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A NOTE ON FIXED POINT THEORY FOR NULL–HOMOTOPIC COMPACT ACYCLIC MAPS

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Abstract. In this paper we establish some fixed point results for compact null–homotopic acyclic maps on completely regular topological spaces.

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1. Introduction

In the literature [17, 18] we established some general Leray–Schauder alternatives in completely regular topological spaces. These results rely on the notion of null–homotopic which gurantee a fixed point result provided some condition is satisfied for compact acyclic self maps (see (1.1) and (1.2) below). In this paper we consider very general spaces (Approximative neighborhood extension spaces) where this condition (namely (1.1) and (1.2)) is shown to be satisfied. Moreover our proof is elementary and relies on a Urysohn type lemma and on the fact that either a compact space is homeomorphic to a closed subset of the Tychonoff cube or that a metrizable compact space is homeomorphic to a closed subset of the Hilbert cube [6].

We first present the following classes of maps from the literature. For notational purposes in a Hausdorff topological vector space E with $A \subseteq E$ we will write $co\, A$ to denote the convex hull of A and $\overline{co}\, A$ to denote the closed convex hull of A. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$ where $f_{\star\, q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:
(i). p is a Vietoris map
and
(ii). $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Górniewicz [9]. A upper semicontinuous map $\phi : X \to Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. A upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y. Let X and Y be subsets of Hausdorff topological spaces and let $F : X \to K(Y)$ i.e. F has nonempty compact values. Recall a nonempty topological space is said to be a acyclic if all its reduced Čech homology groups over the rationals are trivial. Also recall the acyclic maps, namely $F : X \to Ac(Y)$ i.e. $F : X \to K(Y)$ with acyclic values (i.e. F has nonempty acyclic compact values). We say $F \in AC(X, Y)$ if $F : X \to Ac(Y)$ is upper semicontinuous.

Next we consider a general class of maps (which include Kak, AC and Ad maps), namely the PK maps of Park [19]. Let X and Y be Hausdorff topological spaces. Given a class $\mathbf{X}$ of maps, $\mathbf{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathbf{X}$, and $\mathbf{X}_c$ the set of finite compositions of maps in X. We let

$$\mathbf{F}(\mathbf{X}) = \{Z :\ Fix\,F \neq \emptyset \ \text{ for all } \ F \in \mathbf{X}(Z, Z)\}$$

where $Fix\,F$ denotes the set of fixed points of F.

The class $\mathbf{U}$ of maps is defined by the following properties:

(i). $\mathbf{U}$ contains the class C of single valued continuous functions;

(ii). each $F \in \mathbf{U}_c$ is upper semicontinuous and compact valued; and

(iii). $B^n \in \mathbf{F}(\mathbf{U}_c)$ for all $n \in \{1, 2, ...\}$; here $B^n = \{x \in \mathbf{R}^n :\ \|x\| \leq 1\}$.

We say $F \in PK(X, Y)$ if for any compact subset K of X there is a $G \in \mathbf{U}_c(K, G(K))$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Recall PK is closed under compositions. To see this let X, Y, Z be Hausdorff topological spaces. Let $F_1 \in PK(X, Y)$ and $F_2 \in PK(Y, Z)$. Suppose K is a compact subset of X. Then there exists a $G_1 \in \mathbf{U}_c(K, G_1(K))$ with $G_1(x) \subseteq F_1(x)$ for each $x \in K$. Note $G_1(K)$ is compact so there exists a $G_2 \in \mathbf{U}_c(G_1(K), G_2(G_1(K)))$ with

$G_2(y) \subseteq F_2(y)$ for each $y \in G_1(K)$. As a result $G_2\, G_1(x) \subseteq F_2\, G_1(x) \subseteq F_2\, F_1(x)$ for $x \in K$ and note $G_2\, G_1 \in \mathbf{U}_c(K, G_2(G_1(K)))$. Let $G = G_2\, G_1$. Thus $F_2\, F_1 \in PK(X, Z)$.

Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z, W)$ [12] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup \{ \, int\, S^{-1}(w) : \; w \in W \}$; here $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\, S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. These maps are related to the DKT maps in the literature and $F \in DKT(Z, W)$ [5] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note these maps were motivated from the $\Phi^\star$ maps. We say $G \in \Phi^\star(Z, W)$ [5] if W is convex and there exists a map $S : Z \to W$ with $S(x) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ and has convex values for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$.

Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 and let F be a multifunction. We say $F \in W(X, Y)$ if $F : X \to 2^Y$ and there exists a $\theta : X \to 2^Y$ which is lower semicontinuous with $\overline{co}\,(\theta(x)) \subseteq F(x)$ for each $x \in X$. The following result was established in [1, 20].

Theorem 1.1 *Let X be a paracompact subset of a Hausdorff topological vector space E_1 and Y a metrizable, complete subset of a Hausdorff locally convex linear topological space E_2. If $F \in W(X, Y)$ then there exists an upper semicontinuous map $G : X \to CK(Y)$ with $G(x) \subseteq F(x)$ for $x \in X$; here $CK(Y)$ denotes the family of nonempty convex compact subsetes of Y.*

Theorem 1.2 *[2, 15, 16] The Hilbert cube I^∞ (subset of l^2 consisting of points $(x_1, x_2, ...)$ with $|x_i| \leq \frac{1}{2^i}$ for all i) and the Tychonoff cube T (Cartesian product of copies of the unit interval) are in $\mathbf{F}(PK)$.*

For a subset K of a topological space X, we denote by $Cov_X\,(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov\,(K) = Cov_X\,(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov\,(X)$, a point $x \in X$ is said to be an α–fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$.

Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov\,(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$. Of course, given two single valued maps $f, g : X \to Y$ and $\alpha \in Cov\,(Y)$, then f and g are α–close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$.

Theorem 1.3 [3] *Let X be a regular topological space, $F : X \to 2^X$ an upper semicontinuous map with closed values and suppose there exists a cofinal covering $\theta \subseteq Cov_X\,(\overline{F(X)})$ such that F has an α–fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

Remark 1.4 From Theorem 1.3 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices [4, page

298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [11, page 199], so such refinements form a cofinal family of open covers). Note also that uniform spaces are regular (in fact completely regular [7]). Also note in Theorem 1.3 if F is compact valued, then the assumption that X is regular can be removed. We note here that when we apply Theorem 1.3 we will assume the space is uniform. Of course one could consider other appropriate spaces (like regular (Hausdorff) spaces) as well.

We now state some Leray–Schauder alternatives established in [17, 18]. In this paper E will be a completely regular topological space (i.e. a Tychonoff space).

Definition 1.5 We say $F \in ACW(E,E)$ if $F : E \to Ac(E)$ is a upper semicontinuous compact map (i.e. $F \in ACW(E,E)$ if $F \in AC(E,E)$ is a compact map).

Definition 1.6 We say $F \in AW(E,E)$ if $F : E \to 2^E$ is a compact map and there exists a upper semicontinuous selection $G : E \to Ac(E)$ of F (i.e. $F \in AW(E,E)$ if F is a compact map and there exists a selector $G \in AC(E,E)$ of F).

Remark 1.7 (i). Note G in Definition 1.6 is a compact map since $G(E) \subseteq F(E)$ and F is a compact map, so $G \in ACW(E,E)$ in Definition 1.6. Thus we say $F \in AW(E,E)$ if F is a compact map and there exists a selector $G \in ACW(E,E)$ of F.

(ii). If E is a Fréchet space (so E is metrizable so paracompact) and $F \in W(E,E)$ is a compact map, then from Theorem 1.1 there exists an upper semicontinuous map $G : X \to CK(Y)$ of F so G has convex (so acyclic) values (i.e. $G \in ACW(E,E)$).

Definition 1.8 (i). Suppose $F \in ACW(E,E)$ and $u_0 \in E$. We say $F \cong \{u_0\}$ in $ACW(E,E)$ if there exists a upper semicontinuous compact map $H : E \times [0,1] \to K(E)$ with $H_t \in AC(E,E)$ for each $t \in [0,1]$, $H_1 = F$ and $H_0 = \{u_0\}$; here $H_t(x) = H(x,t)$.

(ii). Suppose $F \in AW(E,E)$ and $u_0 \in E$. We say $F \cong \{u_0\}$ in $AW(E,E)$ if for any selector $G \in ACW(E,E)$ of F there exists a upper semicontinuous compact map $H : E \times [0,1] \to K(E)$ with $H_t \in AC(E,E)$ for each $t \in [0,1]$, $H_1 = G$ and $H_0 = \{u_0\}$.

