function

$$g_1(s) = \sinh(ms)\sinh[m(1-s)] - d_1 s(1-s)m\sinh m.$$

Note that $g_1(0) = g_1(1) = 0$ and study its first and second order derivative

$$g_1'(s) = m\cosh(ms)\sinh[m(1-s)] - m\sinh(ms)\cosh[m(1-s)] - d_1(1-2s)m\sinh m,$$

$$g_1''(s) = -2m^2\cosh[m(2s-1)] + 2d_1 m\sinh m.$$

Requiring the concavity of g_1, we obtain

$$d_1\sinh m \le m\cosh[m(2s-1)]$$

and by minimizing right-hand side we get

$$d_1 = \frac{m}{\sinh m}.$$

To fulfill (A_4) we look for constant d_2 such that

$$s(1-s)m\sinh m - d_2\sinh(ms)\sinh[m(1-s)] \ge 0.$$

Similarly, for $s \in [0,1]$ consider function

$$g_2(s) = s(1-s)m\sinh m - d_2\sinh(ms)\sinh[m(1-s)].$$

Then

$$g_2'(s) = (1-2s)m\sinh m - d_2[m\cosh(ms)\sinh[m(1-s)] - m\sinh(ms)\cosh[m(1-s)]],$$

$$g_2''(s) = -2m\sinh m + 2m^2 d_2\cosh[m(2s-1)].$$

Taking into account that $g_2(0) = g_2(1) = 0$ and assuming the concavity of g_2, we lead to

$$md_2\cosh[m(2s-1)] \le \sinh m,$$

which, in turn, gives

$$d_2 = \frac{\sinh m}{m\cosh m}.$$

For $u_i \in [0, r_2]$, where $i = 1,2$ and $t \in [0,1]$ assumptions (A_5) and (A_6) read

$$2 - t\sin(u_1+u_2) \ge e_1\sqrt{t^2+1}(e^{-u_1} + \sqrt{u_2}),$$

$$\sqrt{t^2+1}(e^{-u_1} + \sqrt{u_2}) \ge e_2(2 - t\sin(u_1+u_2)).$$

By minimizing left-hand side and maximizing right-hand side of above inequalities, we get

$$e_1 = \frac{1}{\sqrt{2}(1+\sqrt{r_2})}, \qquad e_2 = \frac{1}{2e^{r_2}}.$$

Taking $m = 2$, $r_2 = 4$, $I = \left[\frac{1}{4}, \frac{3}{4}\right]$ and after comparing all above constants we select $c = c_2 d_2 e_2$, that is

$$c = \frac{\tanh 2}{16e^4}.$$

To meet conditions (C_3) and (C_4) of Theorem 3.1 we compute

$$\max_{[0,1]\times[0,4]^2}[2 - t\sin(u_1+u_2)]\int_0^1 \frac{\sinh(2s)\sinh[2(1-s)]}{2\sinh 2}ds \approx 0.40299,$$

$$\max_{[0,1]\times[0,4]^2}[\sqrt{t^2+1}(e^{-u_1}+\sqrt{u_2})]\int_0^1 s(1-s)ds = \tfrac{\sqrt{2}}{2}.$$

Obviously for such choice of r_2 we get the required inequalities. To determine r_1 we can exploit condition (C_1), namely

$$c_1 \min_{[\frac{1}{4},\frac{3}{4}]\times[cr_1,r_1]^2}[2 - t\sin(u_1+u_2)]\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{\sinh(2s)\sinh[2(1-s)]}{2\sinh 2}ds \approx 0.01601,$$

thus we take $r_1 = 0.016$. Finally we can conclude that our system (4.1) has a positive solution (u_1, u_2), such that $0.016 \le \|(u_1,u_2)\| \le 4$.

In the next example we present an application of Theorem 3.2 for the system of third order equations with non-local boundary conditions.

Example 4.2

$$\begin{cases} -u_1^{'''} = h_1(t)g(u_1,u_2), \\ -u_2^{'''} = h_2(t)g(u_1,u_2), \\ u_i(0) = u_i^{'}(0) = 0, \\ u_1^{'}(1) = 2u_1^{'}(\frac{1}{3}), \quad u_2^{'}(1) = 3u_2^{'}(\frac{1}{4}) \end{cases} \tag{4.2}$$

for $i = 1, 2$, $h_1(t) = \frac{1}{t+3}$, $h_2(t) = \frac{1}{t^2+8}$ and $g(u_1,u_2) = \sqrt{u_1u_2}\cdot 2^{u_1+u_2}$.

