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ELLIPTIC CONTRACTIONS IN COMPLETE METRIC SPACES

HUAPING HUANG*, MOHAMED JLELI** AND BESSEM SAMET***

*School of Mathematics and Statistics, Chongqing Sanxia University of Science and Technology, Wanzhou 404020, China
E-mail: mathhhp@163.com

**Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia
E-mail: jleli@ksu.edu.sa

***Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia
E-mail: bsamet@ksu.edu.sa

Abstract. In this paper we introduce two classes of single-valued mappings defined on metric spaces satisfying contractions involving the perimeters of ellipses. For each class, we obtain some suitable conditions under which the mapping possesses one and only one fixed point. Our exciting results include the famous Banach's fixed point theorem as a special case. We also provide an interesting example in which our approach can be used while the Banach fixed point theorem is inapplicable. Moreover, we pose some open questions for further investigations. These questions point out the directions for studying fixed point theory and its applications in the upcoming future.
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1. Introduction

A contraction on a metric space (X, d) is a self-mapping F defined on X satisfying the inequality $d(Fu, Fv) \leq \sigma d(u, v)$ for every $u, v \in X$, where $\sigma \in (0, 1)$ is a constant. If (X, d) is complete, then F possesses a unique fixed point in X. Moreover, for any $u_0 \in X$, the Picard iterative sequence $\{F^n u_0\}$ converges to this fixed point. This famous result was established by Banach [2] in 1922, and it is known in the literature as Banach's fixed point theorem (for short, BFPT) or Banach's contraction principle. Despite the importance of BFPT and its numerous applications, see, e.g., [23, 24, 26], there are many cases where this theorem is inapplicable. For instance, a discontinuous mapping $F : (X, d) \to (X, d)$ does not belong to the class of contractions on (X, d). So, BFPT is not applicable for such mappings. Moreover, in certain applications, the underlying space is equipped with a topological structure that is not induced by a metric, see, e.g., [17]. In a such case, BFPT is also inapplicable.

The main goal of the metric fixed point theory is to study the existence and uniqueness of fixed points when BFPT is not applicable. We can find in the existing literature several generalizations and extensions of this theorem. Some of them are concerned with the study of different kinds of contractions. For instance, Kannan [14] considered the class of mappings F satisfying the inequality

$$d(Fu,Fv) \leq \sigma[d(u,Fu)+d(v,Fv)],$$

where $\sigma \in (0,\frac{1}{2})$ is a constant. Reich [21] studied the class of mappings F satisfying

$$d(Fu,Fv) \leq \sigma_1 d(u,v) + \sigma_2\left[d(u,Fu)+d(v,Fv)\right],$$

where $\sigma_1, \sigma_2 \geq 0$ are constants with $\sigma_1 + 2\sigma_2 < 1$. Ćirić [6] introduced and studied the class of quasi-contractions. More recent references can be found in [13, 16, 8, 15, 20, 19, 1]. Another category of contributions is concerned with the study of existence and uniqueness of fixed points when the space X is equipped with a generalized metric space such as G-metric space [18, 10], b-metric space [7, 9, 25], and suprametric space [4] and so on.

Elliptic integrals arise in several problems from geometry and mechanics such as the computation of the perimeter of an ellipse, the study of the rotation of a rigid body, and the calculation of the period of a pendulum, see, e.g., [11, 5]. Motivated by the importance of such kinds of integrals, in this paper we introduce two classes of mappings $F:(X,d)\to(X,d)$ satisfying the contractions involving perimeters of ellipses, where (X,d) is a metric space. The first one includes mappings F satisfying the inequality

$$J_f\left(d(Fu,S(Fu,Fv)),d(S(Fu,Fv),Fv)\right) \leq \sigma J_f\left(d(u,S(u,v)),d(S(u,v),v)\right),$$

where $\sigma\in(0,1)$ is a constant, $S: X\times X\to X$ is a mapping, $f:[0,\infty)\to[0,\infty)$ is a polynomial type function of order $\tau>0$ (see Subsection 2.1), $J(a,b)=\frac{1}{4}L(a,b)$, and $L(a,b)$ is the perimeter of the ellipse with semi-minor axis a and semi-major axis b. Here, by J_f, we mean $f\circ J$. We say this class of mappings, so called the class of elliptic contractions with respect to (τ,f,S). The second introduced class, so called the class of ξ-elliptic contractions with respect to S, includes mappings F satisfying the inequality

$$J\left(d(Fu,S(Fu,Fv)),d(S(Fu,Fv),Fv)\right) \leq J_\xi\left(d(u,S(u,v)),d(S(u,v),v)\right),$$

where $\xi:[0,\infty)\to[0,\infty)$ is a (c)-comparison function and $J_\xi=\xi\circ J$. For both classes, under suitable conditions, we establish the existence and uniqueness of fixed points of F. Moreover, we show that the unique fixed point of F is a fixed point of S. We also show that BFPT is a special case of our obtained results. An interesting example is provided, where our approach can be used while BFPT is inapplicable.

The rest of this article is organized as follows. In Section 2, we state and prove our main results. Namely, we introduce in Subsection 2.1 the notion of polynomial type function. In Subsection 2.2, we introduce the class of elliptic contractions with respect to (τ,f,S) and establish a fixed point theorem for such mappings. In Subsection 2.3, a fixed point result is obtained for the class of ξ-elliptic contractions with respect to S. In Section 3, we summarize some foregoing results obtained in this paper and then give some interesting questions which will be considered in the future.

