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REVISITING SOME FAMOUS FIXED POINT RESULTS IN THE FRAMEWORK OF SUPER METRIC SPACES WITH AN APPLICATION

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Dedicated to Professor Hong-Kun Xu in the Honor of his 65^{th} Birthday

Abstract. Recently, the notion of a super metric space was introduced by Karapınar and Khojasteh (Filomat, 36(10), pp.3545-3549, 2022), which is a generalization of usual metric spaces. In this work, we have revisited some famous fixed point results to obtain some new fixed point results in the context of super metric spaces. We have found some mistakes in some of the published papers which we have corrected in this paper. Examples are given to support our new findings along with an application in fractional thermostat model.
Key Words and Phrases: α-admissible mapping, super metric space, Geraghty contraction, P-contraction, Rhoades open question, fractional thermostat model, .
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1. Introduction and Preliminaries

The study of fixed point theory is one of the crucial branch of pure mathematics because of it's vast applications in non-linear analysis, computer science, engineering, economics etc. The Banach contraction principle (in brief BCP) is the fundamental theorem in metric fixed point theory. Researchers worldwide have modified this contraction mainly into two directions, i.e., either they extended the contraction into some abstract spaces or they extended the BCP by introducing some new contractions. In 1973, Geraghty [21] introduced a new contraction to extend the famous BCP. Let Θ denotes all functions $\theta : \mathbb{R}^+ \to [0, 1)$ (where $\mathbb{R}^+ = [0, \infty)$) such that

$$\lim_{n\to\infty} \theta(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0.$$

Geraghty established the following wonderful fixed point theorem by involving $\theta(\in \Theta)$ function.

Theorem 1.1. [21] *Let $(\mathfrak{X}, d)$ be a complete metric space and $K : \mathfrak{X} \to \mathfrak{X}$ be a function. Assume that $\exists$ a function $\theta \in \Theta$ such that the following holds*

$$d(Kx, Ky) \le \theta(d(x, y))d(x, y), \tag{1.1}$$

for every $x, y \in \mathfrak{X}$. Then the function K has a unique fixed point in $\mathfrak{X}$.

The inequality (1.1) is greatly known as Geraghty contraction. Theorem [1.1] influenced many researchers around the globe and they studied different types of remarkable fixed point theorems in various ways (for example, see [4], [2], [7], [8], [10], [18], [25], [27], [31], [34], [44], [45] and the references cited therein). On the other side, one of the outstanding generalization of the BCP was given by Popescu [38] and the author proved the following theorem.

Theorem 1.2. *Let $(\mathfrak{X}, d)$ be a complete metric space and $K : \mathfrak{X} \to \mathfrak{X}$ be an operator. Assume that $\exists$ a real number $\tilde{c} \in [0, 1)$ such that the following holds*

$$d(Kx, Ky) \le \tilde{c}\big[d(x, y)+ \mid d(x, Kx) - d(y, Ky) \mid \big], \tag{1.2}$$

for every $x, y \in \mathfrak{X}$. Then the operator K has a unique fixed point in $\mathfrak{X}$.

The inequality (1.2) is greatly known as P-contraction. Some authors call inequality (1.2) as E-contraction but we will prefer to call it as P-contraction to honour the author Popescu. In recent times, many interesting fixed point results involving P-contraction have been appeared in the literature. Readers can look into [5], [9], [10], [19], [29] for more details. In 2012, Samet et al. [40] introduced the notion of α-admissible mappings to study some fixed point results. In fixed point theory, the concept of α-admissible mappings is a very powerful tool since it covers many important structures like the structure of a metric space endowed with a partial order, the structure of a metric space endowed with a graph, the structure of cyclic mappings via closed subsets of a metric space etc. Due to the reasons mentioned just, researchers have started to work on α-admissible mappings on a wide scale. To study some recent fixed point results on α-admissible mappings, readers can view [2], [11], [10], [19], [23], [24], [31], [35] and the references cited therein. Now, we move to the definition of α-admissible mapping.

Definition 1.1. [40] Let $\mathfrak{X} \neq \emptyset$ and $K : \mathfrak{X} \to \mathfrak{X}, \alpha : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ be two given mappings. We say K is an α-admissible mapping if

$$\alpha(x, y) \ge 1 \Rightarrow \alpha(Kx, Ky) \ge 1, \ \forall\ x, y \in \mathfrak{X}.$$

Note. The mapping α is said to be symmetric if $\alpha(x,y) = \alpha(y,x)$ holds for all $x, y \in \mathfrak{X}$.
Next, we move to the definition of weakly α-admissible mapping.

Definition 1.2. [35] Let $\mathfrak{X} \neq \emptyset$ and $K, L : \mathfrak{X} \to \mathfrak{X}, \alpha : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ be three given mappings. We say the pair (K, L) is weakly α-admissible if

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(K(x), LK(x)) \geq 1, \alpha(L(y), KL(y)) \geq 1, \ \forall\, x, y \ \in \mathfrak{X}.$$

Like weakly α-admissible mapping, in a similar way, we can define weakly μ-subadmissible mapping.

Definition 1.3. Let $\mathfrak{X} \neq \emptyset$ and $K, L : \mathfrak{X} \to \mathfrak{X}, \mu : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ be three given mappings. We say the pair (K, L) is weakly μ-subadmissible if

$$\mu(x,y) \leq 1 \Rightarrow \mu(K(x), LK(x)) \leq 1, \mu(L(y), KL(y)) \leq 1, \ \forall\, x, y \ \in \mathfrak{X}.$$

Now we recall some important definitions, results, concepts which will be required in the proof of our main results.

