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APPROXIMATION OF A COMMON FIXED POINT FOR A COUNTABLE FAMILY OF MAPPINGS IN COMPLETE GEODESIC SPACES WITH CURVATURE BOUNDED ABOVE

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Abstract. In this paper, we introduce a Halpern iteration of a countable family of strongly quasinonexpansive and delta-demiclosed mappings and prove convergence to a common fixed point. Further, we prove its convergence theorem with the combining projection method of balanced type.
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1. Introduction

Approximation of a fixed point of a mapping has been studied by many researchers in various spaces. In 1992, Wittmann [24] proved that an approximation sequence generated by using a Halpern iterative scheme converges to a fixed point of a nonexpansive mapping in a Hilbert space. In 2010, Saejung [21] proved a convergence theorem of a Halpern iterative scheme using a nonexpansive mapping in a complete CAT(0) space. In 2013, Kimura and Satô [16] proved this using a strongly quasinonexpansive and delta-demiclosed mapping in a complete CAT(1) space. In 2021, Kimura and Sasaki [13] proved convergence to a common fixed point of a finite family of strongly quasinonexpansive and delta-demiclosed mappings using a balanced mapping in a complete CAT(1) space.

On the other hand, a projection method is a famous technique of an approximation of a fixed point. In 2003, Nakajo and Takahashi [20] introduced the *Nakajo–Takahashi projection method* in a Hilbert space. In 2008, Takahashi, Takeuchi and Kubota [23] introduced *the shrinking projection method* in a Hilbert space. Further, in 2013, Kimura, Takahashi and Yao [19] introduced another projection method, which is called *the combining projection method* in a Hilbert space. This projection method is generated by a Halpern iteration using a countable family of a metric projections.

Several approximation methods for common fixed points of a countable family of mappings have been proposed in the literature; see, for example, [18, 9, 2]. Unlike many classical results that deal with finitely many mappings, these studies consider approximation problems involving countably many mappings. Although the treatment of a countable family may be difficult from a computational point of view, it is still an interesting topic from the theoretical viewpoint in fixed point theory and nonlinear analysis.

In particular, as shown in this paper, considering a countable family of mappings makes it possible to derive several approximation results by using projection methods. This provides a useful approach for studying common fixed point problems in nonlinear spaces.

Motivated by these observations, in this paper we study approximation methods for common fixed points of a countable family of quasinonexpansive and delta-demiclosed mappings.

In 2023, the authors [10] introduced *the combining projection method of balanced type* and proved its convergence to a common fixed point of a finite family of mappings in a complete CAT(0) space.

Theorem 1.1 (Kimura and Ogihara [10])**.** *Let X be a complete* CAT(0) space*, T a quasinonexpansive and continuous mapping of X into itself such that* $\operatorname{Fix} T \neq \varnothing$. *Let $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0,1]$, $\{\beta_n^k \mid n,k \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \beta_n^k = 1$ for $n \in \mathbb{N}$, and $\{\gamma_n \mid n \in \mathbb{N}\} \subset [0,1]$. Let $u \in X$ and define sequences $\{x_n\}$ and $\{y_n\}$, a sequence $\{U_n\}$ of mappings of X into itself, and a sequence $\{C_n\}$ of a subset of X by $x_1 \in X$ and*

$$
\begin{aligned}
&y_n = \alpha_n x_n \oplus (1-\alpha_n) T x_n;\\
&C_n = \{z \in X \mid d(y_n, z) \leq d(x_n, z)\};\\
&U_n x = \operatorname*{Argmin}_{y \in X} \sum_{k=1}^{n} \beta_n^k d(P_{C_k} x_n, y)^2 \text{ for } x \in X;\\
&x_{n+1} = \delta_n u \oplus (1-\delta_n) U_n x_n
\end{aligned}
$$

for each $n \in \mathbb{N}$, where a mapping P_K is the metric projection of X into a nonempty closed convex subset K of X. Suppose the following conditions hold:

(i) *$\{z \in X \mid d(w,z) \leq d(w',z)\}$ is convex for $w, w' \in X$;*

(ii) $\liminf_{n\to\infty} \alpha_n < 1$*;*

(iii) $\lim_{n\to\infty} \beta_n^k > 0$ *for $k \in \mathbb{N}$ and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$*;*

(iv) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$*.*

Then, $\{x_n\}$ is convergent to $P_{\operatorname{Fix} T} u$.

To prove this theorem, we applied the following theorem.

Theorem 1.2 (Hasegawa [5], Kimura and Ogihara [10])**.** *Let X be a complete* CAT(0) *space, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that*

$F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \varnothing$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset [0,1]$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for all $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset [0,1]$. Let

$$U_n x = \operatorname*{Argmin}_{y \in X} \sum_{k=1}^{n} \alpha_n^k d(T_k x, y)^2$$

for all $x \in X$ and $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ by $u, x_1 \in X$ and

$$x_{n+1} = \delta_n u \oplus (1 - \delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

(a) $\lim_{n\to\infty} \alpha_n^k > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$;

(b) $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then, $\{x_n\}$ is convergent to $P_F u$, where P_F is the metric projection of X onto F.

In this paper, we introduce a Halpern iteration using a balanced mapping of a countable family of strongly quasinonexpansive and delta-demiclosed mappings and prove a convergence to a common fixed point of mappings in a complete CAT(1) space. Further, we introduce the combining projection method of balanced type in a complete CAT(1) space.

2. Preliminaries

Let (X, d) be a metric space and T a mapping of X into itself. The set of all fixed points of T is denoted by $\operatorname{Fix} T$. Let f be a function of X into $\mathbb{R}$. Then, $\operatorname{Argmax}_{x\in X} f(x)$ is the set of all maximizers of f and $\operatorname{Argmin}_{x\in X} f(x)$ is the set of all minimizers of f.

Let $\{x_n\} \subset X$ be a bounded sequence. Then, $\{x_n\}$ *is delta-convergent to* $x_0 \in X$ if for every subsequences $\{x_{n_i}\} \subset \{x_n\}$,

$$\{x_0\} = \operatorname*{Argmin}_{y \in X} \left(\limsup_{i\to\infty} d(x_{n_i}, y) \right),$$

which is denoted by $x_n \xrightarrow{\Delta} x_0$.

Let $x, y \in X$ and γ be a mapping of $[0, d(x,y)]$ into X. A mapping γ is said to be a *geodesic with endpoints x and y* if $\gamma(0) = x$, $\gamma(d(x,y)) = y$ and $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [0, d(x,y)]$. For $D \in \,]0, \infty]$, we call X a *D-geodesic space* if a geodesic with endpoints x and y exists for all $x, y \in X$ with $d(x,y) < D$. In what follows, when X is a D-geodesic space, every geodesic in X whose length is less than D is always supposed to be unique. We call such X a *uniquely D-geodesic space*. Let X be a uniquely D-geodesic space. The image of a geodesic joining x and y is denoted by $[x, y]$. For $t \in [0,1]$, there exists unique $z \in [x, y]$ such that

$$d(x, z) = (1-t) d(x, y) \text{ and } d(y, z) = t d(x, z),$$

which is denoted by $z = tx \oplus (1-t)y$.