Throughout this paper, the following notations and concept will be used:

- $\mathbb{N}$ denotes the set of nonnegative integers, that is, $\mathbb{N} = \{0, 1, \cdots\}$.
- $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.
- For a nonempty set X and a mapping $F : X \to X$ and $u \in X$, we denote by $\{F^n u\}$ the sequence in X defined by
$$F^0 u = u, \quad F^{n+1} u = F(F^n u), \quad n \in \mathbb{N}.$$
We denote by $\mathrm{Fix}(F)$ the set of fixed points of F, that is,
$$\mathrm{Fix}(F) = \{u \in X : Fu = u\}.$$
- For a mapping $S : X \times X \to X$, we denote by $\mathrm{Fix}(S)$ the set of fixed points of S, that is,
$$\mathrm{Fix}(S) = \{u \in X : S(u, u) = u\}.$$

Definition 1.1 [12] We say that a mapping $F : X \to X$ is weakly orbitally continuous on a metric space (X, d) provided that the following condition holds: for every $u, v \in X$, if
$$\lim_{n\to\infty} d(F^n u, v) = 0,$$
then there exists a subsequence $\{F^{n_j} u\}$ of $\{F^n u\}$ such that
$$\lim_{j\to\infty} d(F(F^{n_j} u), Fv) = 0.$$

Remark 1.1 It can be easily seen that, if $F : X \to X$ is continuous on a metric space (X, d), then F is weakly orbitally continuous on (X, d). However, the converse is not necessarily true. Kindly see [12, Example 2.4].

2. Main results

In this section, we introduce two types of elliptic contractions for which we study the existence and uniqueness of fixed points.

2.1. **Polynomial type functions of order τ.** For some $\tau > 0$, we denote by P_τ the set of functions $f : [0, \infty) \to [0, \infty)$ satisfying:
$$c_\tau := \inf_{x>0} \frac{f(x)}{x^\tau} > 0. \tag{2.1}$$

Definition 2.1 We say that $f : [0, \infty) \to [0, \infty)$ is a polynomial type function of order $\tau > 0$, if $f \in P_\tau$.

Remark 2.1 From (2.1), we deduce that, if $f \in P_\tau$ for some $\tau > 0$, then
$$f(x) > 0, \quad x > 0.$$

Example 2.1 Let $f : [0, \infty) \to [0, \infty)$ be the function defined by
$$f(x) = ax^2 + bx + c, \quad x \geq 0,$$

where $a > 0$ and $b, c \geq 0$ are constants. Then, $f \in P_2$, that is, f is a polynomial type function of order 2.

Example 2.2 Let $f : [0, \infty) \to [0, \infty)$ be the function defined by

$$f(x) = \mathrm{e}^{\sqrt{x}} - 1, \quad x \geq 0.$$

Then, $f \in P_{\frac{1}{2}}$, that is, f is a polynomial type function of order $\frac{1}{2}$.

Example 2.3 Let $f : [0, \infty) \to [0, \infty)$ be the function defined by

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1, \\ \dfrac{1}{2} x^{\frac{3}{2}}, & \text{if } x > 1. \end{cases}$$

Then, $f \in P_{\frac{3}{2}}$, that is, f is a polynomial type function of order $\dfrac{3}{2}$.

2.2. **The class of elliptic contractions.** Let $\mathcal{E}(a, b)$ be an ellipse with semi-minor axis a and semi-major axis b. Let $L(a, b)$ be the perimeter of $\mathcal{E}(a, b)$. It is well known that (see, e.g., [3])

$$L(a, b) = 4J(a, b),$$

where $J(a, b)$ is the elliptic integral given by

$$J(a, b) = \int_0^{\frac{\pi}{2}} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, \mathrm{d}\theta. \tag{2.2}$$

It can be easily seen that (substitute $\theta = \frac{\pi}{2} - \varphi$)

$$J(a, b) = J(b, a). \tag{2.3}$$

For $f : [0, \infty) \to [0, \infty)$, we set

$$J_f(a, b) = f\left(J(a, b)\right).$$

Definition 2.2 Let (X, d) be a metric space, $S : X \times X \to X$ be a mapping, $\tau > 0$ be a constant and $f \in P_\tau$. A mapping $F : X \to X$ is called an elliptic contraction with respect to (τ, f, S), if there exists a constant $\sigma \in (0, 1)$ such that

$$J_f\left(d(Fu, S(Fu, Fv)), d(S(Fu, Fv), Fv)\right) \leq \sigma J_f\left(d(u, S(u, v)), d(S(u, v), v)\right) \tag{2.4}$$

for every $u, v \in X$.

Our first main result is the following fixed point theorem.

Theorem 2.1 *Let (X, d) be a complete metric space, $S : X \times X \to X$ be a mapping, $\tau > 0$ be a constant and $f \in P_\tau$. Suppose that $F : X \to X$ is a weakly orbitally continuous mapping and it is also an elliptic contraction with respect to (τ, f, S). Then:*

(I) *There exists $u^* \in X$ such that* $\mathrm{Fix}(F) = \{u^*\}$*;*
(II) *For every $u_0 \in X$, the sequence $\{F^n u_0\}$ converges to u^*;*
(III) $u^* \in \mathrm{Fix}(S)$.

Proof. We first show that F admits at most one fixed point. Indeed, let $u, v \in \mathrm{Fix}(F)$, then by (2.4), we have

$$J_f\left(d(Fu, S(Fu, Fv)), d(S(Fu, Fv), Fv)\right) \leq \sigma J_f\left(d(u, S(u, v)), d(S(u, v), v)\right),$$

that is,

$$J_f\left(d(u, S(u, v)), d(S(u, v), v)\right) \leq \sigma J_f\left(d(u, S(u, v)), d(S(u, v), v)\right).$$

Since $\sigma \in (0, 1)$, the above inequality implies that

$$J_f\left(d(u, S(u, v)), d(S(u, v), v)\right) = 0,$$

that is,

$$f\left[J\left(d(u, S(u, v)), d(S(u, v), v)\right)\right] = 0.$$

Hence, taking into consideration Remark 2.1, we obtain

$$J\left(d(u, S(u, v)), d(S(u, v), v)\right) = 0,$$

which implies by (2.2) that

$$d(u, S(u, v)) = d(S(u, v), v) = 0.$$

Consequently, we obtain $u = v$. This shows that F admits at most one fixed point.