Definition 1.4. [6] A function $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be a subclass function of type I if

$$x \geq 1 \Rightarrow h(1,y) \leq h(x,y), \ \forall\, x, y \in \mathbb{R}^+.$$

Definition 1.5. [6] Let $h, F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be two given functions. Then the pair (F, h) is said to be a upper class of type I if h is a subclass function of type I and

$$0 \leq s \leq 1 \Rightarrow F(s,t) \leq F(1,t),$$
$$h(1,y) \leq F(s,t) \Rightarrow y \leq st, \ \forall\, s, t, x, y \in \mathbb{R}^+.$$

Definition 1.6. [6] Let $h, F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be two given functions. Then the pair (F, h) is said to be a special upper class of type I if h is a subclass function of type I and

$$0 \leq s \leq 1 \Rightarrow F(s,t) \leq F(1,t),$$
$$h(1,y) \leq F(1,t) \Rightarrow y \leq t, \ \forall\, s, t, x, y \in \mathbb{R}^+.$$

Next, we move to the definition of C-class function.

Definition 1.7. [22] A continuous mapping $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a C-class function if it satisfies the following two properties:

(1) $\mathcal{F}(s,t) \leq s$;
(2) $\mathcal{F}(s,t) = s$ implies either $s = 0$ or $t = 0$, $\forall\, s, t \in \mathbb{R}^+$.

From now we write " $\mathcal{C}$ " to denote the collection of all C-class functions. Now, we give the definition of altering distance function.

Definition 1.8. [32] A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an altering distance function if the following two properties are satisfied:

(1) ψ is non-decreasing and continuous;
(2) $\psi(t) = 0 \Leftrightarrow t = 0$.

From now, we will write " Ψ " to denote the collection of all altering distance functions. In 2022, Karapınar and Khojasteh [30] introduced the notion of super metric spaces, which is one of the recent and marvellous extension of usual metric spaces.

Definition 1.9. [30] Let $\mathfrak{X} \neq \emptyset$. A mapping $\mathfrak{m} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ is said to be a super metric space (abbreviated as **SMS**) if $\mathfrak{m}$ satisfies the following three conditions:
(C_1) if $\mathfrak{m}(x,y) = 0 \Rightarrow x = y,\ \forall\ x, y\ \in \mathfrak{X}$;
(C_2) $\mathfrak{m}(x,y) = \mathfrak{m}(y,x),\ \forall\ x, y\ \in \mathfrak{X}$;
(C_3) there exists texta "$\omega \geq 1$" such that for every $y \in \mathfrak{X}$, $\exists$ two distinct sequences $\{x_n\}, \{y_n\} \subset \mathfrak{X}$ with $\lim\limits_{n\to\infty} \mathfrak{m}(x_n, y_n) = 0$, such that

$$\limsup_{n\to\infty} \mathfrak{m}(y_n, y) \leq \omega \limsup_{n\to\infty} \mathfrak{m}(x_n, y).$$

Then the pair $(\mathfrak{X}, \mathfrak{m})$ is called a SMS.

Note. From (C_1) of Definition 1.9, it is clear that if $x = y$ then $\mathfrak{m}(x,y)$ may not be equals to 0.
For examples on SMS, we refer the reader to see [28], [30].

Definition 1.10. [30] Let $(\mathfrak{X}, \mathfrak{m})$ be a SMS. Then a sequence $\{x_n\}$ is said to be

(1) convergent and converges to x in $\mathfrak{X}$ iff $\lim\limits_{n\to\infty} \mathfrak{m}(x_n, x) = 0$.
(2) a Cauchy sequence in $\mathfrak{X}$ if $\limsup\limits_{n\to\infty}\{\mathfrak{m}(x_n, x_p) : p > n\} = 0$.

Proposition 1.1. [28] *The limit of a convergent sequence is unique in a SMS.*

Definition 1.11. [28] Let $(\mathfrak{X}, \mathfrak{m})$ be a SMS. Then $(\mathfrak{X}, \mathfrak{m})$ is said to be complete iff every Cauchy sequence in $\mathfrak{X}$ is convergent in $\mathfrak{X}$.

From now, we will write **CSMS** to mean "complete super metric space". Next, we introduce the following definition on SMSs.

Definition 1.12. Let $(\mathfrak{X}_1, \mathfrak{m}_1)$ and $(\mathfrak{X}_2, \mathfrak{m}_2)$ be two SMSs and $K : \mathfrak{X}_1 \to \mathfrak{X}_2$ be a function. Then K is continuous at $x \in \mathfrak{X}_1$ iff $\{Kx_n\}$ converges to a point Kx whenever $\{x_n\}$ converges to a point x.

Before going to our main results, first we observe that in general a SMS is not necessarily continuous as seen in the following example.

Example 1.1. Let $\mathfrak{X} = \mathbb{R}^+$ and denote $\Delta = \{(x,x) : x \in \mathfrak{X}\}$. Define a mapping $\mathfrak{m} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ in the following way.

$$\mathfrak{m}(x,y) = x^2 + y^2,\ \text{if } x \neq y, x \neq 0, y \neq 0;$$
$$\mathfrak{m}(x,x) = 0,\ \text{if } (x,x) \in \Delta\backslash(0,0);$$
$$\mathfrak{m}(0,0) = 1;$$
$$\mathfrak{m}(x,0) = \mathfrak{m}(0,x) = \frac{x^3}{2},\ \text{if } x \in (0,\infty).$$

Clearly, conditions $(C_1), (C_2)$ of Definition 1.9 are satisfied. To check condition (C_3), we now do the following two cases.