The following result was established in [18].

Theorem 1.9 *Let E be a completely regular topological space, U an open subset of E and $u_0 \in U$.*

(i). Let $F \in ACW(E,E)$ and suppose there exists a upper semicontinuous compact map $H : E \times [0,1] \to K(E)$ with $H_t \in AC(E,E)$ for each $t \in [0,1]$, $H_1 = F$ and $H_0 = \{u_0\}$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume the following condition is satisfied:

$$\begin{cases} \text{for any } \ \theta \in ACW(E,E) \ \text{ with } \ \theta \cong \{u_0\} \ \text{ in } \ ACW(E,E) \\ \text{we have that } \ \theta \ \text{ has a fixed point in } \ E. \end{cases} \tag{1.1}$$

Then F has a fixed point in U.

(ii). Let $F \in AW(E,E)$ and for any selector $G \in ACW(E,E)$ of F suppose there exists a upper semicontinuous compact map $H : E \times [0,1] \to K(E)$ with $H_t \in AC(E,E)$ for each $t \in [0,1]$, $H_1 = G$ and $H_0 = \{u_0\}$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume (1.1) *holds. Then F has a fixed point in U.*

In applications one is usually interested in maps $F : \overline{U} \to 2^E$. One can adjust the statement of Theorem 1.9 and establish the following result (see [18]).

Definition 1.10 We say $F \in ACW(\overline{U},E)$ if $F : \overline{U} \to Ac(E)$ is a upper semicontinuous compact map (i.e. $F \in ACW(\overline{U},E)$ if $F \in AC(\overline{U},E)$ is a compact map).

Definition 1.11 We say $F \in AW(\overline{U},E)$ if $F : \overline{U} \to 2^E$ is a compact map and there exists a upper semicontinuous selection $G : \overline{U} \to Ac(E)$ of F (i.e. $F \in AW(\overline{U},E)$ if F is a compact map and there exists a selector $G \in AC(\overline{U},E)$ of F).

Theorem 1.12 *Let E be a completely regular topological space, U an open subset of E and $u_0 \in U$.*

(i). Let $F \in ACW(\overline{U},E)$ and suppose there exists a upper semicontinuous compact map $\Lambda : \overline{U} \times [0,1] \to K(E)$ with $\Lambda_t \in AC(\overline{U},E)$ for each $t \in [0,1]$, $\Lambda_1 = F$ and $\Lambda_0 = \{u_0\}$ and $x \notin \Lambda_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume (1.1) *holds. Then F has a fixed point in U.*

(ii). Let $F \in AW(\overline{U},E)$ and for any selector $G \in ACW(\overline{U},E)$ of F suppose there exists a upper semicontinuous compact map $\Lambda : \overline{U} \times [0,1] \to K(E)$ with $\Lambda_t \in AC(\overline{U},E)$ for each $t \in [0,1]$, $\Lambda_1 = G$ and $\Lambda_0 = \{u_0\}$ and $x \notin \Lambda_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume (1.1) *holds. Then F has a fixed point in U.*

Definition 1.13 We say $F \in A(E,E)$ if $F : E \to 2^E$ is a compact map and there exists a selection $f \in C(E,E)$ (i.e. $f : E \to E$ is a single valued continuous map) of F.

Remark 1.14 If $F \in DKT(E,E)$ is a compact map and E is paracompact and E is a Hausdorff topological vector space (so Tychonoff) then from [5] there exists a selection $f \in C(E,E)$ of F. Similarly if $F \in HLPY(E,E)$ is a compact map and E is paracompact and E is a Hausdorff topological vector space then from [12] there exists a selection $f \in C(E,E)$ of F. Also if $F : E \to 2^E$ is a lower semicontinuous compact map then [13, 14] give examples of spaces where there exists a selection $f \in C(E,E)$ of F.

Definition 1.15 Suppose $f \in C(E,E)$ is a compact map and $u_0 \in E$. We say $f \cong u_0$ in $C(E,E)$ if there exists a continuous compact map $h : E \times [0,1] \to E$ with $h_1 = f$ and $h_0 = u_0$; here $h_t(x) = h(x,t)$.