We next show that $\mathrm{Fix}(F) \neq \emptyset$. For an arbitrary $u_0 \in X$, let $\{u_n\} \subset X$ be the sequence defined by

$$u_n = F^n u_0, \quad n \in \mathbb{N}.$$

Taking $(u, v) = (u_0, u_1)$ in (2.4), we obtain

$$\begin{aligned} &J_f\left(d(Fu_0, S(Fu_0, Fu_1)), d(S(Fu_0, Fu_1), Fu_1)\right) \\ &\leq \sigma J_f\left(d(u_0, S(u_0, u_1)), d(S(u_0, u_1), u_1)\right), \end{aligned}$$

that is,

$$J_f\left(d(u_1, S(u_1, u_2)), d(S(u_1, u_2), u_2)\right) \leq \sigma J_f\left(d(u_0, S(u_0, u_1)), d(S(u_0, u_1), u_1)\right). \tag{2.5}$$

Proceeding as above with $(u, v) = (u_1, u_2)$ in (2.4), we obtain

$$\begin{aligned} &J_f\left(d(Fu_1, S(Fu_1, Fu_2)), d(S(Fu_1, Fu_2), Fu_2)\right) \\ &\leq \sigma J_f\left(d(u_1, S(u_1, u_2)), d(S(u_1, u_2), u_2)\right), \end{aligned}$$

that is,

$$J_f\left(d(u_2, S(u_2, u_3)), d(S(u_2, u_3), u_3)\right) \leq \sigma J_f\left(d(u_1, S(u_1, u_2)), d(S(u_1, u_2), u_2)\right). \tag{2.6}$$

Thus, in view of (2.5) and (2.6), it holds that

$$J_f\left(d(u_2, S(u_2, u_3)), d(S(u_2, u_3), u_3)\right) \leq \sigma^2 J_f\left(d(u_0, S(u_0, u_1)), d(S(u_0, u_1), u_1)\right).$$

Continuing in the same way, we obtain by induction that

$$\begin{aligned} &J_f\left(d(u_n, S(u_n, u_{n+1})), d(S(u_n, u_{n+1}), u_{n+1})\right) \\ &\leq \sigma^n J_f\left(d(u_0, S(u_0, u_1)), d(S(u_0, u_1), u_1)\right) \end{aligned} \tag{2.7}$$

for every $n \in \mathbb{N}$.

On the other hand, if there exists $n_0 \in \mathbb{N}$ such that

$$J(d(u_{n_0}, S(u_{n_0}, u_{n_0+1})), d(S(u_{n_0}, u_{n_0+1}), u_{n_0+1})) = 0,$$

then

$$d(u_{n_0}, S(u_{n_0}, u_{n_0+1})) = d(S(u_{n_0}, u_{n_0+1}), u_{n_0+1}) = 0,$$

which follows that

$$u_{n_0} = S(u_{n_0}, u_{n_0+1}) = u_{n_0+1} = Fu_{n_0}.$$

That is to say, $u_{n_0} \in \mathrm{Fix}(F)$, the claim holds.

Without loss of generality, let $J(d(u_n, S(u_n, u_{n+1})), d(S(u_n, u_{n+1}), u_{n+1})) > 0$ hold for every $n \in \mathbb{N}$. Since f is a polynomial type function of order τ, by (2.1), for every $n \in \mathbb{N}$, we have

$$\begin{aligned} &J_f\left(d(u_n, S(u_n, u_{n+1})), d(S(u_n, u_{n+1}), u_{n+1})\right) \\ &= f\left[J\left(d(u_n, S(u_n, u_{n+1})), d(S(u_n, u_{n+1}), u_{n+1})\right)\right] \\ &\geq c_\tau \left[J\left(d(u_n, S(u_n, u_{n+1})), d(S(u_n, u_{n+1}), u_{n+1})\right)\right]^\tau . \end{aligned} \tag{2.8}$$

Furthermore, by (2.2), we have

$$\begin{aligned} &J\left(d(u_n, S(u_n, u_{n+1})), d(S(u_n, u_{n+1}), u_{n+1})\right) \\ &= \int_0^{\frac{\pi}{2}} \sqrt{d^2(u_n, S(u_n, u_{n+1}))\cos^2\theta + d^2(S(u_n, u_{n+1}), u_{n+1})\sin^2\theta}\,\mathrm{d}\theta. \end{aligned} \tag{2.9}$$

We now introduce the vectors $\overrightarrow{W_n}, \overrightarrow{Z_n} \in \mathbb{R}^2$ given by

$$\overrightarrow{W_n} = \begin{pmatrix} d(u_n, S(u_n, u_{n+1})) \\ d(S(u_n, u_{n+1}), u_{n+1}) \end{pmatrix}, \quad \overrightarrow{Z_n} = \begin{pmatrix} d(u_n, S(u_n, u_{n+1}))\cos\theta \\ d(S(u_n, u_{n+1}), u_{n+1})\sin\theta \end{pmatrix}.$$