Case 1. Suppose $y \neq 0$ and $\exists$ two distinct sequences $\{x_n\}, \{y_n\}$ such that $\lim_{n\to\infty} \mathfrak{m}(x_n, y_n) = 0$, i.e., $\lim_{n\to\infty}(x_n^2 + y_n^2) = 0$ implies $x_n \to 0$ and $y_n \to 0$ whenever $n \to \infty$. Consequently, we have

$$\overline{\lim_{n\to\infty}} \mathfrak{m}(y_n, y) = \overline{\lim_{n\to\infty}}(y_n^2 + y^2) = y^2 \leq \omega y^2 = \omega \overline{\lim_{n\to\infty}}(x_n^2 + y^2) = \omega \overline{\lim_{n\to\infty}} \mathfrak{m}(x_n, y).$$

Case 2. Now for $y = 0$, we are taking $x_n = \frac{1}{2n}, y_n = \frac{1}{2n+1}, n \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of all natural numbers). Then, $\mathfrak{m}(\frac{1}{2n}, \frac{1}{2n+1}) = \left(\frac{1}{2n}\right)^2 + \left(\frac{1}{2n+1}\right)^2 \to 0$ whenever $n \to \infty$.

$$\overline{\lim_{n\to\infty}} \mathfrak{m}(y_n, 0) = \overline{\lim_{n\to\infty}} \frac{1}{2(2n+1)^3} = 0 \leq \omega \overline{\lim_{n\to\infty}} \frac{1}{2(2n)^3} = \omega \overline{\lim_{n\to\infty}} \mathfrak{m}(x_n, 0).$$

Hence $(\mathfrak{X}, \mathfrak{m})$ is a SMS which is neither a b-metric space [16] nor a b-metric-like space [3] nor a JS-metric space [26], as we observe the following. Suppose $\exists$ a real number $b \in [1, \infty)$ for which the following inequality holds.

$$\mathfrak{m}(0, x) \leq b[\mathfrak{m}(0, 1) + \mathfrak{m}(1, x)] \Rightarrow \frac{x^3}{2} \leq b[\frac{1}{2} + 1 + x^2] = b[1.5 + x^2] \Rightarrow \frac{x^3}{2x^2 + 3} \leq b.$$

Thus $(\mathfrak{X}, \mathfrak{m})$ is neither a b-metric space nor a b-metric-like space since $\mathfrak{X}$ is unbounded. Next consider a sequence $\{x_n\}$ given in Case 2. Then $\lim_{n\to\infty} \mathfrak{m}(x_n, 0) = 0$. Suppose $\exists$ a $C \in (0, \infty)$ for which the following inequality holds

$$\mathfrak{m}(0, y) \leq C \limsup_{n\to\infty} \mathfrak{m}(x_n, y) \Rightarrow \frac{y^3}{2} \leq C \limsup_{n\to\infty}[\left(\frac{1}{2n}\right)^2 + y^2]$$

$$\Rightarrow \frac{y^3}{2} \leq C[0 + y^2] \Rightarrow \frac{y}{2} \leq C.$$

Thus $(\mathfrak{X}, \mathfrak{m})$ is not a JS-metric space since $\mathfrak{X}$ is unbounded. Notice that, here we have $\lim_{n\to\infty} \mathfrak{m}(x_n, 0) = \lim_{n\to\infty} \mathfrak{m}(y_n, 0) = 0$, which means $x_n \to 0 (= x^*)$ and $y_n \to 0 (= y^*)$ whenever $n \to \infty$, but

$$\lim_{n\to\infty} \mathfrak{m}(x_n, y_n) = \lim_{n\to\infty} \left[\left(\frac{1}{2n}\right)^2 + \left(\frac{1}{2n+1}\right)^2 \right] = 0 \neq \mathfrak{m}(x^*, y^*).$$

Consequently, we conclude that the SMS $(\mathfrak{X}, \mathfrak{m})$ is not continuous.

Motivated from the current ongoing research on Geraghty contractions, α-admissible mappings, (F, h)-contractions, C-class functions and results on SMSs, next we are going to extend and generalize Theorem 1.1 as well as P-contraction proposed by Popescu [38] in the setting of SMSs.

2. Main results

Let $(\mathfrak{X}, \mathfrak{m})$ be a SMS. Suppose $K, L : \mathfrak{X} \to \mathfrak{X}$ be two operators. For any $x, y \in \mathfrak{X}$, we write

$$\mathcal{M}_{x,y} = \max\Big\{\mathfrak{m}(x, y), \mathfrak{m}(x, Ky), \mathfrak{m}(y, Ly), \frac{\mathfrak{m}(x, Kx)\mathfrak{m}(y, Ly)}{1 + \mathfrak{m}(x, y)}, \frac{\mathfrak{m}(x, Kx)\mathfrak{m}(y, Ly)}{1 + \mathfrak{m}(Kx, Ly)}\Big\},$$

and

$$\mathcal{M}^*_{x,y} = \max\Big\{\mathfrak{m}(x, y), \mathfrak{m}(x, Kx), \mathfrak{m}(y, Ky), \frac{\mathfrak{m}(x, Kx)\mathfrak{m}(y, Ky)}{1 + \mathfrak{m}(x, y)}, \frac{\mathfrak{m}(x, Kx)\mathfrak{m}(y, Ky)}{1 + \mathfrak{m}(Kx, Ky)}\Big\}.$$

We now define two important subsets of $\mathfrak{X}$ as follows.

$$C(K,L)=\{x\in\mathfrak{X}:Kx=Lx\},$$

i.e., the set of all common coincidence points of the pair (K,L), and

$$Fix(K,L)=\{x\in\mathfrak{X}:x=Kx=Lx\},$$

i.e., the set of all common fixed points (abbreviated as **CFP**) of the pair (K,L). Now, we state and prove our first main result.