For $\kappa \in \mathbb{R}$, the 2-dimensional model space M_κ^2 with a curvature κ is defined by

$$M_\kappa^2 = \begin{cases} \dfrac{1}{\sqrt{\kappa}}\mathbb{S}^2 & (\kappa > 0); \\ \mathbb{E}^2 & (\kappa = 0); \\ \dfrac{1}{\sqrt{-\kappa}}\mathbb{H}^2 & (\kappa < 0), \end{cases}$$

where $\mathbb{E}^2$ is the 2-dimensional Euclidean space, $\mathbb{H}^2$ is the 2-dimensional hyperbolic space and $\mathbb{S}^2$ is the 2-dimensional unit sphere.

We define $D_\kappa \in]0, \infty]$ by

$$D_\kappa = \begin{cases} \dfrac{\pi}{\sqrt{\kappa}} & (\kappa > 0); \\ \infty & (\kappa \leq 0). \end{cases}$$

For $x, y, z \in X$, a *geodesic triangle* $\triangle(x, y, z)$ *with vertices* $x, y, z \in X$ is defined by $[x, y] \cup [y, z] \cup [z, x]$. For $x, y, z \in X$ with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, *a comparison triangle* to $\triangle(x, y, z) \subset X$ of vertices $\bar{x}, \bar{y}, \bar{z} \in M_\kappa^2$ is defined by $[\bar{x}, \bar{y}] \cup [\bar{y}, \bar{z}] \cup [\bar{z}, \bar{x}]$ with $d(x, y) = d_{M_\kappa^2}(\bar{x}, \bar{y})$, $d(y, z) = d_{M_\kappa^2}(\bar{y}, \bar{z})$ and $d(z, x) = d_{M_\kappa^2}(\bar{z}, \bar{x})$. It is denoted by $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z}) \subset M_\kappa^2$. A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called *a comparison point* for $p \in [x, y]$ if $d(x, p) = d_{M_\kappa^2}(\bar{x}, \bar{p})$. If for all triangles in X and for all $p, q \in \triangle(x, y, z)$ and their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$, the inequality $d(p, q) \leq d_{M_\kappa^2}(\bar{p}, \bar{q})$ holds, then we call X a CAT(κ) space. We call X *an admissible* CAT(κ) space if X is a CAT(κ) space and $d(x, y) < D_\kappa/2$ for every $x, y \in X$. In a CAT(1) space, the following inequality holds:

Lemma 2.1 (Kimura and Satô [15])**.** *Let X be an admissible* CAT(1) *space, $x, y, z \in X$ and $t \in [0, 1]$. Then, the inequality holds:*

$$\cos d(tx \oplus (1-t)y, z) \geq t \cos d(x, z) + (1-t) \cos d(y, z).$$

Let X be an admissible CAT(1) space, and T a mapping of X into itself. Then, a mapping T is said to be *quasinonexpansive* if $\operatorname{Fix} T \neq \varnothing$ and the inequality

$$d(Tx, z) \leq d(x, z)$$

holds for all $x \in X$ and $z \in \operatorname{Fix} T$. We know that the set of fixed points of a quasinonexpansive mapping is closed and convex.

A mapping T is said to be *delta-demiclosed* if $x_0 \in \operatorname{Fix} T$ whenever a sequence $\{x_n\} \subset X$ satisfies

$$x_n \xrightharpoonup{\Delta} x_0 \in X \text{ and } \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

A mapping T is said to be *strongly quasinonexpansive* if T is quasinonexpansive and for every $z \in \operatorname{Fix} T$ and $\{x_n\} \subset X$ satisfying $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and

$$\lim_{n\to\infty} (d(x_n, p) - d(Tx_n, p)) = 0,$$

it follows that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. A mapping $T\colon X \to X$ satisfies *condition (D)* if $x_n \to x_0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ implies $x_0 \in \operatorname{Fix} T$. Notice that if T is continuous, then it satisfies the condition (D). Moreover, it is easy to see that the delta-demiclosedness also implies the condition (D).

Let X be an admissible complete CAT(1) space, C a nonempty closed convex subset of X. We know that there exists a unique point $p_x \in X$ such that

$$p_x = \operatorname*{Argmin}_{y \in C} d(x, y)$$

for each $x \in X$, we define $P_C\colon X \to C$ by $P_C x = p_x$ for $x \in X$, and it is called *the metric projection of X onto C*. We know that P_C is strongly quasinonexpansive and delta-demiclosed [16].

Let X be an admissible complete CAT(1) space, T_k a mapping of X into itself for $k \in \{1, 2, \dots, N\}$ and $\{\alpha^k \mid k \in \{1, 2, \dots, N\}\} \subset \,]0, 1[$ such that $\sum_{k=1}^N \alpha^k = 1$. Then, a multivalued mapping $U\colon X \to 2^X$ is defined by

$$Ux \in \operatorname*{Argmax}_{y \in X} \sum_{k=1}^{N} \alpha^k \cos d(T_k x, y)$$

for $x \in X$, which is called *a balanced mapping of $\{T_k\}$ and $\{\alpha^k\}$*. The following theorem shows important properties of balanced mappings.

Theorem 2.1 (Kajimura, Kasahara, Kimura and Nakagawa [8])**.** *Let X be an admissible complete* CAT(1) *space. Let T_k be a strongly quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \{1, 2, \dots, N\}$ such that $\bigcap_{k=1}^N \operatorname{Fix} T_k \neq \varnothing$, $\{\alpha^k \mid k \in \{1, 2, \dots, N\}\} \subset \,]0, 1[$ such that $\sum_{k=1}^N \alpha^k = 1$, and U a balanced mapping of $\{T_k\}$ and $\{\alpha^k\}$. Then, the following hold:*

- *U is single-valued;*
- *the inequality*
$$\sum_{k=1}^{N} \alpha^k \cos d(T_k x, Ux) \cos d(Ux, Uy) \geq \sum_{k=1}^{N} \alpha^k \cos d(T_k x, Uy)$$
holds for $x, y \in X$;
- $\operatorname{Fix} U = \bigcap_{k=1}^N \operatorname{Fix} T_k$*;*
- *U is a strongly quasinonexpansive and delta-demiclosed mapping.*

Let X be an admissible complete CAT(1) space, $u, v \in X$ with $u \neq v$ and $\alpha \in [0, 1]$. Define a 1-*convex combination* of u and v by

$$\alpha u \overset{1}{\oplus} (1 - \alpha) v = \operatorname*{Argmax}_{x \in X} \{\alpha \cos d(u, x) + (1 - \alpha) \cos d(v, x)\}.$$

Let $t \in [0, 1]$. Then, we know that $\alpha u \overset{1}{\oplus} (1 - \alpha) v = tu \oplus (1 - t) v$, where

$$\alpha = \frac{\sin t d(u, v)}{\sin t d(u, v) + \sin(1 - t) d(u, v)};$$

see [12]. The following lemmas are properties of a 1-convex combination.