Making use of Cauchy-Schwarz inequality, we obtain

$$\left\langle \overrightarrow{W_n}, \overrightarrow{Z_n} \right\rangle \leq \left\| \overrightarrow{W_n} \right\| \left\| \overrightarrow{Z_n} \right\|,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^2$ and $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^2$. The above inequality is equivalent to

$$\begin{aligned} &d(u_n, S(u_n, u_{n+1}))\left[d(u_n, S(u_n, u_{n+1}))\cos\theta\right] \\ &\quad + d(S(u_n, u_{n+1}), u_{n+1})\left[d(S(u_n, u_{n+1}), u_{n+1})\sin\theta\right] \\ &\leq \sqrt{d^2(u_n, S(u_n, u_{n+1})) + d^2(S(u_n, u_{n+1}), u_{n+1})} \\ &\quad \cdot \sqrt{d^2(u_n, S(u_n, u_{n+1}))\cos^2\theta + d^2(S(u_n, u_{n+1}), u_{n+1})\sin^2\theta}. \end{aligned}$$

Integrating the above inequality over $\theta \in \left[0, \frac{\pi}{2}\right]$, we obtain

$$\begin{aligned} &d^2(u_n, S(u_n, u_{n+1})) \int_0^{\frac{\pi}{2}} \cos\theta\,\mathrm{d}\theta + d^2(S(u_n, u_{n+1}), u_{n+1}) \int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta \\ &\leq \sqrt{d^2(u_n, S(u_n, u_{n+1})) + d^2(S(u_n, u_{n+1}), u_{n+1})} \\ &\quad \cdot \int_0^{\frac{\pi}{2}} \sqrt{d^2(u_n, S(u_n, u_{n+1}))\cos^2\theta + d^2(S(u_n, u_{n+1}), u_{n+1})\sin^2\theta}\,\mathrm{d}\theta, \end{aligned}$$

that is,

$$\begin{aligned}&\sqrt{d^2(u_n,S(u_n,u_{n+1}))+d^2(S(u_n,u_{n+1}),u_{n+1})}\\&\le\int_0^{\frac{\pi}{2}}\sqrt{d^2(u_n,S(u_n,u_{n+1}))\cos^2\theta+d^2(S(u_n,u_{n+1}),u_{n+1})\sin^2\theta}\,\mathrm{d}\theta.\end{aligned}$$

Then, by (2.9), it holds that

$$\begin{aligned}&J\left(d(u_n,S(u_n,u_{n+1})),d(S(u_n,u_{n+1}),u_{n+1})\right)\\&\ge\sqrt{d^2(u_n,S(u_n,u_{n+1}))+d^2(S(u_n,u_{n+1}),u_{n+1})},\quad n\in\mathbb{N}.\end{aligned}\tag{2.10}$$

Besides, due to the concavity of the function $x\mapsto\sqrt{x}$ on $[0,\infty)$, we have

$$\begin{aligned}&\sqrt{d^2(u_n,S(u_n,u_{n+1}))+d^2(S(u_n,u_{n+1}),u_{n+1})}\\&=\sqrt{2}\sqrt{\frac{1}{2}d^2(u_n,S(u_n,u_{n+1}))+\frac{1}{2}d^2(S(u_n,u_{n+1}),u_{n+1})}\\&\ge\sqrt{2}\left(\frac{1}{2}\sqrt{d^2(u_n,S(u_n,u_{n+1}))}+\frac{1}{2}\sqrt{d^2(S(u_n,u_{n+1}),u_{n+1})}\right)\\&=\frac{\sqrt{2}}{2}\left(d(u_n,S(u_n,u_{n+1}))+d(S(u_n,u_{n+1}),u_{n+1})\right),\end{aligned}$$

which implies by the triangle inequality that

$$\sqrt{d^2(u_n,S(u_n,u_{n+1}))+d^2(S(u_n,u_{n+1}),u_{n+1})}\ge\frac{\sqrt{2}}{2}d(u_n,u_{n+1}).\tag{2.11}$$

Thus, in view of (2.10) and (2.11), it holds that

$$J\left(d(u_n,S(u_n,u_{n+1})),d(S(u_n,u_{n+1}),u_{n+1})\right)\ge\frac{\sqrt{2}}{2}d(u_n,u_{n+1}),\quad n\in\mathbb{N}.\tag{2.12}$$

Next, it follows from (2.8) and (2.12) that

$$J_f\left(d(u_n,S(u_n,u_{n+1})),d(S(u_n,u_{n+1}),u_{n+1})\right)\ge c_\tau\left(\frac{\sqrt{2}}{2}\right)^\tau d^\tau(u_n,u_{n+1}),\quad n\in\mathbb{N}.$$

The above inequality together with (2.7) implies that

$$c_\tau\left(\frac{\sqrt{2}}{2}\right)^\tau d^\tau(u_n,u_{n+1})\le\sigma^n J_f\left(d(u_0,S(u_0,u_1)),d(S(u_0,u_1),u_1)\right),$$

that is,

$$d(u_n,u_{n+1})\le\eta\sigma_\tau^n,\quad n\in\mathbb{N},\tag{2.13}$$

where

$$\sigma_\tau=\sigma^{\frac{1}{\tau}}\in(0,1)$$

and

$$\eta=\sqrt{2}c_\tau^{-\frac{1}{\tau}}\left[J_f\left(d(u_0,S(u_0,u_1)),d(S(u_0,u_1),u_1)\right)\right]^{\frac{1}{\tau}}.$$

Then, using (2.13) and the triangle inequality, we claim that for all $n \in \mathbb{N}$ and any $p \in \mathbb{N}^*$ that

$$\begin{aligned} d(u_n, u_{n+p}) &\leq d(u_n, u_{n+1}) + \cdots + d(u_{n+p-1}, u_{n+p}) \\ &\leq \eta\sigma_\tau^n \left(1 + \sigma_\tau + \cdots + \sigma_\tau^{p-1}\right) \\ &= \eta \frac{1-\sigma_\tau^p}{1-\sigma_\tau}\sigma_\tau^n \\ &\leq \frac{\eta}{1-\sigma_\tau}\sigma_\tau^n \to 0 \text{ as } n \to \infty. \end{aligned}$$