Theorem 2.1. *Let $(\mathfrak{X},\mathfrak{m})$ be a CSMS and $K,L:\mathfrak{X}\to\mathfrak{X}$ be two operators. Let $\psi,\eta:\mathbb{R}^+\to\mathbb{R}^+$ be two mappings such that $\psi\in\Psi$ and $\eta(x)\leq\psi(x)$, whenever $x>0$. Also, let $\alpha,\mu:\mathfrak{X}\times\mathfrak{X}\to\mathbb{R}^+$ be two given mappings such that they are symmetric. Assume that for $\theta\in\Theta$, the pair (K,L) satisfies the following inequality*

$$h(\alpha(x,y),\psi(\mathfrak{m}(Kx,Ly)))\leq F(\mu(x,y),\theta(\mathcal{M}_{x,y})\eta(\mathcal{M}_{x,y})), \tag{2.1}$$

$\forall\ x,y\in\mathfrak{X}$ with $\alpha(x,y)\geq1,\mu(x,y)\leq1$, where the pair (F,h) is a special upper class of type I. Further assume that the pair (K,L) satisfies the following assertions.

1. *The pair (K,L) is a weakly α-admissible and weakly μ-subadmissible mappings.*
2. *There exists a point $x_0\in\mathfrak{X}$ such that $\alpha(x_0,Kx_0)\geq1$ and $\mu(x_0,Kx_0)\leq1$.*
3. *Either K or L is continuous.*

Then we can construct a sequence $\{x_n\}$ in $\mathfrak{X}$ such that $x_n\to a^*(\in\mathfrak{X})$. If $\mathfrak{m}(a^*,a^*)=0$ with $\alpha(a^*,a^*)\geq1$ and $\mu(a^*,a^*)\leq1$, then a^* is a CFP of the pair (K,L). Moreover, if $a^*,b^*\in Fix(K,L)$ such that $\mathfrak{m}(a^*,a^*)=\mathfrak{m}(b^*,b^*)=0$ together with $\alpha(a^*,b^*)\geq1$ and $\mu(a^*,b^*)\leq1$. Then the CFP of the pair (K,L) is unique.

Proof. According to assertion 2, $\exists$ a $x_0\in\mathfrak{X}$ such that $\alpha(x_0,Kx_0)\geq1$ and $\mu(x_0,Kx_0)\leq1$. We now construct an iterative sequence $\{x_n\}$ of points in $\mathfrak{X}$ such that $x_1=Kx_0,x_2=Lx_1$ and in this way, we can write $x_{2n+1}=Kx_{2n}$ and $x_{2n+2}=Lx_{2n+1}$, $\forall\ n\in\mathbb{N}_0$(where $\mathbb{N}_0=\{0\}\cup\mathbb{N}$). Now, by assertion 1, we have $\alpha(x_0,Kx_0)=\alpha(x_0,x_1)\geq1$ implies $\alpha(Kx_0,LKx_0)=\alpha(x_1,Lx_1)=\alpha(x_1,x_2)\geq1$ and $\alpha(Lx_1,KLx_1)=\alpha(x_2,x_3)\geq1$. Again $\alpha(x_2,x_3)\geq1$ implies that $\alpha(Kx_2,LKx_2)=\alpha(x_3,x_4)\geq\alpha(x_3,x_4)\geq1$ and $\alpha(Lx_3,KLx_3)=\alpha(x_4,x_5)\geq1$. Continuing in this way, we can obtain $\alpha(Kx_{2n},LKx_{2n})=\alpha(x_{2n+1},x_{2n+2})\geq1$ and $\alpha(Lx_{2n-1},KLx_{2n-1})=\alpha(x_{2n},x_{2n+1})\geq1$. Hence $\alpha(x_n,x_{n+1})\geq1$, $\forall\ n\in\mathbb{N}_0$. In a similar way, using weakly μ-subadmissible property we can show that $\mu(x_n,x_{n+1})\leq1$, $\forall\ n\in\mathbb{N}_0$. Suppose $\exists$ a $c\in\mathbb{N}$ such that $\mathfrak{m}(x_{2c},x_{2c+1})=0$. Then $x_{2c}=x_{2c+1}=Kx_{2c}$, i.e., x_{2c} is a fixed point of K. Now to show that x_{2c} is a fixed point of L, it is sufficient to show that $x_{2c}=x_{2c+1}=x_{2c+2}$. Consider $\mathfrak{m}(x_{2c+1},x_{2c+2})>0$. Since $\alpha(x_{2c},x_{2c+1})\geq1$ and $\mu(x_{2c},x_{2c+1})\leq1$ thus by inequality 2.1, we have

$$\begin{aligned}
&h(1,\psi(\mathfrak{m}(x_{2c+1},x_{2c+2})))\\
&=h(1,\psi(\mathfrak{m}(Kx_{2c},Lx_{2c+1})))\\
&\leq h(\alpha(x_{2c},x_{2c+1}),\psi(\mathfrak{m}(Kx_{2c},Lx_{2c+1})))\\
&\leq F(\mu(x_{2c},x_{2c+1}),\theta(\mathcal{M}_{x_{2c},x_{2c+1}})\eta(\mathcal{M}_{x_{2c},x_{2c+1}}))\\
&\leq F(1,\theta(\mathcal{M}_{x_{2c},x_{2c+1}})\eta(\mathcal{M}_{x_{2c},x_{2c+1}})).
\end{aligned}$$

Since the pair (F,h) is a special upper class of type I, so we have

$$\psi(\mathfrak{m}(x_{2c+1},x_{2c+2}))\leq\theta(\mathcal{M}_{x_{2c},x_{2c+1}})\eta(\mathcal{M}_{x_{2c},x_{2c+1}}).\tag{2.2}$$