Lemma 2.2 (Kimura and Sasaki [12])**.** *Let X be an admissible* CAT(1) *space. Then, for $x, y, z \in X$ and $\alpha \in]0, 1[$, the inequality*

$$\cos d(\alpha x \overset{1}{\oplus} (1-\alpha)y, z) \geq \alpha \cos d(x, z) + (1-\alpha) \cos d(y, z)$$

holds.

Lemma 2.3 (Kimura and Sasaki [12])**.** *Let X be an admissible* CAT(1) *space. Then for $u, y, z \in X$ and $\alpha \in]0, 1[$, the inequality*

$$\begin{aligned} &1 - \cos d(\alpha u \overset{1}{\oplus} (1-\alpha)y, z) \\ &\leq (1-\beta)(1 - \cos d(y, z)) \\ &\quad + \beta \left(1 - \frac{\left(1 - \alpha + \sqrt{\alpha^2 + 2\alpha(1-\alpha)\cos d(u, y) + (1-\alpha)^2}\right) \cos d(u, z)}{\alpha + 2(1-\alpha)\cos d(u, y)} \right) \end{aligned}$$

holds where

$$\beta = 1 - \frac{1-\alpha}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cos d(u, y) + (1-\alpha)^2}}.$$

The following lemmas are important for the proof of Halpern's type approximation theorem.

Lemma 2.4 (Aoyama, Kimura and Kohsaka [1], Saejung and Yotkaew [22])**.** *Let $\{a_n\} \subset [0, \infty[$, $\{t_n\} \subset \mathbb{R}$ and $\{\beta_n\} \subset]0, 1[$ such that $\sum_{n=1}^{\infty} \beta_n = \infty$. Suppose that*

$$a_{n+1} \leq (1 - \beta_n)a_n + \beta_n t_n$$

for all $n \in \mathbb{N}$. If $\liminf_{i\to\infty} \left(a_{\varphi(i)+1} - a_{\varphi(i)}\right) \geq 0$ implies $\limsup_{i\to\infty} t_{\varphi(i)} \leq 0$ for all nondecreasing function φ of $\mathbb{N}$ into itself such that $\lim_{i\to\infty} \varphi(i) = \infty$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 (He, Fang, López and Li [7])**.** *Let X be an admissible* CAT(1) *space and $p \in X$. If a sequence $\{x_n\}$ of X satisfies that $\limsup_{n\to\infty} d(x_n, p) < \pi/2$ and that $x_n \overset{\Delta}{\rightharpoonup} x \in X$, then $d(x, p) \leq \liminf_{n\to\infty} d(x_n, p)$.*

3. A convergence theorem of Halpern iteration

In this section, we introduce a Halpern iterative scheme of a countable family of mappings in a complete CAT(1) space. First, we consider the properties of a balanced mapping.

Lemma 3.1. *Let X be an admissible complete* CAT(1) *space, $\{u_1, u_2, \ldots, u_{n+1}\} \subset X$, and $\{\alpha^k \mid k = 1, 2, \ldots, n\}, \{\beta_k \mid k = 1, 2, \ldots, n+1\} \subset]0, 1[$ such that*

$$\sum_{k=1}^{n} \alpha^k = \sum_{k=1}^{n+1} \beta^k = 1.$$

Let

$$g_1 = \operatorname*{Argmax}_{y \in X} \sum_{k=1}^{n} \alpha^k \cos d(u_k, y) \quad \text{and} \quad g_2 = \operatorname*{Argmax}_{y \in X} \sum_{k=1}^{n+1} \beta^k \cos d(u_k, y).$$

Let

$$M = \max_{k=1,2,\dots,n,n+1}\{d(u_k,g_1),d(u_k,g_2)\} < \frac{\pi}{2}.$$

Then,

$$\begin{aligned} d(g_1,g_2) \le \frac{4}{\cos M}\sum_{k=1}^{n}|\beta^k-\alpha^k| \\ \times\left(\sin\frac{d(u_k,g_1)+d(u_k,g_2)}{4}+\sin\frac{d(u_{n+1},g_1)+d(u_{n+1},g_2)}{4}\right). \end{aligned}$$

Proof. If $g_1 = g_2$, we get the result obviously. Suppose $g_1 \neq g_2$. By Lemma 2.1, we get

$$\sum_{k=1}^{n}\alpha^k\cos d(u_k,g_1)\cos d(g_1,g_2) \ge \sum_{k=1}^{n}\alpha^k\cos d(u_k,g_2)$$

and hence

$$\begin{aligned} 1-\cos d(g_1,g_2) &\le 1-\frac{\sum_{k=1}^{n}\alpha^k\cos d(u_k,g_2)}{\sum_{k=1}^{n}\alpha^k\cos d(u_k,g_1)} \\ &= \frac{\sum_{k=1}^{n}\alpha^k(\cos d(u_k,g_1)-\cos d(u_k,g_2))}{\sum_{k=1}^{n}\alpha^k\cos d(u_k,g_1)} \\ &\le \frac{\sum_{k=1}^{n}\alpha^k(\cos d(u_k,g_1)-\cos d(u_k,g_2))}{\cos M}. \end{aligned}$$

and hence

$$1-\cos d(g_1,g_2) \le \frac{\sum_{k=1}^{n}\alpha^k(\cos d(u_k,g_1)-\cos d(u_k,g_2))}{\cos M}. \tag{3.1}$$

Similarly, we get

$$1-\cos d(g_1,g_2) \le \frac{\sum_{k=1}^{n+1}\beta^k(\cos d(u_k,g_2)-\cos d(u_k,g_1))}{\cos M}. \tag{3.2}$$

Adding (3.1) and (3.2), we have

$$\begin{aligned} 2(1-\cos d(g_1,g_2)) &\le \frac{\sum_{k=1}^{n}\alpha^k(\cos d(u_k,g_1)-\cos d(u_k,g_2))}{\cos M} \\ &\quad + \frac{\sum_{k=1}^{n+1}\beta^k(\cos d(u_k,g_2)-\cos d(u_k,g_1))}{\cos M} \\ &= \frac{\sum_{k=1}^{n}(\alpha^k-\beta^k)(\cos d(u_k,g_1)-\cos d(u_k,g_2))}{\cos M} \\ &\quad + \frac{\beta^{n+1}(\cos d(u_{n+1},g_2)-\cos d(u_{n+1},g_1))}{\cos M} \\ &\le \frac{\sum_{k=1}^{n}|\beta^k-\alpha^k||\cos d(u_k,g_1)-\cos d(u_k,g_2)|}{\cos M} \\ &\quad + \frac{(1-\sum_{k=1}^{n}\beta^k)(\cos d(u_{n+1},g_2)-\cos d(u_{n+1},g_1))}{\cos M} \end{aligned}$$

$$\leq \frac{\sum_{k=1}^{n} |\beta^k - \alpha^k| |\cos d(u_k, g_1) - \cos d(u_k, g_2)|}{\cos M} + \frac{\sum_{k=1}^{n} |\beta^k - \alpha^k| |\cos d(u_{n+1}, g_1) - \cos d(u_{n+1}, g_2)|}{\cos M}$$