Definition 1.16 Suppose $F \in A(E,E)$ and $u_0 \in E$. We say $F \cong \{u_0\}$ in $A(E,E)$ if for any selection $f \in C(E,E)$ of F there exists a continuous compact map $h : E \times [0,1] \to E$ with $h_1 = f$ and $h_0 = u_0$.

The following results were established in [17].

Theorem 1.17 *Let E be a completely regular topological space, U an open subset of E with $u_0 \in U$, and $F \in A(E,E)$. For any selection $f \in C(E,E)$ of F suppose there exists a continuous compact map $h : E \times [0,1] \to E$ with $h_1 = f$ and $h_0 = u_0$ (i.e. suppose $F \cong \{u_0\}$ in $A(E,E)$) and $x \neq h_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume the following condition is satisfied:*

$$\begin{cases} \text{for any compact } g \in C(E,E) \text{ with } g \cong u_0 \text{ in } C(E,E) \\ \text{we have that } g \text{ has a fixed point in } E. \end{cases} \tag{1.2}$$

Then F has a fixed point in U.

Corollary 1.18 *Let E be a completely regular topological space, U an open subset of E with $u_0 \in U$, and $\theta \in C(E,E)$ a compact map. Suppose there exists a continuous compact map $h : E \times [0,1] \to E$ with $h_1 = \theta$, $h_0 = u_0$ and $x \neq h_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume* (1.2) *holds. Then θ has a fixed point in U.*

Definition 1.19 We say $F \in A(\overline{U},E)$ if $F : \overline{U} \to 2^E$ is a compact map and there exists a selection $f \in C(\overline{U},E)$ of F.

Theorem 1.20 *Let E be a completely regular topological space, U an open subset of E with $u_0 \in U$, and $F \in A(\overline{U},E)$. For any selection $f \in C(\overline{U},E)$ of F suppose there exists a continuous compact map $\lambda : \overline{U} \times [0,1] \to E$ with $\lambda_1 = f$, $\lambda_0 = u_0$ and $x \neq \lambda_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume* (1.2) *holds. Then F has a fixed point in U.*

Corollary 1.21 *Let E be a completely regular topological space, U an open subset of E with $u_0 \in U$, and $\theta \in C(\overline{U},E)$ a compact map. Suppose there exists a continuous compact map $\lambda : \overline{U} \times [0,1] \to E$ with $\lambda_1 = \theta$, $\lambda_0 = u_0$ and $x \neq \lambda_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume* (1.2) *holds. Then θ has a fixed point in U.*

2. Fixed point results

The aim in this section is to present very general spaces E where (1.1) (and (1.2)) hold. Let Ω be a class of topological spaces. A space E is called a $NES(\Omega)$ (written $E \in NES(\Omega)$) if any continuous map $f : K \to E$ defined on a closed subset K of the space $X \in \Omega$ has a continuous extension $f_0 : U \to E$ to an open set $U \subseteq X$ containing K. Now E is called an Approximative $NES(\Omega)$ (written $E \in$ Approximative $NES(\Omega)$) if given any continuous map $f : K \to E$ defined on a closed subset K of the space $X \in \Omega$ there exists for any covering $\alpha \in Cov\,(E)$ an open set $U_\alpha \subseteq X$ containing K and a continuous map $f_\alpha : U_\alpha \to E$ such that $f_\alpha|_K$ and $f|_K$ are α–close. We refer the reader to [4, 8] for more details on these spaces.

Theorem 2.1 *Let $E \in NES(compact)$. Then* (1.1) *holds.*
Proof. Let $\theta \in ACW(E,E)$ be a map with $\theta \cong \{u_0\}$ in $ACW(E,E)$; here $u_0 \in E$. Then there exists a upper semicontinuous compact map $H : E \times [0,1] \to K(E)$ with $H_t \in AC(E,E)$ for each $t \in [0,1]$, $H_1 = \theta$ and $H_0 = \{u_0\}$. Let $K = \overline{H(E \times [0,1])}$. Recall [6, 10] every compact space is homeomorphic to a closed subset of the Tychonoff cube T, so as a result K can be embedded as a closed subset $K^\star$ of T; let $s : K \to K^\star$