Now,

$$\begin{aligned}\mathcal{M}_{x_{2c},x_{2c+1}}=\max\Big\{&\mathfrak{m}(x_{2c},x_{2c+1}),\mathfrak{m}(x_{2c},x_{2c+1}),\mathfrak{m}(x_{2c+1},x_{2c+2}),\\ &\frac{\mathfrak{m}(x_{2c},x_{2c+1})\mathfrak{m}(x_{2c+1},x_{2c+2})}{1+\mathfrak{m}(x_{2c},x_{2c+1})},\frac{\mathfrak{m}(x_{2c},x_{2c+1})\mathfrak{m}(x_{2c+1},x_{2c+2})}{1+\mathfrak{m}(x_{2c+1},x_{2c+2})}\Big\}.\end{aligned}$$

Then, from 2.2, we have

$$\begin{aligned}\psi(\mathfrak{m}(x_{2c+1},x_{2c+2}))\leq\theta(\max\{\mathfrak{m}(x_{2c},x_{2c+1}),\mathfrak{m}(x_{2c+1},x_{2c+2})\})\eta(\max\{\mathfrak{m}(x_{2c},x_{2c+1}),\\ \mathfrak{m}(x_{2c+1},x_{2c+2})\}).\end{aligned}$$

Since we have taken $\mathfrak{m}(x_{2c},x_{2c+1})=0$, consequently we have

$$\begin{aligned}\psi(\mathfrak{m}(x_{2c+1},x_{2c+2}))&\leq\theta(\mathfrak{m}(x_{2c+1},x_{2c+2}))\eta(\mathfrak{m}(x_{2c+1},x_{2c+2}))\\ &<\eta(\mathfrak{m}(x_{2c+1},x_{2c+2}))\leq\psi(\mathfrak{m}(x_{2c+1},x_{2c+2})),\end{aligned}$$

which is a contradiction, since we have assumed $\mathfrak{m}(x_{2c+1},x_{2c+2})>0$. Thus, we must have $\mathfrak{m}(x_{2c+1},x_{2c+2})=0$, i.e., $x_{2c}=x_{2c+1}=x_{2c+2}$. Hence, x_{2c} is a CFP of K and L. Now suppose if $\mathfrak{m}(x_{2c-1},x_{2c})=0\Rightarrow x_{2c-1}=x_{2c}=L(x_{2c-1})$, then x_{2c-1} is a fixed point of the operator L. Now, we will show that x_{2c-1} is also a fixed point of K, i.e., it is sufficient to show that $x_{2c}=x_{2c+1}$. Suppose on the contrary, i.e., $\mathfrak{m}(x_{2c},x_{2c+1})>0$. Then $\mathfrak{m}(x_{2c},x_{2c+1})=\mathfrak{m}(Lx_{2c-1},Kx_{2c})=\mathfrak{m}(Kx_{2c},Lx_{2c-1})$. Now by using inequality 2.1 and keeping in mind that the functions α,μ both are symmetric, we have the following.

$$\psi(\mathfrak{m}(x_{2c},x_{2c+1}))=\psi(\mathfrak{m}(Kx_{2c},Lx_{2c-1}))\leq\theta(\mathcal{M}_{x_{2c},x_{2c-1}})\eta(\mathcal{M}_{x_{2c},x_{2c-1}}),\tag{2.3}$$

where $\mathcal{M}_{x_{2c},x_{2c-1}}=\max\{\mathfrak{m}(x_{2c},x_{2c-1}),\mathfrak{m}(x_{2c},x_{2c+1})\}=\mathfrak{m}(x_{2c},x_{2c+1})$. Clearly, from 2.3, we arrive at a contradiction, since we have assumed $\mathfrak{m}(x_{2c},x_{2c+1})>0$. Thus, we have $\mathfrak{m}(x_{2c},x_{2c+1})=0\Rightarrow x_{2c}=x_{2c+1}$. Hence, we assume that $\mathfrak{m}(x_n,x_{n+1})\neq0$, $\forall\ n\in\mathbb{N}$, i.e., $x_n\neq x_{n+1},\ \forall\ n\in\mathbb{N}$. Suppose n is a positive even integer, i.e., $n=2e$, for some $e\in\mathbb{N}$. Since α,μ both are symmetric so $\alpha(x_{2e},x_{2e-1})=\alpha(x_{2e-1},x_{2e})\geq1$ and $\mu(x_{2e},x_{2e-1})=\mu(x_{2e-1},x_{2e})\leq1$. Thus we have

$$\begin{aligned}&h(1,\psi(\mathfrak{m}(x_n,x_{n+1})))\\ &=h(1,\psi(\mathfrak{m}(x_{2e},x_{2e+1})))\\ &=h(1,\psi(\mathfrak{m}(Kx_{2e},Lx_{2e-1})))\\ &\leq h(\alpha(x_{2e},x_{2e-1}),\psi(\mathfrak{m}(Kx_{2e},Lx_{2e-1})))\\ &\leq F(\mu(x_{2e},x_{2e-1}),\theta(\mathcal{M}_{x_{2e},x_{2e-1}}))\eta(\mathcal{M}_{x_{2e},x_{2e-1}}))\\ &\leq F(1,\theta(\mathcal{M}_{x_{2e},x_{2e-1}}))\eta(\mathcal{M}_{x_{2e},x_{2e-1}})),\end{aligned}$$

implies

$$\psi(\mathfrak{m}(x_{2e},x_{2e+1}))\leq\theta(\mathcal{M}_{x_{2e},x_{2e-1}})\eta(\mathcal{M}_{x_{2e},x_{2e-1}})<\eta(\mathcal{M}_{x_{2e},x_{2e-1}}).\tag{2.4}$$