and hence

$$\begin{aligned}
&4\sin^2 \frac{d(g_1, g_2)}{2} \\
&\leq \frac{1}{\cos M} \sum_{k=1}^{n} |\beta^k - \alpha^k| \left| -2\sin \frac{d(u_k, g_1) + d(u_k, g_2)}{2} \sin \frac{d(u_k, g_1) - d(u_k, g_2)}{2} \right| \\
&\quad + \frac{1}{\cos M} \sum_{k=1}^{n} |\beta^k - \alpha^k| \\
&\qquad \times \left| -2\sin \frac{d(u_{n+1}, g_1) + d(u_{n+1}, g_2)}{2} \sin \frac{d(u_{n+1}, g_1) - d(u_{n+1}, g_2)}{2} \right| \\
&= \frac{2}{\cos M} \sum_{k=1}^{n} |\beta^k - \alpha^k| \sin \frac{d(u_k, g_1) + d(u_k, g_2)}{2} \sin \frac{|d(u_k, g_1) - d(u_k, g_2)|}{2} \\
&\quad + \frac{2}{\cos M} \sum_{k=1}^{n} |\beta^k - \alpha^k| \\
&\qquad \times \sin \frac{d(u_{n+1}, g_1) + d(u_{n+1}, g_2)}{2} \sin \frac{|d(u_{n+1}, g_1) - d(u_{n+1}, g_2)|}{2} \\
&\leq \frac{2}{\cos M} \sum_{k=1}^{n} |\beta^k - \alpha^k| \sin \frac{d(u_k, g_1) + d(u_k, g_2)}{2} \sin \frac{d(g_1, g_2)}{2} \\
&\quad + \frac{2}{\cos M} \sum_{k=1}^{n} |\beta^k - \alpha^k| \sin \frac{d(u_{n+1}, g_1) + d(u_{n+1}, g_2)}{2} \sin \frac{d(g_1, g_2)}{2}.
\end{aligned}$$

Dividing $2\sin(d(g_1, g_2)/2) > 0$, we get

$$\begin{aligned}
2\sin \frac{d(g_1, g_2)}{2} \leq{}& \frac{1}{\cos M} \sum_{k=1}^{n} |\beta^k - \alpha^k| \\
&\times \left(\sin \frac{d(u_k, g_1) + d(u_k, g_2)}{2} + \sin \frac{d(u_{n+1}, g_1) + d(u_{n+1}, g_2)}{2} \right).
\end{aligned}$$

and hence

$$\begin{aligned}
\sin \frac{d(g_1, g_2)}{2} &\leq \frac{1}{2\cos M} \sum_{k=1}^{n} |\beta^k - \alpha^k| \\
&\qquad \times \left(\sin \frac{d(u_k, g_1) + d(u_k, g_2)}{2} + \sin \frac{d(u_{n+1}, g_1) + d(u_{n+1}, g_2)}{2} \right) \\
&\leq \frac{1}{\cos M} \sum_{k=1}^{n} |\beta^k - \alpha^k|
\end{aligned}$$

$$\times \left(\sin \frac{d(u_k, g_1) + d(u_k, g_2)}{4} + \sin \frac{d(u_{n+1}, g_1) + d(u_{n+1}, g_2)}{4} \right)$$

Since $t/2 \le \sin t$ for $t \in [0, \pi/2[$, it follows that

$$\begin{aligned} \frac{d(g_1, g_2)}{4} \le & \frac{1}{\cos M} \sum_{k=1}^{n} |\beta^k - \alpha^k| \\ & \times \left(\sin \frac{d(u_k, g_1) + d(u_k, g_2)}{4} + \sin \frac{d(u_{n+1}, g_1) + d(u_{n+1}, g_2)}{4} \right). \end{aligned}$$

Then, we get the desired result. □

The following lemma is an important property to prove a convergence theorem of a Halpern iteration using a countable family of mappings.

Lemma 3.2. *Let X be a complete* CAT(1) *space with $\sup_{w,w' \in X} d(w, w') < \pi/2$, T_k a quasinonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \varnothing$, and $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \le n\} \subset]0, 1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$. Let*

$$U_n x = \operatorname*{Argmax}_{y \in X} \sum_{k=1}^{n} \alpha_n^k \cos d(T_k x, y)$$

for each $n \in \mathbb{N}$ and $x \in X$. If $\sum_{n=1}^{\infty} \sum_{k=1}^{N} |\alpha_{n+1}^k - \alpha_n^k| < \infty$, then

$$\sum_{n=1}^{\infty} \sup_{x \in X} d(U_{n+1}x, U_n x) < \infty.$$

Proof. Let $x \in X$ and $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n+1} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n} \operatorname{Fix} T_k$. Put $M = \sup_{w,w' \in X} d(w, w') < \pi/2$. By Lemma 3.1, we get

$$\begin{aligned} & d(U_n x, U_{n+1} x) \\ & \le \frac{4}{\cos M} \sum_{k=1}^{N} |\alpha_{n+1}^k - \alpha_n^k| \\ & \quad \times \left(\sin \frac{d(T_k x, U_n x) + d(T_k x, U_{n+1} x)}{4} + \sin \frac{d(T_{n+1} x, U_n x) + d(T_{n+1} x, U_{n+1} x)}{4} \right) \\ & \le \frac{8}{\cos M} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| \sin d(x, z) \\ & \le \frac{8}{\cos M} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k|. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$, it follows that

$$\sum_{n=1}^{\infty} \sup_{x \in X} d(U_{n+1}x, U_n x) < \infty$$

and thus we get the desired result. □

We can prove the following corollary easily by Lemma 3.2.

Corollary 3.1. *Let X be a complete* CAT(1) *space with* $\sup_{w,w'\in X} d(w,w') < \pi/2$*,* T_k *a quasinonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that* $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \varnothing$*,* $\{\alpha_n^k \mid n,k \in \mathbb{N}, k \leq n\} \subset]0,1[$ *with* $\sum_{k=1}^{n} \alpha_n^k = 1$ *for $n \in \mathbb{N}$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. If* $\sum_{n=1}^{\infty}\sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$*, then,*

$$\sum_{n=1}^{\infty} d(U_{n+1}x, U_n x) < \infty$$

and $\{U_n x\}$ is a Cauchy sequence for each $x \in X$.

By Corollary 3.1, there exists a limit of $\{U_n x\}$ for each $x \in X$. In the following lemma, we consider its properties.