This shows that $\{u_n\}$ is a Cauchy sequence in (X, d). Then, due to the completeness of (X, d), we infer that there exists $u^* \in X$ such that

$$\lim_{n\to\infty} d(F^n u_0, u^*) = \lim_{n\to\infty} d(u_n, u^*) = 0. \tag{2.14}$$

Since F is weakly orbitally continuous, then there exists a subsequence $\{F^{n_j}u_0\}$ of $\{F^n u_0\}$ such that

$$\lim_{j\to\infty} d(u_{n_j+1}, Fu^*) = \lim_{j\to\infty} d(F(F^{n_j}u_0), Fu^*) = 0. \tag{2.15}$$

Hence, it follows from (2.14) and (2.15) that $u^* = Fu^*$, that is, $u^* \in \mathrm{Fix}(F)$. Consequently, due to the previous step, we deduce that

$$\mathrm{Fix}(F) = \{u^*\},$$

which proves parts (I) and (II).

The last step is to show that $u^* \in \mathrm{Fix}(S)$. Indeed, making use of (2.4) with $(u, v) = (u^*, u^*)$, we have

$$\begin{aligned} &J_f\left(d(Fu^*, S(Fu^*, Fu^*)), d(S(Fu^*, Fu^*), Fu^*)\right) \\ &\leq \sigma J_f\left(d(u^*, S(u^*, u^*)), d(S(u^*, u^*), u^*)\right), \end{aligned}$$

that is,

$$J_f\left(d(u^*, S(u^*, u^*)), d(S(u^*, u^*), u^*)\right) \leq \sigma J_f\left(d(u^*, S(u^*, u^*)), d(S(u^*, u^*), u^*)\right).$$

Since $\sigma \in (0, 1)$, the above inequality yields

$$J_f\left(d(u^*, S(u^*, u^*)), d(S(u^*, u^*), u^*)\right) = 0,$$

that is,

$$f\left[J\left(d(u^*, S(u^*, u^*)), d(S(u^*, u^*), u^*)\right)\right] = 0.$$

Then, taking into consideration Remark 2.1, we obtain

$$J\left(d(u^*, S(u^*, u^*)), d(S(u^*, u^*), u^*)\right) = 0,$$

which implies by (2.2) that

$$d(u^*, S(u^*, u^*)) = 0.$$

Consequently, we have

$$u^* = S(u^*, u^*),$$

that is, $u^* \in \mathrm{Fix}(S)$. This proves part (III). The proof of Theorem 2.1 is then completed. $\square$

By Remark 1.1 and Theorem 2.1, we deduce the following result.

Corollary 2.1 *Let (X,d) be a complete metric space, $S : X \times X \to X$ be a mapping, $\tau > 0$ be a constant and $f \in P_\tau$. Suppose that $F : X \to X$ is a continuous mapping and it is also an elliptic contraction with respect to (τ, f, S). Then:*

(I) *There exists $u^* \in X$ such that $\mathrm{Fix}(F) = \{u^*\}$;*
(II) *For every $u_0 \in X$, the sequence $\{F^n u_0\}$ converges to u^*;*
(III) *$u^* \in \mathrm{Fix}(S)$.*

We now show that by Theorem 2.1, we recover BFPT.

Corollary 2.2 (BFPT) *Let (X,d) be a complete metric space and $F : X \to X$ a mapping such that*

$$d(Fu, Fv) \leq \sigma d(u,v) \tag{2.16}$$

for every $u, v \in X$, where $\sigma \in (0,1)$ is a constant. Then:

(I) *There exists $u^* \in X$ such that $\mathrm{Fix}(F) = \{u^*\}$;*
(II) *For every $u_0 \in X$, the sequence $\{F^n u_0\}$ converges to u^*.*

Proof. Let us introduce the projection mapping $S : X \times X \to X$ defined by

$$S(u,v) = v, \quad (u,v) \in X \times X.$$

Let $f : [0,\infty) \to [0,\infty)$ be the identity function, that is,

$$f(x) = x, \quad x \geq 0.$$

Clearly, f is a polynomial type function of order $\tau = 1$. Then, by (2.2), for all $u, v \in X$, we have

$$\begin{aligned}
&J_f\left(d(u, S(u,v)), d(S(u,v), v)\right)\\
&= J\left(d(u,v), d(v,v)\right)\\
&= J\left(d(u,v), 0\right)\\
&= \int_0^{\frac{\pi}{2}} \sqrt{d^2(u,v)\cos^2\theta}\,\mathrm{d}\theta\\
&= d(u,v)\int_0^{\frac{\pi}{2}} \cos\theta\,\mathrm{d}\theta\\
&= d(u,v).
\end{aligned}$$

Hence, (2.16) is equivalent to

$$J_f\left(d(Fu, S(Fu,Fv)), d(S(Fu,Fv), Fv)\right) \leq \sigma J_f\left(d(u, S(u,v)), d(S(u,v), v)\right)$$

for every $u, v \in X$, that is, F is an elliptic contraction with respect to (τ, f, S). Furthermore, due to (2.16), F is continuous on (X,d). Then, applying Theorem 2.1 (or, more precisely, Corollary 2.1), we obtain (I) and (II). □

2.3. **The class of ξ-elliptic contractions.** We recall below the notion of (c)-comparison functions, see, e.g., [22].

Definition 2.3 Let $\xi : [0,\infty) \to [0,\infty)$ be a given function. We say that ξ is a (c)-comparison function, if it satisfies the following properties:

(a) ξ is nondecreasing;

(b) For every $x > 0$, it holds $\displaystyle\sum_{n=0}^{\infty} \xi^n(x) < \infty$.