Now

$$\mathcal{M}_{x_{2e},x_{2e-1}}=\max\{\mathfrak{m}(x_{2e},x_{2e-1}),\mathfrak{m}(x_{2e},x_{2e+1})\}.$$

If we choose $\max\{\mathfrak{m}(x_{2e}, x_{2e-1}), \mathfrak{m}(x_{2e}, x_{2e+1})\} = \mathfrak{m}(x_{2e}, x_{2e+1})$, then from 2.4, we arrive at a contradiction. Hence, we have

$$\max\{\mathfrak{m}(x_{2e}, x_{2e-1}), \mathfrak{m}(x_{2e}, x_{2e+1})\} = \mathfrak{m}(x_{2e-1}, x_{2e}).$$

Consequently, from 2.4, we obtain

$$\psi(\mathfrak{m}(x_{2e}, x_{2e+1})) < \psi(\mathfrak{m}(x_{2e-1}, x_{2e})).$$

Now, ψ is an altering distance function, so we have

$$\mathfrak{m}(x_{2e}, x_{2e+1}) < \mathfrak{m}(x_{2e-1}, x_{2e}).$$

Thus, in general, we can write

$$\mathfrak{m}(x_{2n}, x_{2n+1}) < \mathfrak{m}(x_{2n-1}, x_{2n}),\ \forall\ n \in \mathbb{N}. \tag{2.5}$$

Next, we suppose that n is an odd positive integer, i.e., $n = 2r + 1$, for some $r \in \mathbb{N}$. Now, we have

$$\begin{aligned}
&h(1, \psi(\mathfrak{m}(x_n, x_{n+1})))\\
&= h(1, \psi(\mathfrak{m}(x_{2r+1}, x_{2r+2})))\\
&= h(\alpha(x_{2r}, x_{2r+1}), \psi(\mathfrak{m}(Kx_{2r}, Lx_{2r+1})))\\
&\leq F(\mu(x_{2r}, x_{2r+1}), \theta(\mathcal{M}_{x_{2r},x_{2r+1}}))\eta(\mathcal{M}_{x_{2r},x_{2r+1}}))\\
&\leq F(1, \theta(\mathcal{M}_{x_{2r},x_{2r+1}}))\eta(\mathcal{M}_{x_{2r},x_{2r+1}})),
\end{aligned}$$

implies

$$\psi(\mathfrak{m}(x_{2r+1}, x_{2r+2})) \leq \theta(\mathcal{M}_{x_{2r},x_{2r+1}})\eta(\mathcal{M}_{x_{2r},x_{2r+1}}) < \eta(\mathcal{M}_{x_{2r},x_{2r+1}}). \tag{2.6}$$

Now

$$\mathcal{M}_{x_{2r},x_{2r+1}} = \max\{\mathfrak{m}(x_{2r}, x_{2r+1}), \mathfrak{m}(x_{2r+1}, x_{2r+2})\}.$$

If we choose $\max\{\mathfrak{m}(x_{2r}, x_{2r+1}), \mathfrak{m}(x_{2r+1}, x_{2r+2})\} = \mathfrak{m}(x_{2r+1}, x_{2r+2})$, then from 2.6, we arrive at a contradiction. Thus

$$\max\{\mathfrak{m}(x_{2r}, x_{2r+1}), \mathfrak{m}(x_{2r+1}, x_{2r+2})\} = \mathfrak{m}(x_{2r}, x_{2r+1}).$$

Consequently, from 2.6, we have

$$\psi(\mathfrak{m}(x_{2r+1}, x_{2r+2})) < \psi(\mathfrak{m}(x_{2r}, x_{2r+1}))$$

implies

$$\mathfrak{m}(x_{2r+1}, x_{2r+2}) < \mathfrak{m}(x_{2r}, x_{2r+1})$$

(by using the property of altering distance function ψ). So, in general, we can write

$$\mathfrak{m}(x_{2n+1}, x_{2n+2}) < \mathfrak{m}(x_{2n}, x_{2n+1}),\ \forall\ n \in \mathbb{N}. \tag{2.7}$$

Thus, combining inequality 2.5 and 2.7, we obtain

$$\mathfrak{m}(x_n, x_{n+1}) < \mathfrak{m}(x_{n-1}, x_n),\ \forall\ n \in \mathbb{N}. \tag{2.8}$$

Hence, the sequence $\{\mathfrak{m}(x_n, x_{n+1})\}_{n=0}^{\infty}$ is a decreasing sequence which is bounded by 0. Consequently, $\exists$ a non negative real number, say $l^* \in \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \mathfrak{m}(x_n, x_{n+1}) = l^*.$$

Our goal is to show that $l^* = 0$. Suppose on the contrary, i.e., $l^* > 0$. Then, from 2.4, 2.6, we have

$$\psi(\mathfrak{m}(x_n, x_{n+1})) \leq \theta(\mathfrak{m}(x_{n-1}, x_n))\eta(\mathfrak{m}(x_{n-1}, x_n)).$$