Lemma 3.3. *Let X be a complete* CAT(1) *space with* $\sup_{w,w'\in X} d(w,w') < \pi/2$*,* T_k *a quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that* $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \varnothing$*, and* $\{\alpha_n^k \mid n,k \in \mathbb{N}, k \leq n\} \subset]0,1[$ *such that* $\sum_{k=1}^{n} \alpha_n^k = 1$ *for $n \in \mathbb{N}$. Let*

$$U_n x = \operatorname*{Argmax}_{y \in X} \sum_{k=1}^{n} \alpha_n^k \cos d(T_k x, y)$$

for each $n \in \mathbb{N}$ and $x \in X$. Suppose the following conditions hold:

(a) $\lim_{n\to\infty} \alpha_n^k > 0$ *for $k \in \mathbb{N}$;*

(b) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Put $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Then, the following conditions hold:

(i) $\lim_{n\to\infty} \sup_{x\in X} d(U_n x, Ux) = 0$*;*

(ii) $\operatorname{Fix} U = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$*;*

(iii) *U is quasinonexpansive and delta-demiclosed.*

Proof. (i) Let $m, n \in \mathbb{N}$ such that $n \leq m$ and $x \in X$. Then, we get

$$\begin{aligned} d(U_m x, U_n x) &\leq d(U_m x, U_{n+1} x) + d(U_{n+1} x, U_n x) \\ &\leq d(U_m x, U_{n+2} x) + d(U_{n+2} x, U_{n+1} x) + d(U_{n+1} x, U_n x) \\ &\leq \cdots \\ &\leq \sum_{l=n}^{m-1} d(U_l x, U_{l+1} x) \leq \sum_{l=n}^{\infty} d(U_l x, U_{l+1} x) \end{aligned}$$

and hence

$$d(U_m x, U_n x) \leq \sum_{l=n}^{\infty} d(U_l x, U_{l+1} x).$$

Letting $m \to \infty$, we get

$$\sup_{x \in X} d(Ux, U_n x) \le \sum_{l=n}^{\infty} \sup_{x \in X} d(U_l x, U_{l+1} x).$$

By Lemma 3.2, letting $n \to \infty$, we get $\lim_{n\to\infty} \sup_{x\in X} d(Ux, U_n x) = 0$.

(ii) Let $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n} \operatorname{Fix} T_k = \operatorname{Fix} U_n$. Then, we get

$$Uz = \lim_{n\to\infty} U_n z = \lim_{n\to\infty} z = z$$

and hence $z \in \operatorname{Fix} U$. Inversely, let $z \in \operatorname{Fix} U$ and $w \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$. By Theorem 2.1, we have

$$\sum_{k=1}^{n} \alpha_n^k \cos d(T_k z, U_n z) \ge \frac{\sum_{k=1}^{n} \alpha_n^k \cos d(T_k z, w)}{\cos d(U_n z, w)} \ge \frac{\cos d(z, w)}{\cos d(U_n z, w)}$$

and hence

$$\sum_{k=1}^{n} \alpha_n^k (1 - \cos d(T_k z, U_n z)) \le 1 - \frac{\cos d(z, w)}{\cos d(U_n z, w)}.$$

Fix $i \in \mathbb{N}$ arbitrarily. Then, it follows that

$$0 \le \alpha_n^i (1 - \cos d(T_i z, U_n z)) \le 1 - \frac{\cos d(z, w)}{\cos d(U_n z, w)}.$$

Since $\lim_{n\to\infty} \alpha_n^i > 0$, we put $\lim_{n\to\infty} \alpha_n^i = \alpha^i \in]0, 1]$. Letting $n \to \infty$, we get

$$0 \le \alpha^i (1 - \cos d(T_i z, z)) \le 1 - \frac{\cos d(z, w)}{\cos d(Uz, w)} = 0$$

and hence $d(T_i z, z) = 0$. Since $i \in \mathbb{N}$ is arbitrary, we get $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$.

(iii) Let $x \in X$ and $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n} \operatorname{Fix} T_k$. Since U_n is quasinonexpansive for $n \in \mathbb{N}$, we get

$$d(Ux, z) = \lim_{n\to\infty} d(U_n x, z) \le \lim_{n\to\infty} d(x, z) = d(x, z)$$

and thus U is quasinonexpansive. We next show U is delta-demiclosed. Let $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$, and $\{x_i\} \subset X$ with $x_i \overset{\Delta}{\rightharpoonup} x_0 \in X$ and $\lim_{i\to\infty} d(x_i, Ux_i) = 0$. By Theorem 2.1, we get

$$\sum_{k=1}^{n} \alpha_n^k \cos d(T_k x_i, U_n x_i) \ge \frac{\cos d(x_j, z)}{\cos d(U_n x_i, z)}$$

and hence

$$0 \le \sum_{k=1}^{n} \alpha_n^k (1 - \cos d(T_k x_i, U_n x_i)) \le 1 - \frac{\cos d(x_i, z)}{\cos d(U_n x_i, z)}.$$

Fix $j \in \mathbb{N}$ arbitrarily. Then, it follows that

$$0 \le \alpha_n^j (1 - \cos d(T_j x_i, U_n x_i)) \le 1 - \frac{\cos d(x_i, z)}{\cos d(U_n x_i, z)}.$$

Since $\lim_{n\to\infty} \alpha_n^j > 0$, put $\lim_{n\to\infty} \alpha_n^j = \alpha^j \in]0, 1]$. Letting $n \to \infty$, we obtain

$$0 \le \alpha^j (1 - \cos d(T_j x_i, U x_i))$$

$$
\begin{aligned}
&\le 1-\frac{\cos d(x_i,z)}{\cos d(Ux_i,z)}\\
&=\frac{\cos d(Ux_i,z)-\cos d(x_i,z)}{\cos d(Ux_i,z)}\\
&=\frac{2}{\cos d(Ux_i,z)}\sin\frac{d(x_i,z)+d(Ux_i,z)}{2}\sin\frac{d(x_i,z)-d(Ux_i,z)}{2}\\
&\le\frac{2}{\cos d(Ux_i,z)}\sin\frac{d(x_i,z)+d(Ux_i,z)}{2}\sin\frac{d(x_i,Ux_i)}{2}\\
&\le\frac{2}{\cos M}\sin\frac{d(x_i,Ux_i)}{2},
\end{aligned}
$$

where $M=\sup_{w,w'\in X} d(w,w')$. Letting $i\to\infty$, we have $\lim_{i\to\infty} d(T_jx_i,Ux_i)=0$. Since $\lim_{i\to\infty} d(x_i,Ux_i)=0$, we get $\lim_{i\to\infty} d(x_i,T_jx_i)=0$. By the delta-demiclosedness of T_j, we obtain $x_0\in \operatorname{Fix}T_j$. Since $j\in\mathbb{N}$ is arbitrary, it follows that $x_0\in\bigcap_{k=1}^{\infty}\operatorname{Fix}T_k=\operatorname{Fix}U$ and U is delta-demiclosed. Consequently, we complete the proof. □

In the following theorem, we introduce a Halpern iterative scheme using a balanced mapping of a countable family of strongly quasinonexpansive and delta-demiclosed mappings in a complete CAT(1) space.