The explicit form of the Green's function for the associated linear boundary value problem is given in [4]. Using the results from this work, we have met the assumption (A_1) for

$$\Phi_1(s) = 9s(1-s) \quad \text{and} \quad \Phi_2(s) = 16s(1-s).$$

while (A_2) is satisfied for

$$c_1 = \tfrac{1}{216} \text{ for } t \in [\tfrac{1}{6}, \tfrac{1}{3}] \quad \text{and} \quad c_2 = \tfrac{1}{864}, \text{ for } t \in [\tfrac{1}{12}, \tfrac{1}{4}].$$

Therefore set $I = [\frac{1}{6}, \frac{1}{3}] \cap [\frac{1}{12}, \frac{1}{4}] = [\frac{1}{6}, \frac{1}{4}]$. Clearly (A_3) and (A_4) are fulfilled for

$$d_1 = \tfrac{9}{16}, \quad d_2 = \tfrac{16}{9}.$$

There exist also constants

$$e_1 = 2, \quad e_2 = \tfrac{1}{3}$$

that satisfy (A_5) and (A_6), respectively. After computations we can conclude that $c = c_2 d_2 e_2 = \frac{1}{1458}$. To apply Theorem 3.2, we have to check that both (D_3) and (D_4) hold. Hence

$$\max_{[0,1]\times[0,r_1]^2}\left[\frac{1}{t+3}\sqrt{u_1u_2}\cdot 2^{u_1+u_2}\right]\int_0^1 9s(1-s)ds = \frac{1}{3}\cdot r_1\cdot 2^{2r_1}\cdot\frac{3}{2} = \frac{4^{r_1}r_1}{2},$$

$$\max_{[0,1]\times[0,r_1]^2}\left[\frac{1}{t^2+8}\sqrt{u_1u_2}\cdot 2^{u_1+u_2}\right]\int_0^1 16s(1-s)ds = \frac{1}{8}\cdot r_1\cdot 2^{2r_1}\cdot\frac{8}{3} = \frac{4^{r_1}r_1}{3}.$$

This gives us restriction $0 < r_1 \leq \frac{1}{2}$. Choosing (D_1) we obtain

$$c_1\min_{I\times[cr_2,r_2]^2}\left[\frac{1}{t+3}\sqrt{u_1u_2}\cdot 2^{u_1+u_2}\right]\int_I 9s(1-s)ds = \frac{1}{216}\cdot\frac{4}{13}\cdot cr_2\cdot 2^{2cr_2}\frac{71}{576},$$

which leads to inequality

$$4^{cr_2} \geq \frac{404352}{71c}.$$

Thus we can take $r_2 = 16757$ and we can state that our system (4.2) has a positive solution (u_1, u_2), such that $0.5 \leq \|(u_1, u_2)\| \leq 16757$. Observe that the system (4.2) has also a trivial solution.

References

[1] F. Cianciaruso, G. Infante, P. Pietramala, *Solutions of perturbed Hammerstein integral equations with applications*, Nonlinear Anal. Real World Appl., **33**(2017), 317-347.

[2] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, New York, 1991.

[3] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[4] L.J. Guo, J.P. Sun, Y. Zhao, *Existence of positive solutions for nonlinear third-order three-point boundary value problems*, Nonlinear Anal., **68**(2008), no. 10, 3151-3158.

[5] G. Infante, G. Mascali, J. Rodríguez-López, *A hybrid Krasnosel'skiĭ-Schauder fixed point theorem for systems*, Nonlinear Anal. Real World Appl., **80**(2024), 104165.

[6] G. Infante, F. Minhós, *Nontrivial solutions of systems of Hammerstein integral equations with first derivative dependence*, Mediterr. J. Math., **14**(2017), no. 6, 242.

[7] K. Lan, *Coexistence fixed point theorems in product Banach spaces and applications*, Math. Methods Appl. Sci., **44**(2021), no. 5, 3960-3984.

[8] F. Minhós, R. de Sousa, *On coupled systems of Hammerstein integral equations*, Bound. Value Probl., **7**(2019).

[9] J.D. Murray, *Mathematical Biology. II: Spatial Models and Biomedical Applications*, Third edition. Interdisciplinary Applied Mathematics, 18, Springer-Verlag, New York, 2003.

[10] D. Oz, I.Y. Karaca, *Eigenvalue intervals for iterative systems of second-order nonlinear equations with impulses and m-point boundary conditions*, Fixed Point Theory, **25**(2024), no. 1, 289-308.

[11] R. Precup, *A vector version of Krasnosel'skiĭ's fixed point theorem in cones and positive periodic solutions of nonlinear systems*, J. Fixed Point Theory Appl., **2**(2007), no. 1, 141-151.

[12] J. Rodríguez-López, *Hybrid localization for nonlinear systems: Lower/upper solution and Krasnosel'skiĭ fixed point theorem techniques*, Monatsh. Math., https://doi.org/10.1007/s00605-024-02026-1.

[13] G. Szajnowska, M. Zima, *Positive solutions to a third order nonlocal boundary value problem with a parameter*, Opuscula Math., **44**(2024), no. 2, 267-283.

[14] G. Szajnowska, M. Zima, *A fixed point index approach to a third order nonlocal boundary value problem*, Topol. Methods Nonlinear Anal., **65**(2025), no. 2, 673-686.
[15] Z. Yang, D. O'Regan, *Positive solvability of systems of nonlinear Hammerstein integral equations*, J. Math. Anal. Appl., **311**(2005), no. 2, 600-614.
[16] Z. Yang, Z. Zhang, *Positive solutions for a system of nonlinear singular Hammerstein integral equations via nonnegative matrices and applications*, Positivity, **16**(2012), no. 4, 783-200.

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