Now considering limit as $n \to \infty$, we have

$$\psi(l^*) \leq \theta(l^*)\eta(l^*) < \eta(l^*) \leq \psi(l^*),$$

a contradiction (since $l^* > 0$). Hence, we must have $l^* = 0$, i.e.,

$$\lim_{n\to\infty} \mathfrak{m}(x_n, x_{n+1}) = 0. \tag{2.9}$$

Our next aim is to show that $x_{2n} \neq x_{2p}$ and $x_{2n+1} \neq x_{2p+1}$ for $n \neq p$. we will show it by the method of contradiction. Without loss of generality, we may suppose that
Case A. $\exists\, p, n \in \mathbb{N}_0$ such that $x_{2p} = x_{2n}$, with $p > n$.
Case B. $\exists\, p, n \in \mathbb{N}_0$ such that $x_{2p+1} = x_{2n+1}$, with $p > n$.
Now in Case A, utilizing 2.8, we have

$$\mathfrak{m}(x_{2n}, x_{2n+1}) = \mathfrak{m}(x_{2n}, Kx_{2n}) = \mathfrak{m}(x_{2p}, Kx_{2p}) = \mathfrak{m}(x_{2p}, x_{2p+1}) < \mathfrak{m}(x_{2n}, x_{2n+1}),$$

which is a contradiction. Again in Case B, utilizing 2.8, we have

$$\begin{aligned}\mathfrak{m}(x_{2n+2}, x_{2n+1}) = \mathfrak{m}(Lx_{2n+1}, x_{2n+1}) = \mathfrak{m}(Lx_{2p+1}, x_{2p+1}) = \mathfrak{m}(x_{2p+2}, x_{2p+1})\\ < \mathfrak{m}(x_{2n+2}, x_{2n+1}),\end{aligned}$$

which is a contradiction. Thus, we have the following observation. For any $n \in \mathbb{N}$, there is at most one $p \neq n$ such that $x_p = x_n$. In fact, if $\exists$ three different integers a, b, c such that $x_a = x_b = x_c$ then there are two distinct numbers $q, r \in \{a, b, c\}$ such that $q \equiv r \bmod 2$. Now, if we take $q = a$ and $r = b$, then $x_a \neq x_b$ by our assumption in Case A and Case B, but that is a contradiction. Next, our aim is to show that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since we have already shown that $\lim_{n\to\infty} \mathfrak{m}(x_n, x_{n+1}) = 0$, so it is sufficient to show that the sub sequence $\{x_{2n}\}$ is a Cauchy sequence in order to prove that the sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Therefore, by using the property of super metric spaces, we have

$$\overline{\lim_{n\to\infty}} \mathfrak{m}(x_{2n}, x_{2n+2}) \leq \omega \overline{\lim_{n\to\infty}} \mathfrak{m}(x_{2n+1}, x_{2n+2}).$$

By using 2.9, we have

$$\overline{\lim_{n\to\infty}} \mathfrak{m}(x_{2n}, x_{2n+2}) = 0. \tag{2.10}$$

Now, by using 2.10 and Case A, we have

$$\overline{\lim_{n\to\infty}} \mathfrak{m}(x_{2n}, x_{2n+4}) \leq \omega \overline{\lim_{n\to\infty}} \mathfrak{m}(x_{2n+2}, x_{2n+4}). \tag{2.11}$$

But, we have

$$\overline{\lim_{n\to\infty}} \mathfrak{m}(x_{2n+2}, x_{2n+4}) \leq \omega \overline{\lim_{n\to\infty}} \mathfrak{m}(x_{2n+3}, x_{2n+4}).$$

By using 2.9, we have $\overline{\lim_{n\to\infty}} \mathfrak{m}(x_{2n+3}, x_{2n+4}) = 0$. Consequently, from 2.11,

$$\overline{\lim_{n\to\infty}} \mathfrak{m}(x_{2n}, x_{2n+4}) = 0.$$

Now, proceeding in this way, one can show that

$$\overline{\lim_{n\to\infty}} \mathfrak{m}(x_{2n}, x_{2n+2p}) = 0,\ \forall\, n, p \in \mathbb{N}.$$

It means $\{x_{2n}\}_{n=0}^{\infty}$ is a Cauchy sequence, i.e., $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since, $(\mathfrak{X}, \mathfrak{m})$ is a CSMS, consequently the sequence $\{x_n\}$ will convergence to some point in $\mathfrak{X}$, say a^*, which means $x_{2n+1} \to a^*$ and $x_{2n+2} \to a^*$ whenever $n \to \infty$. First, we suppose that K is continuous. Then, we have

$$a^* = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} K(x_{2n}) = K(\lim_{n\to\infty} x_{2n}) = Ka^*.$$

Now if $\mathfrak{m}(a^*, La^*) > 0$ with $\mathfrak{m}(a^*, a^*) = 0$ and $\alpha(a^*, a^*) \geq 1, \mu(a^*, a^*) \leq 1$, then by 2.1, we have

$$\begin{aligned}
&h(1, \psi(\mathfrak{m}(Ka^*, La^*)))\\
&\leq h(\alpha(a^*, a^*), \psi(\mathfrak{m}(Ka^*, La^*)))\\
&\leq F(\mu(a^*, a^*), \theta(\mathcal{M}_{a^*,a^*})\eta(\mathcal{M}_{a^*,a^*}))\\
&\leq F(1, \theta(\mathcal{M}_{a^*,a^*})\eta(\mathcal{M}_{a^*,a^*})).
\end{aligned}$$

Since the pair (F, h) is a special upper class of type I, so we have

$$\psi(\mathfrak{m}(Ka^*, La^*)) \leq \theta(\mathcal{M}_{a^*,a^*})\eta(\mathcal{M}_{a^*,a^*}). \tag{2.12}$$

Now,

$$\begin{aligned}
\mathcal{M}_{a^*,a^*} = \max \Big\{ &\mathfrak{m}(a^*, a^*), \mathfrak{m}(a^*, a^*), \mathfrak{m}(a^*, La^*),\\
&\frac{\mathfrak{m}(a^*, a^*)\mathfrak{m}(a^*, La^*)}{1 + \mathfrak{m}(a^*, a^*)}, \frac{\mathfrak{m}(a^*, a^*)\mathfrak{m}(a^*, La^*)}{1 + \mathfrak{m}(Ka^*, La^*)} \Big\}.
\end{aligned}$$