Theorem 3.1. *Let X be a complete* CAT(1) *space with $\sup_{w,w'\in X} d(w,w')<\pi/2$, T_k a strongly quasinonexpansive and delta-demiclosed mapping of X into itself for $k\in\mathbb{N}$ such that $F=\bigcap_{k=1}^{\infty}\operatorname{Fix}T_k\neq\varnothing$, and $\{\alpha_n^k\mid n,k\in\mathbb{N},k\le n\}\subset\,]0,1[$ such that $\sum_{k=1}^{n}\alpha_n^k=1$ for $n\in\mathbb{N}$, $\{\delta_n\mid n\in\mathbb{N}\}\subset\,]0,1[$. Let*

$$
U_nx=\operatorname*{Argmax}_{y\in X}\sum_{k=1}^{n}\alpha_n^k\cos d(T_kx,y)
$$

for each $n\in\mathbb{N}$ and $x\in X$. Let $u\in X$ and define a sequence $\{x_n\}\subset X$ by $x_1\in X$ and

$$
x_{n+1}=\delta_nu\overset{1}{\oplus}(1-\delta_n)U_nx_n
$$

for each $n\in\mathbb{N}$. Suppose the following conditions hold:

(a) $\lim_{n\to\infty}\alpha_n^k>0$ *for $k\in\mathbb{N}$ and* $\sum_{n=1}^{\infty}\sum_{k=1}^{n}|\alpha_{n+1}^k-\alpha_n^k|<\infty$;

(b) $\lim_{n\to\infty}\delta_n=0$ *and* $\sum_{n=1}^{\infty}\delta_n=\infty$.

Then, $\{x_n\}$ converges to P_Fu.

Proof. Put $p=P_Fu$. Then, it follows that

$$
\begin{aligned}
\cos d(x_{n+1},p)&=\cos d(\delta_nu\overset{1}{\oplus}(1-\delta_n)U_nx_n)\\
&\ge\delta_n\cos d(u,p)+(1-\delta_n)\cos d(U_nx_n,p)\\
&\ge\delta_n\cos d(u,p)+(1-\delta_n)\cos d(x_n,p)\\
&\ge\min\{\cos d(u,p),\cos d(x_n,p)\}\\
&\ge\min\{\cos d(u,p),\cos d(x_1,p)\}
\end{aligned}
$$

for all $n \in \mathbb{N}$ and hence

$$\sup_{n\in\mathbb{N}} d(x_n, p) \le \max\{d(u,p), d(x_1,p)\} < \frac{\pi}{2}.$$

Put

$$\begin{aligned}
a_n &= 1 - \cos d(x_n, p),\\
b_n &= 1 - \frac{\left(1-\delta_n + \sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos d(u, U_n x_n) + (1-\delta_n)^2}\right)\cos d(u,p)}{\delta_n + 2(1-\delta_n)\cos d(u, U_n x_n)},\\
\sigma_n &= 1 - \frac{1-\delta_n}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos d(u, U_n x_n) + (1-\delta_n)^2}}
\end{aligned}$$

for all $n \in \mathbb{N}$. By Lemma 2.3, we get

$$a_{n+1} = 1 - \cos d(\delta_n u \overset{1}{\oplus} (1-\delta_n) U_n x_n, p) \le (1-\sigma_n) a_n + \sigma_n b_n$$

for all $n \in \mathbb{N}$. We first show $\sum_{n=1}^{\infty} \sigma_n = \infty$. Put $M = \sup_{w,w' \in X} d(w, w')$. Since $\delta_n^2 + (1-\delta_n)^2 \le 1/2$ and $\delta_n(1-\delta_n) \le 1/4$, we get

$$\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M + (1-\delta_n)^2} \le \sqrt{\frac{1+\cos M}{2}}.$$

Since $\delta_n \to 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $1 - \delta_n \ge 1/2$. Then, it follows that

$$\begin{aligned}
\sigma_n &\ge 1 - \frac{1-\delta_n}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M + (1-\delta_n)^2}}\\
&= \frac{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M + (1-\delta_n)^2} - (1-\delta_n)}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M + (1-\delta_n)^2}}\\
&= \frac{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M + (1-\delta_n)^2} + (1-\delta_n)}\\
&\quad \times \frac{1}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M + (1-\delta_n)^2} - (1-\delta_n)}\\
&\ge \frac{\delta_n + 2(1-\delta_n)\cos M}{\left(\sqrt{\frac{1+\cos M}{2}} + 1\right)\sqrt{\frac{1+\cos M}{2}}}\delta_n \ge \frac{\cos M}{\left(\sqrt{\frac{1+\cos M}{2}} + 1\right)\sqrt{\frac{1+\cos M}{2}}}\delta_n
\end{aligned}$$

for all $n \ge n_0$. Since $\sum_{n=n_0}^{\infty} \delta_n = \infty$, we get $\sum_{n=1}^{\infty} \sigma_n = \infty$.

Let $\varphi\colon \mathbb{N} \to \mathbb{N}$ be a nondecreasing function such that $\lim_{i\to\infty} \varphi(i) = \infty$. Suppose

$$\liminf_{i\to\infty} \left(a_{\varphi(i)+1} - a_{\varphi(i)}\right) \ge 0$$

and put $n_i = \varphi(i)$. Then, we get

$$\begin{aligned}
0 &\le \liminf_{i\to\infty}(a_{n_i+1} - a_{n_i})\\
&= \liminf_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(x_{n_i+1}, p))
\end{aligned}$$

$$
\begin{aligned}
&\leq \liminf_{i\to\infty}(\cos d(x_{n_i},p) - \delta_{n_i}\cos d(u,p) - (1-\delta_{n_i})\cos d(U_{n_i}x_{n_i},p)) \\
&= \liminf_{i\to\infty}(\cos d(x_{n_i},p) - \cos d(U_{n_i}x_{n_i},p)) \\
&\leq \limsup_{i\to\infty}(\cos d(x_{n_i},p) - \cos d(U_{n_i}x_{n_i},p)) \leq 0
\end{aligned}
$$

and hence $\lim_{i\to\infty}(\cos d(x_{n_i},p) - \cos d(U_{n_i}x_{n_i},p)) = 0$. By Theorem 2.1, we get

$$
\sum_{k=1}^{n_i}\alpha_{n_i}^k \cos d(T_k x_{n_i}, U_{n_i}x_{n_i})\cos d(U_{n_i}x_{n_i},p) \geq \sum_{k=1}^{n_i}\alpha_{n_i}\cos d(T_k x_{n_i},p) \geq \cos d(x_{n_i},p)
$$

and hence

$$
\begin{aligned}
0 &\leq \alpha_{n_i}^1(1-\cos d(T_1 x_{n_i}, U_{n_i}x_{n_i})) \\
&\leq \sum_{k=1}^{n_i}\alpha_{n_i}^k(1-\cos d(T_k x_{n_i}, U_{n_i}x_{n_i})) \leq 1 - \frac{\cos d(x_{n_i},p)}{\cos d(U_{n_i}x_{n_i},p)}.
\end{aligned}
$$

Since $\lim_{i\to\infty}\alpha_{n_i}^1 > 0$, letting $i\to\infty$, we have $\lim_{i\to\infty} d(T_1x_{n_i}, U_{n_i}x_{n_i}) = 0$. Then, it follows that

$$
0 \leq d(x_{n_i},p) - d(T_1x_{n_i},p) \leq d(x_{n_i},p) - (d(U_{n_i}x_{n_i},p) + d(T_1x_{n_i},U_{n_i}x_{n_i})).
$$