Remark 2.2 It can be easily seen that, if $\xi : [0,\infty) \to [0,\infty)$ is a (c)-comparison function, then

(i) For every $x > 0$, we have $\lim\limits_{n\to\infty} \xi^n(x) = 0$;

(ii) For every $x > 0$, we have $\xi(x) < x$;

(iii) $\xi(0) = 0$.

Example 2.4 Let $\xi : [0,\infty) \to [0,\infty)$ be the function defined by

$$\xi(x) = \sigma x, \quad x \geq 0,$$

where $\sigma \in (0,1)$ is a constant. Then, ξ is a (c)-comparison function.

Example 2.5 Let $\xi : [0,\infty) \to [0,\infty)$ be the function defined by

$$\xi(x) = \sigma \arctan x, \quad x \geq 0,$$

where $\sigma \in (0,1)$ is a constant. Then, ξ is a (c)-comparison function.

Definition 2.4 Let (X,d) be a metric space, $S : X \times X \to X$ be a mapping and $\xi : [0,\infty) \to [0,\infty)$ be a (c)-comparison function. A mapping $F : X \to X$ is called a ξ-elliptic contraction with respect to S, if

$$J\left(d(Fu, S(Fu,Fv)), d(S(Fu,Fv),Fv)\right) \leq J_\xi\left(d(u,S(u,v)), d(S(u,v),v)\right) \tag{2.17}$$

for every $u, v \in X$.

Our second main result is the following fixed point theorem.

Theorem 2.2 *Let (X,d) be a complete metric space, $S : X \times X \to X$ be a mapping and ξ be a (c)-comparison function. Suppose that $F : X \to X$ is a weakly orbitally continuous mapping and it is also a ξ-elliptic contraction with respect to S. Then:*

(I) *There exists $u^* \in X$ such that $\mathrm{Fix}(F) = \{u^*\}$;*

(II) *For every $u_0 \in X$, the sequence $\{F^n u_0\}$ converges to u^*;*

(III) *$u^* \in \mathrm{Fix}(S)$.*

Proof. We first prove that F admits at most one fixed point. Indeed, if $u, v \in \mathrm{Fix}(F)$, then by (2.17), we have

$$J\left(d(Fu, S(Fu,Fv)), d(S(Fu,Fv),Fv)\right) \leq J_\xi\left(d(u,S(u,v)), d(S(u,v),v)\right),$$

that is,

$$J\left(d(u,S(u,v)),d(S(u,v),v)\right)\leq \xi\left(J\left(d(u,S(u,v)),d(S(u,v),v)\right)\right).$$

If $J\left(d(u,S(u,v)),d(S(u,v),v)\right)>0$, then by Remark 2.2 (ii) and the above inequality, we reach a contradiction. Consequently, we have

$$J\left(d(u,S(u,v)),d(S(u,v),v)\right)=0,$$

which implies by (2.2) that

$$d(u,S(u,v))=d(S(u,v),v)=0.$$

This shows that $u=v$. Then, F admits at most one fixed point.

We next show that $\mathrm{Fix}(F)\neq\emptyset$. For an arbitrary $u_0\in X$, let $\{u_n\}\subset X$ be the sequence defined by

$$u_n=F^n u_0,\quad n\in\mathbb{N}.$$

Taking $(u,v)=(u_0,u_1)$ in (2.17), we obtain

$$\begin{aligned}&J\left(d(Fu_0,S(Fu_0,Fu_1)),d(S(Fu_0,Fu_1),Fu_1)\right)\\ &\leq J_\xi\left(d(u_0,S(u_0,u_1)),d(S(u_0,u_1),u_1)\right),\end{aligned}$$

that is,

$$J\left(d(u_1,S(u_1,u_2)),d(S(u_1,u_2),u_2)\right)\leq J_\xi\left(d(u_0,S(u_0,u_1)),d(S(u_0,u_1),u_1)\right).$$

Since ξ is a nondecreasing function, then the above inequality implies that

$$J_\xi\left(d(u_1,S(u_1,u_2)),d(S(u_1,u_2),u_2)\right)\leq J_{\xi^2}\left(d(u_0,S(u_0,u_1)),d(S(u_0,u_1),u_1)\right).\tag{2.18}$$

Proceeding as above with $(u,v)=(u_1,u_2)$ in (2.17), we obtain

$$\begin{aligned}&J\left(d(Fu_1,S(Fu_1,Fu_2)),d(S(Fu_1,Fu_2),Fu_2)\right)\\ &\leq J_\xi\left(d(u_1,S(u_1,u_2)),d(S(u_1,u_2),u_2)\right),\end{aligned}$$

that is,

$$J\left(d(u_2,S(u_2,u_3)),d(S(u_2,u_3),u_3)\right)\leq J_\xi\left(d(u_1,S(u_1,u_2)),d(S(u_1,u_2),u_2)\right).\tag{2.19}$$

Hence, in view of (2.18) and (2.19), we have

$$J\left(d(u_2,S(u_2,u_3)),d(S(u_2,u_3),u_3)\right)\leq J_{\xi^2}\left(d(u_0,S(u_0,u_1)),d(S(u_0,u_1),u_1)\right).$$