be a homeomorphism. Now let $i : K \hookrightarrow E$ and $j : K^\star \hookrightarrow T$ be inclusions and note $i\,s^{-1} : K^\star \to E$. Since $E \in NES(\text{compact})$ then $i\,s^{-1}$ can be extended to a neighborhood U of T containing $K^\star$ and we call the extension $h : U \to E$. Now $K^\star$ compact, T completely regular (recall any product of completely regular spaces is completely regular) and $U \subseteq T$ so there exists an open set V with $K^\star \subseteq V \subseteq \overline{V} \subseteq U$. Let $\eta : T \to [0,1]$ be a continuous map with $\eta(T\backslash V) = 0$ and $\eta(K^\star) = 1$ (note T is completely regular, $T\backslash V$ is closed and $K^\star$ is compact). Now (recall $H(E\times[0,1]) \subseteq K$) let

$$Q(x) = \begin{cases} s\,H(h(x), \eta(x)), & x \in \overline{V} \\ \{s\,u_0\}, & x \in T\backslash V; \end{cases}$$

note for $x \in \partial V$ we have $Q(x) = s\,H(h(x), \eta(x)) = s\,H(h(x), 0) = s\,H_0(h(x)) = \{s\,u_0\}$. Note $Q : T \to K(T)$ is a upper semicontinuous ([21]) compact map and $Q \in AC(T,T)$ (recall homeomorphic spaces have isomorphic groups so for a fixed $x \in T$, $Q(x)$ is acyclic valued). Now Theorem 1.2 guarantees that there exists a $x \in T$ with $x \in Q(x)$. If $x \in T\backslash V$ then $x \in \{s\,u_0\}$, which is a contradiction since $\{s\,u_0\} \subseteq K^\star \subseteq V$. Thus $x \in \overline{V}$. If $x \in \overline{V}\backslash K^\star$ then since $Q : T \to K(K^\star)$ (note $H(E \times [0,1]) \subseteq K$ and $s : K \to K^\star$) and $x \in Q(x)$ we have $x \in K^\star$, a contradiction. Thus $x \in K^\star$ so as a result $h(x) = i\,s^{-1}(x) = s^{-1}(x)$ and $\eta(x) = 1$ so $x \in Q(x) = s\,H_1(s^{-1}(x))$. Let $y = s^{-1}(x)$ and note $y \in K \subseteq E$ and $y \in H_1(y)$ i.e. (1.1) holds. □

Corollary 2.2 *Let $E \in NES(compact)$. Then* (1.2) *holds.*

Theorem 2.3 *Let E be a metrizable $NES(compact\ metric)$. Then* (1.1) *(and* (1.2)*) holds.*

Proof. Note the proof of Theorem 2.1 (with T replaced by I^∞) immediately guarantees that if E is a metrizable $NES(\text{compact metric})$ then (1.1) (and (1.2)) holds. To see this note [6] any metrizable compact space is homeomorphic to a closed subset of the Hilbert cube I^∞. □

Theorem 2.4 *Let $E \in$ Approximative $NES(compact)$ and let E be a uniform space. Then* (1.1) *holds.*

Proof. Let $\theta \in ACW(E,E)$ be a map with $\theta \cong \{u_0\}$ in $ACW(E,E)$; here $u_0 \in E$. Then there exists a upper semicontinuous compact map $H : E \times [0,1] \to K(E)$ with $H_t \in AC(E,E)$ for each $t \in [0,1]$, $H_1 = \theta$ and $H_0 = \{u_0\}$. Let $K = \overline{H(E \times [0,1])}$ and let $\beta \in Cov_E(\overline{H_1(E)})\,(= Cov_E(\overline{\theta(E)}))$. Note $\alpha = \beta \cup \left\{E\backslash\ \overline{\theta(E)}\right\}$ is an open covering of E. Now let $K^\star$, s, i and j be as in Theorem 2.1 and note $i\,s^{-1} : K^\star \to E$. Since $E \in$ Approximative $NES(\text{compact})$ there exists an open set U_α of T containing $K^\star$ and a continuous extension $h_\alpha : U_\alpha \to E$ such that $h_\alpha|_{K^\star}$ and $i\,s^{-1}|_{K^\star}$ are α–close. Now since $K^\star$ is compact, T is completely regular, $U_\alpha \subseteq T$ then there exists an open set V_α with $K^\star \subseteq V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$. Let $\eta_\alpha : T \to [0,1]$ be a continuous map with $\eta_\alpha(T\backslash V_\alpha) = 0$ and $\eta_\alpha(K^\star) = 1$ and let

$$Q_\alpha(x) = \begin{cases} s\,H(h_\alpha(x), \eta_\alpha(x)), & x \in \overline{V_\alpha} \\ \{s\,u_0\}, & x \in T\backslash V_\alpha. \end{cases}$$