Letting $i\to\infty$, we get $\lim_{i\to\infty}(d(x_{n_i},p) - d(T_1x_{n_i},p)) = 0$. Since T_1 is strongly quasinonexpansive, we get $\lim_{i\to\infty} d(x_{n_i},T_1x_{n_i}) = 0$. Then, we obtain that

$$
d(x_{n_i},U_{n_i}x_{n_i}) \leq d(x_{n_i},T_1x_{n_i}) + d(T_1x_{n_i},U_{n_i}x_{n_i})
$$

and thus $\lim_{i\to\infty} d(x_{n_i},U_{n_i}x_{n_i}) = 0$. Further, it follows that

$$
0 \leq |d(x_{n_i},u) - d(U_{n_i}x_{n_i},u)| \leq d(x_{n_i},U_{n_i})
$$

and thus

$$
\liminf_{i\to\infty} d(x_{n_i},u) = \liminf_{i\to\infty} d(U_{n_i}x_{n_i},u).
$$

Take a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that

$$
\lim_{j\to\infty} d(x_{n_{i_j}},u) = \liminf_{i\to\infty} d(x_{n_i},u).
$$

Further, take a subsequence $\{x_{n_{i_{j_l}}}\}$ of $\{x_{n_{i_j}}\}$ such that $x_{n_{i_{j_l}}} \overset{\Delta}{\rightharpoonup} x_0 \in X$. Put $w_l = n_{i_{j_l}}$. Then, we get $\lim_{l\to\infty} d(x_{w_l}, U_{w_l}x_{w_l}) = 0$. By Corollary 3.1, $\{U_{w_l}x\}$ is a Cauchy sequence for each $x\in X$ and put $Ux = \lim_{l\to\infty} U_{w_l}x$. Then, we get

$$
\begin{aligned}
d(x_{w_l}, Ux_{w_l}) &\leq d(x_{w_l}, U_{w_l}x_{w_l}) + d(U_{w_l}x_{w_l}, Ux_{w_l}) \\
&\leq d(x_{w_l}, U_{w_l}x_{w_l}) + \sup_{x\in X} d(U_{w_l}x, Ux).
\end{aligned}
$$

By Lemma 3.3, letting $l\to\infty$, we get $\lim_{l\to\infty} d(x_{w_l}, Ux_{w_l}) = 0$. Since U is delta-demiclosed, we get $x_0 \in \operatorname{Fix} U = \bigcap_{k=1}^{\infty}\operatorname{Fix} T_k$. By Lemma 2.5, we have

$$
\liminf_{i\to\infty} d(u, U_{n_i}x_{n_i}) = \lim_{i\to\infty} d(u, x_{n_i}) = \lim_{l\to\infty} d(u, x_{w_l}) \geq d(u,x_0) \geq d(u,p)
$$

and hence

$$
\limsup_{i\to\infty} b_{n_i} = \limsup_{i\to\infty}\left(1 - \frac{\cos d(u,p)}{\cos d(U_{n_i}x_{n_i},p)}\right)
$$

$$= 1 - \frac{\cos d(u,p)}{\cos(\liminf_{i\to\infty} d(U_{n_i}x_{n_i}, p))} \le 0.$$

Using Lemma 2.4, we obtain $x_n \to P_F u$ and get the desired result. □

Corollary 3.2. *Let X be a complete* CAT(1) *space with $\sup_{w,w'\in X} d(w,w') < \pi/2$, T_k a strongly quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \varnothing$, and $\{\alpha_n^k \mid n,k \in \mathbb{N}, k \le n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$. Let*

$$U_n x = \operatorname*{Argmax}_{y\in X} \sum_{k=1}^{n} \alpha_n^k \cos d(T_k x, y)$$

for each $n \in \mathbb{N}$ and $x \in X$. Let $u \in X$ and define a sequence $\{x_n\} \subset X$ by $x_1 \in X$ and

$$x_{n+1} = \delta_n u \oplus (1-\delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

(a) $\lim_{n\to\infty} \alpha_n^k > 0$ *for $k \in \mathbb{N}$ and* $\sum_{n=1}^{\infty}\sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$*;*

(b) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$*.*

Then, $\{x_n\}$ converges to $P_F u$.

Proof. Let

$$\delta_n u \oplus (1-\delta_n) U_n x_n = \gamma_n u \overset{1}{\oplus} (1-\gamma_n) U_n x_n$$

for each $n \in \mathbb{N}$. By definition of 1-convex combination, it follows that

$$\gamma_n = \frac{\sin \delta_n d(u, U_n x_n)}{\sin \delta_n d(u, U_n x_n) + \sin(1-\delta_n) d(u, U_n x_n)}$$

for $n \in \mathbb{N}$. Then, we get

$$\begin{aligned}
\gamma_n &= \frac{\sin \dfrac{\delta_n d(u,U_n x_n)}{2} \cos \dfrac{\delta_n d(u,U_n x_n)}{2}}{\sin \dfrac{d(u,U_n x_n)}{2} \cos\left(\left(\delta_n - \dfrac{1}{2}\right) d(u,U_n x_n)\right)} \\
&\ge \frac{\delta_n \cos \dfrac{\delta_n d(u,U_n x_n)}{2}}{\cos\left(\left(\delta_n - \dfrac{1}{2}\right) d(u,U_n x_n)\right)} \ge \delta_n \cos \frac{\delta_n d(u,U_n x_n)}{2} \\
&\ge \frac{\delta_n}{\sqrt{2}}
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} \delta_n = \infty$, we get $\sum_{n=1}^{\infty} \gamma_n = \infty$. Since $\delta_n \to 0$, we obtain $\gamma_n \to 0$. By Theorem 3.1, we get the desired result. □

4. Applications

In this section, we prove a convergence theorem with the combining projection method of balanced type to apply Corollary 3.2 in the unit sphere of a Hilbert space.

Let H be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and a norm $\|\cdot\|$. We denote the unit sphere of H by

$$S_H = \{x \in H \mid \|x\| = 1\}.$$

We define $\rho\colon S_H \times S_H \to [0,\pi]$ by $\rho(x,y) = \arccos\langle x,y\rangle$ for $x, y \in S_H$. Then, we know that (S_H, ρ) is an example of complete CAT(1) spaces. Let C be a nonempty closed convex subset of S_H satisfying $d(u,v) < \pi/2$ for all $u, v \in C$. A half space

$$\{z \in C \mid \rho(x,z) \leq \rho(y,z)\}$$

is convex for $x, y \in C$; see [17].

Let S_H be a unit sphere of a Hilbert space H with $\rho(u,v) < \pi$ for $u, v \in S_H$. For $x, y \in S_H$ and $\alpha \in [0,1]$, we know that

$$\alpha x \oplus (1-\alpha)y = \frac{\sin \alpha d(x,y)}{\sin d(x,y)}x + \frac{\sin(1-\alpha)d(x,y)}{\sin d(x,y)}y$$

and

$$\alpha x \overset{1}{\oplus} (1-\alpha)y = \frac{\alpha x + (1-\alpha)y}{\|\alpha x + (1-\alpha)y\|},$$

and for $\{x_1, x_2, \dots, x_N\} \subset S_H$ and $\{\alpha^i\} \subset\,]0,1[$ with $\sum_{i=1}^N \alpha^i = 1$, we know that

$$\operatorname*{Argmax}_{y \in S_H} \sum_{k=1}^{N} \alpha^i \cos \rho(x_i, y) = \frac{\sum_{i=1}^N \alpha^i x_i}{\|\sum_{i=1}^N \alpha^i x_i\|};$$

see [3, 14].