Continuing this process, we obtain by induction that

$$J\left(d(u_n,S(u_n,u_{n+1})),d(S(u_n,u_{n+1}),u_{n+1})\right)\leq J_{\xi^n}\left(d(u_0,S(u_0,u_1)),d(S(u_0,u_1),u_1)\right)\tag{2.20}$$

for every $n\in\mathbb{N}$. Then, using (2.12) and (2.20), we obtain

$$\frac{\sqrt{2}}{2}d(u_n,u_{n+1})\leq \xi^n\left(J\left(d(u_0,S(u_0,u_1)),d(S(u_0,u_1),u_1)\right)\right),$$

that is,

$$d(u_n,u_{n+1})\leq\sqrt{2}\xi^n(\gamma),\quad n\in\mathbb{N},\tag{2.21}$$

where

$$\gamma=J\left(d(u_0,S(u_0,u_1)),d(S(u_0,u_1),u_1)\right).$$

Next, using (2.21), the triangle inequality, and the property (b), for all $n \in \mathbb{N}$ and any $p \in \mathbb{N}^*$, we obtain

$$\begin{aligned} d(u_n, u_{n+p}) &\leq \sum_{i=n}^{n+p-1} d(u_i, u_{i+1}) \\ &\leq \sqrt{2} \sum_{i=n}^{n+p-1} \xi^i(\gamma) \\ &\leq \sqrt{2} \sum_{i=n}^{\infty} \xi^i(\gamma) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This shows that $\{u_n\}$ is a Cauchy sequence in (X, d). Then, by the completeness of (X, d), there exists $u^* \in X$ such that $\lim_{n\to\infty} d(u_n, u^*) = 0$. Next, proceeding as in the proof of Theorem 2.1, we obtain that $u^* \in \mathrm{Fix}(F)$. This proves parts (I) and (II).

Finally, we have to show that u^* is also a fixed point of S. Indeed, taking $(u, v) = (u^*, u^*)$ in (2.17), we obtain

$$\begin{aligned} &J\left(d(Fu^*, S(Fu^*, Fu^*)), d(S(Fu^*, Fu^*), Fu^*)\right) \\ &\leq J_\xi\left(d(u^*, S(u^*, u^*)), d(S(u^*, u^*), u^*)\right), \end{aligned}$$

that is,

$$J\left(d(u^*, S(u^*, u^*)), d(S(u^*, u^*), u^*)\right) \leq \xi\left(J\left(d(u^*, S(u^*, u^*)), d(S(u^*, u^*), u^*)\right)\right).$$

If $J\left(d(u^*, S(u^*, u^*)), d(S(u^*, u^*), u^*)\right) > 0$, then by Remark 2.2 (ii), the above inequality yields a contradiction. Consequently, we have

$$J\left(d(u^*, S(u^*, u^*)), d(S(u^*, u^*), u^*)\right) = 0,$$

which implies by (2.2) that

$$d(u^*, S(u^*, u^*)) = 0.$$

Thus, we have $u^* = S(u^*, u^*)$, that is, $u^* \in \mathrm{Fix}(S)$. This proves part (III). The proof of Theorem 2.2 is then completed. □

By using Remark 1.1, we deduce from Theorem 2.2 that the following result.

Corollary 2.3 *Let (X, d) be a complete metric space, $S : X \times X \to X$ be a mapping and ξ be a* (c)*-comparison function. Suppose that $F : X \to X$ is a continuous mapping and it is also a ξ-elliptic contraction with respect to S. Then:*

(I) *There exists $u^* \in X$ such that* $\mathrm{Fix}(F) = \{u^*\}$*;*
(II) *For every $u_0 \in X$, the sequence $\{F^n u_0\}$ converges to u^*;*
(III) $u^* \in \mathrm{Fix}(S)$.

We end the paper with the following example where BFPT is inapplicable.

Example 2.6 Let $X = \{x_1, x_2, x_3\}$ and d be the discrete metric on X, that is,

$$d(x_i, x_j) = \begin{cases} 1, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

We consider the mapping $F : X \to X$ defined by

$$Fx_1 = x_1,\ Fx_2 = x_1,\ Fx_3 = x_2.$$

Notice that F is not a contraction in sense of Banach. Indeed, we have

$$\frac{d(Fx_1, Fx_3)}{d(x_1, x_3)} = \frac{d(x_1, x_2)}{d(x_1, x_3)} = 1,$$

which shows that there is no $\sigma \in (0, 1)$ such that (2.16) holds for every $u, v \in X$.

Let us introduce the mapping $S : X \times X \to X$ defined by

$$S(x_i, x_i) = x_i,\ S(x_i, x_j) = S(x_j, x_i), \quad i, j \in \{1, 2, 3\}$$

and

$$S(x_1, x_2) = x_1,\ S(x_1, x_3) = x_2,\ S(x_2, x_3) = x_1.$$

We claim that

$$\begin{aligned} & J\left(d(Fx_i, S(Fx_i, Fx_j)), d(S(Fx_i, Fx_j), Fx_j)\right) \\ & \leq \frac{2}{\pi} J\left(d(x_i, S(x_i, x_j)), d(S(x_i, x_j), x_j)\right) \end{aligned} \tag{2.22}$$

for every $i, j \in \{1, 2, 3\}$.

As a matter of fact, remark that for every $i \in \{1, 2, 3\}$, by (2.2), we have

$$\begin{aligned} J\left(d(Fx_i, S(Fx_i, Fx_i)), d(Fx_i, S(Fx_i, Fx_i)\right) &= J\left(d(Fx_i, Fx_i), d(Fx_i, Fx_i)\right) \\ &= J(0, 0) \\ &= 0, \end{aligned}$$

which shows that (2.22) holds for every $i = j \in \{1, 2, 3\}$.

On the other hand, due to the symmetry of J (see (2.3)) and the symmetry of S, we just have to show that (2.22) holds for every $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$.