Now $Q_\alpha : T \to K(T)$ (in fact $Q_\alpha : T \to K^\star$) is an upper semicontinuous compact map and $Q_\alpha \in AC(T,T)$. Now Theorem 1.2 guarantees a $x \in T$ with $x \in Q_\alpha(x)$. If $x \in T \backslash \overline{V_\alpha}$ then $x \in \{s\, u_0\}$, a contradiction since $\{s\, u_0\} \subseteq K^\star \subseteq V_\alpha$. Thus $x \in \overline{V_\alpha}$. If $x \in \overline{V_\alpha} \backslash K^\star$ then since $Q_\alpha : T \to K(K^\star)$ and $x \in Q_\alpha(x)$ we have $x \in K^\star$, a contradiction. Thus $x \in K^\star$ so $\eta_\alpha(x) = 1$ and $x \in Q_\alpha(x) = s\, H_1(h_\alpha(x))$. Now since $h_\alpha|_{K^\star}$ and $i\, s^{-1}|_{K^\star}$ are α–close then there exists a $U_x \in \alpha$ with $h_\alpha(x) \in U_x$ and $i\, s^{-1}(x) \in U_x$. Note $x \in s\, H_1(h_\alpha(x))$ (i.e. $s^{-1}(x) \in H_1(h_\alpha(x))$) implies $s^{-1}(x) \in \overline{H_1(E)}$ so $U_x \neq E \setminus \overline{H_1(E)}$, and as a result $U_x \in \beta$. Thus we have shown that there exists a $U_x \in \beta$ with $h_\alpha(x) \in U_x$, $i\, s^{-1}(x) \in U_x$ and $x \in s\, H_1(h_\alpha(x))$. Let $y = h_\alpha(x)$. Then $y \in U_x$ and $H_1(y) \cap U_x \neq \emptyset$ (since $s^{-1}(x) \in U_x$ and $s^{-1}(x) \in H_1(h_\alpha(x)) = H_1(y)$). Thus H_1 has a β–fixed point (i.e. y) and we can do this argument for every $\beta \in Cov_E(\overline{H_1(E)})$. Now Remark 1.4 guarantees that H_1 has a fixed point, so (1.1) holds. $\square$

Remark 2.5 In particular if E is as in Theorem 2.4 then (1.2) holds.

Now the argument in Theorem 2.4, with T replaced by I^∞, guarantees the following result.

Theorem 2.6 *Let E be a metrizable Approximative NES(compact metric) space. Then* (1.1) *(and* (1.2)*) holds.*

Remark 2.7 The results presented above involve only elementary arguments. It is of interest to note that one could also make use of homology theory i.e. it is possible to consider (1.1) (and (1.2)) if E is a Lefschetz space [8, 9, 16]. Recall E is a Lefschetz space if for any $F \in ACW(E,E)$ the Lefschetz number $\Lambda(F)$ (see [8, 9]) is well defined and $\Lambda(F) \neq 0$ implies F has a fixed point. Suppose E is a Lefschetz space and $\theta \in ACW(E,E)$ with $\theta \cong \{u_0\}$ in $ACW(E,E)$. Let $\Psi = \{u_0\}$ and note from [9, Corollary 32.7, pp161] (or [8]) that $\Lambda(\theta) = \Lambda(\Psi) \neq 0$. Now since E is a Lefschetz space then θ has a fixed point, so (1.1) holds.

Example 2.8 Many periodic problems for differential equations and inclusions in an abstract setting [9, Section 70] reduce to the study of the Poincare operator (which is the composition of the evaluation map and the solution operator P associated with the Cauchy problem). This operator P is usually upper semicontinuous with R_δ (so acyclic) values. As a result it is of interest to obtain fixed point results for these self map types (i.e. to obtain spaces where (1.1) hold for these maps).

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