In the following theorem, we introduce the combining projection method of balanced type and prove convergence of a sequence to a fixed point of a mapping to apply Theorem 3.1 in a complete CAT(1) space.

Theorem 4.1. *Let X be a complete* CAT(1) *space with $\sup_{w,w' \in X} d(w,w') < \pi/2$, T a quasinonexpansive mapping of X into itself satisfying condition (D), $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0,1]$, $\{\beta_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset\,]0,1[$ such that $\sum_{k=1}^n \beta_n^k = 1$ for $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset\,]0,1[$. Let $u \in X$. Define sequences $\{y_n\}$ and $\{x_n\}$, a sequence of subsets $\{C_n\}$ of X, and a mapping $\{U_n\}$ by $x_1 \in X$ and*

$$\begin{aligned}
&y_n = \alpha_n x_n \oplus (1-\alpha_n)Tx_n;\\
&C_n = \{z \in X \mid d(y_n, z) \leq d(x_n, z)\};\\
&U_n x_n = \operatorname*{Argmax}_{y \in X} \sum_{k=1}^{n} \beta_n^k \cos d\left(P_{C_k} x_n, y\right);\\
&x_{n+1} = \delta_n u \overset{1}{\oplus} (1-\delta_n) U_n x_n
\end{aligned}$$

for each $n \in \mathbb{N}$. Suppose that the following conditions hold:

(a) *$\{z \in X \mid d(z,v) \leq d(z,v')\}$ is convex for $v, v' \in X$;*

(b) $\liminf_{n\to\infty} \alpha_n < 1$;

(c) $\lim_{n\to\infty} \beta_n^k > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$;

(d) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n^2 = \infty$.

Then, $\{x_n\}$ *is convergent to* $P_{\operatorname{Fix} T} u$.

Proof. Let $p \in \operatorname{Fix} T$. Then, we get

$$\begin{aligned}\cos d(y_n, p) &= \cos d(\alpha_n x_n \overset{1}{\oplus} (1-\alpha_n) T x_n, p) \\ &\geq \alpha_n \cos d(x_n, p) + (1-\alpha_n) \cos d(T x_n, p) \\ &\geq \cos d(x_n, p)\end{aligned}$$

and hence $d(y_n, p) \leq d(x_n, p)$ for all $n \in \mathbb{N}$. This implies that $\operatorname{Fix} T \subset \bigcap_{n=1}^{\infty} C_n$. Since C_n is a nonempty closed convex set, the metric projection P_{C_n} is well-defined for $n \in \mathbb{N}$. Then, we get

$$\bigcap_{k=1}^{\infty} \operatorname{Fix} P_{C_k} = \bigcap_{k=1}^{\infty} C_k \supset \operatorname{Fix} T \neq \varnothing.$$

Since the metric projection P_{C_k} is strongly quasinonexpansive and delta-demiclosed for $k \in \mathbb{N}$, we get U_n is strongly quasinonexpansive and delta-demiclosed for each $n \in \mathbb{N}$. By Theorem 3.1, we get $x_n \to P_{\bigcap_{k=1}^{\infty} C_k} u$. Put $x_0 = P_{\bigcap_{k=1}^{\infty} C_k} u$. Since $x_0 \in \bigcap_{k=1}^{\infty} C_k$, we have $y_n \to x_0$. Since $\liminf_{n\to\infty} \alpha_n < 1$, we can take a subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ such that $\lim_{i\to\infty} \alpha_{n_i} \in [0, 1[$. Then, we have

$$d(x_{n_i}, T x_i) = \frac{1}{1-\alpha_{n_i}} d(x_{n_i}, y_{n_i}) \leq \frac{1}{1-\alpha_{n_i}} (d(x_{n_i}, x_0) + d(x_0, y_{n_i}))$$

and hence $\lim_{i\to\infty} d(x_{n_i}, T x_{n_i}) = 0$. Since T satisfies condition (D), we get $x_0 \in \operatorname{Fix} T$. Therefore, we have $x_n \to P_{\operatorname{Fix} T} u$ and we get the desired result. □

By Theorem 4.1, we get the following corollary in the unit sphere of a Hilbett space.

Corollary 4.1. *Let* S_H *be a unit sphere of a Hilbert space* H *with the metric with* ρ, C *a nonempty closed convex subset of* S_H *with* $\sup_{w,w'} \rho(w, w') < \pi/2$. *Let* T *be a quasinonexpansive mapping of* C *into itself satisfies condition (D),* $\{\beta_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0, 1[$ *such that* $\sum_{k=1}^{n} \beta_n^k = 1$ *for* $n \in \mathbb{N}$, *and* $\{\delta_n \mid n \in \mathbb{N}\} \subset]0, 1[$. *Let* $u \in C$. *Define sequences* $\{y_n\}$, $\{x_n\}$, $\{u_n\}$, *and a sequence of subset* $\{C_n\}$ *of* C *by* $x_1 \in C$ *and*

$$\begin{aligned}&y_n = \frac{\sin \alpha_n d(x_n, T x_n)}{\sin d(x_n, T x_n)} x_n + \frac{\sin(1-\alpha_n) d(x_n, T x_n)}{\sin d(x_n, T x_n)} x_n; \\ &C_n = \{z \in C \mid \rho(y_n, z) \leq \rho(x_n, z)\}; \\ &u_n = \frac{1}{\| \sum_{k=1}^{n} \beta_n^k P_{C_k} x_n \|} \sum_{k=1}^{n} \beta_n^k P_{C_k} x_n;\end{aligned}$$

$$x_{n+1} = \frac{\delta_n u + (1-\delta_n)u_n}{\|\delta_n u + (1-\delta_n)u_n\|}$$

for each $n \in \mathbb{N}$. *Suppose that the following conditions hold:*

(a) $\lim_{n\to\infty} \beta_n^k > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$*;*

(b) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$*.*

Then, $\{x_n\}$ *is convergent to* $P_{\mathrm{Fix}\, T} u$*.*

Proof. By the properties of convex combination and 1-convex combination in the unit sphere of a Hilbert space, we get

$$y_n = \frac{\sin \alpha_n d(x_n, Tx_n)}{\sin d(x_n, Tx_n)} x_n + \frac{\sin(1-\alpha_n) d(x_n, Tx_n)}{\sin d(x_n, Tx_n)} x_n = \alpha_n x_n \oplus (1-\alpha_n) Tx_n;$$

$$\frac{1}{\|\sum_{k=1}^{n} \beta_n^k P_{C_k} x_n\|} \sum_{k=1}^{n} \beta_n^k P_{C_k} x_n = \operatorname*{Argmax}_{y \in C} \sum_{k=1}^{n} \beta_n^k \cos \rho(P_{C_k} x_n, y);$$

$$\frac{\delta_n u + (1-\delta_n)u_n}{\|\delta_n u + (1-\delta_n)u_n\|} = \delta_n u \overset{1}{\oplus} (1-\delta_n) u_n$$

for each $n \in \mathbb{N}$. By Theorem 4.1, we complete the proof. □

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