Case 1. $(i, j) = (1, 2)$. In this case, we have

$$\begin{aligned} & J\left(d(Fx_i, S(Fx_i, Fx_j)), d(Fx_j, S(Fx_i, Fx_j))\right) \\ & = J\left(d(Fx_1, S(Fx_1, Fx_2)), d(Fx_2, S(Fx_1, Fx_2))\right) \\ & = J\left(d(x_1, S(x_1, x_1)), d(x_1, S(x_1, x_1))\right) \\ & = J\left(d(x_1, x_1), d(x_1, x_1)\right) \\ & = J(0, 0) \\ & = 0, \end{aligned}$$

which shows that (2.22) holds.

Case 2. $(i, j) = (1, 3)$. In this case, we have

$$\begin{aligned}
&\frac{J\left(d(Fx_i, S(Fx_i, Fx_j)), d(Fx_j, S(Fx_i, Fx_j))\right)}{J\left(d(x_i, S(x_i, x_j)), d(x_j, S(x_i, x_j))\right)} \\
&= \frac{J\left(d(Fx_1, S(Fx_1, Fx_3)), d(Fx_3, S(Fx_1, Fx_3))\right)}{J\left(d(x_1, S(x_1, x_3)), d(x_3, S(x_1, x_3))\right)} \\
&= \frac{J\left(d(x_1, S(x_1, x_2)), d(x_2, S(x_1, x_2))\right)}{J\left(d(x_1, S(x_1, x_3)), d(x_3, S(x_1, x_3))\right)} \\
&= \frac{J\left(d(x_1, x_1), d(x_2, x_1)\right)}{J\left(d(x_1, x_2), d(x_3, x_2)\right)} \\
&= \frac{J(0,1)}{J(1,1)}.
\end{aligned}$$

Notice that by (2.2), we have

$$J(0,1) = \int_0^{\frac{\pi}{2}} \sqrt{\sin^2\theta}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta = 1$$

and

$$J(1,1) = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta + \sin^2\theta}\,\mathrm{d}\theta = \frac{\pi}{2}.$$

Thus, we obtain

$$\frac{J\left(d(Fx_i, S(Fx_i, Fx_j)), d(Fx_j, S(Fx_i, Fx_j))\right)}{J\left(d(x_i, S(x_i, x_j)), d(x_j, S(x_i, x_j))\right)} = \frac{2}{\pi}.$$

Case 3. $(i, j) = (2, 3)$. In this case, we have

$$\begin{aligned}
&\frac{J\left(d(Fx_i, S(Fx_i, Fx_j)), d(Fx_j, S(Fx_i, Fx_j))\right)}{J\left(d(x_i, S(x_i, x_j)), d(x_j, S(x_i, x_j))\right)} \\
&= \frac{J\left(d(Fx_2, S(Fx_2, Fx_3)), d(Fx_3, S(Fx_2, Fx_3))\right)}{J\left(d(x_2, S(x_2, x_3)), d(x_3, S(x_2, x_3))\right)} \\
&= \frac{J\left(d(x_1, S(x_1, x_2)), d(x_2, S(x_1, x_2))\right)}{J\left(d(x_2, S(x_2, x_3)), d(x_3, S(x_2, x_3))\right)} \\
&= \frac{J\left(d(x_1, x_1), d(x_2, x_1)\right)}{J\left(d(x_2, x_1), d(x_3, x_1)\right)} \\
&= \frac{J(0,1)}{J(1,1)} = \frac{2}{\pi}.
\end{aligned}$$

Then, from the above calculations, we deduce that (2.22) holds for every $i, j \in \{1, 2, 3\}$. This shows that F is an elliptic contraction with respect to (τ, f, S), where $\tau = 1$ and $f(x) = x$ for every $x \geq 0$. In addition, F is continuous on (X, d). Then, Theorem 2.1 (or, more precisely, Corollary 2.1) applies. Further, in this example, we have

$$\mathrm{Fix}(F) = \{x_1\}, \qquad S(x_1, x_1) = \{x_1\},$$

which confirms Theorem 2.1.

3. Conclusions

Two classes of elliptic contractions defined on a metric space (X, d) involving the perimeters of ellipses are introduced in this paper. The first one is the class of elliptic contractions $F : X \to X$ with respect to (τ, f, S), where $f : [0,\infty) \to [0,\infty)$ is a polynomial type function of order τ and $S : X \times X \to X$ is a mapping. The second one is the class of ξ-elliptic contractions $F : X \to X$ with respect to S, where $\xi : [0,\infty) \to [0,\infty)$ is a (c)-comparison function. For every class of mappings, under suitable conditions, we establish the existence and uniqueness of fixed points of F. Moreover, we prove that the fixed point of F is also a fixed point of S (see Theorems 2.1 and 2.2). We claim that by our obtained results, we recover BFPT. We also provide an example where our approach can be used while BFPT is inapplicable (see Example 2.6).

The proposed approach can be developed for generalizing many other known fixed point theorems. For instance, inspired by the work of Kannan [14], one can consider the class of mappings $F : X \to X$ satisfying the inequality

$$\begin{aligned}&J_f\left(d(Fu, S(Fu, Fv)), d(Fv, S(Fu, Fv))\right)\\ &\leq \sigma\left[J_f\left(d(u, S(u, Fu)), d(Fu, S(u, Fu))\right) + J_f\left(d(v, S(v, Fv)), d(Fv, S(v, Fv))\right)\right]\end{aligned}$$

for every $u, v \in X$, where $\sigma > 0$ is a constant. Indeed, in the particular case $f(x) = x$ and $S(u, v) = v$, the above inequality reduces to

$$J\left(d(Fu, Fv), 0\right) \leq \sigma\left[J\left(d(u, Fu), 0\right) + J\left(d(v, Fv), 0\right)\right],$$

that is,

$$d(Fu, Fv) \leq \sigma\left[d(u, Fu) + d(v, Fv)\right],$$

which is the Kannan contraction for $\sigma \in (0, \frac{1}{2})$.

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