Thus, from 2.12, we have

$$\psi(\mathfrak{m}(a^*, La^*)) \leq \theta(\mathfrak{m}(a^*, La^*))\eta(\mathfrak{m}(a^*, La^*)) < \eta(\mathfrak{m}(a^*, La^*)) \leq \psi(\mathfrak{m}(a^*, La^*)),$$

a contradiction. Thus, we must have $\mathfrak{m}(a^*, La^*) = 0$. Hence $a^* = Ka^* = La^*$. In a similar way, if we take L is continuous then we can show that $a^* = Ka^* = La^*$. Now suppose that the self mappings K, L have two CFP, i.e., a^* and b^* satisfying $\mathfrak{m}(a^*, a^*) = \mathfrak{m}(b^*, b^*) = 0$ together with $\alpha(a^*, b^*) \geq 1$, $\mu(a^*, b^*) \leq 1$. Our goal is to show that $\mathfrak{m}(a^*, b^*) = 0$. Suppose on the contrary, i.e., $\mathfrak{m}(Ka^*, Lb^*) = \mathfrak{m}(a^*, b^*) > 0$. Consequently, from 2.1, we have

$$\begin{aligned}
&h(1, \psi(\mathfrak{m}(Ka^*, Lb^*)))\\
&\leq h(\alpha(a^*, b^*), \psi(\mathfrak{m}(Ka^*, Lb^*)))\\
&\leq F(\mu(a^*, b^*), \theta(\mathcal{M}_{a^*,b^*})\eta(\mathcal{M}_{a^*,b^*}))\\
&\leq F(1, \theta(\mathcal{M}_{a^*,b^*})\eta(\mathcal{M}_{a^*,b^*})),
\end{aligned}$$

implies

$$\psi(\mathfrak{m}(Ka^*, Lb^*)) \leq \theta(\mathcal{M}_{a^*,b^*})\eta(\mathcal{M}_{a^*,b^*}).$$

It can be easily checked that $\mathcal{M}_{a^*,b^*} = \mathfrak{m}(a^*, b^*)$. Hence, we have

$$\psi(\mathfrak{m}(a^*, b^*)) \leq \theta(\mathfrak{m}(a^*, b^*))\eta(\mathfrak{m}(a^*, b^*)) < \eta(\mathfrak{m}(a^*, b^*)) \leq \psi(\mathfrak{m}(a^*, b^*)),$$

a contradiction. Thus, we have $\mathfrak{m}(a^*, b^*) = 0 \Rightarrow a^* = b^*$. $\square$

Note. The inequality given in 2.1 is called as generalized ${}^{(\alpha,\mu)}(F,h)_{(\psi,\eta)}$-Geraghty contraction. Before going to our next result, we now introduce two important definitions on a SMS.

Definition 2.1. Let $(\mathfrak{X},\mathfrak{m})$ be a SMS and let $\alpha,\mu:\mathfrak{X}\times\mathfrak{X}\to\mathbb{R}^+$ be two mappings. Then, the space $(\mathfrak{X},\mathfrak{m})$ is said to be (α,μ)- regular if for any sequence $\{x_n\}\subset\mathfrak{X}$ such that $\alpha(x_n,x_{n+1})\geq 1,\mu(x_n,x_{n+1})\leq 1$ and $x_n\to a^*$ as $n\to\infty$, we have $\alpha(x_n,a^*)\geq 1,\mu(x_n,a^*)\leq 1,\ \forall\, n\in\mathbb{N}$.

Definition 2.2. A SMS $(\mathfrak{X},\mathfrak{m})$ is said to satisfy condition(M) if for any sequence $\{x_n\}\subset\mathfrak{X}$ with $x_n\to a^*$ whenever $n\to\infty$ implies $\mathfrak{m}(a^*,y)\leq\overline{\lim_{n\to\infty}}\mathfrak{m}(x_n,y),\ \forall\, y\ \in\mathfrak{X}$.

The continuity of the mappings K or L in Theorem 2.1 can be replaced by the (α,μ)-regularity property of the SMS $(\mathfrak{X},\mathfrak{m})$ provided the super metric $\mathfrak{m}$ satisfies the condition(M).

Theorem 2.2. *Let $(\mathfrak{X},\mathfrak{m})$ be a CSMS together with condition(M). Suppose $K,L:\mathfrak{X}\to\mathfrak{X}$ be two mappings satisfying all the hypotheses of Theorem 2.1 except assertion 3. Suppose in place of assertion 3, the following assertion is satisfied.*

$$\textit{Assertion } 3^*:\ \textit{The SMS } (\mathfrak{X},\mathfrak{m}) \textit{ is } (\alpha,\mu)-\textit{regular.}$$

Then we can construct a sequence $\{x_n\}$ in $\mathfrak{X}$ such that $x_n\to a^*(\in\mathfrak{X})$ and that a^* is a CFP of the pair (K,L). Moreover, if $a^*,b^*\in Fix(K,L)$ such that $\mathfrak{m}(a^*,a^*)=\mathfrak{m}(b^*,b^*)=0$ together with $\alpha(a^*,b^*)\geq 1$ and $\mu(a^*,b^*)\leq 1$. Then the CFP of the pair (K,L) is unique.

Proof. Following the proof of Theorem 2.1, we can construct a sequence $\{x_n\}\subset\mathfrak{X}$ such that $x_n\to a^*(\in\mathfrak{X})$ with $\alpha(x_n,x_{n+1})\geq 1,\mu(x_n,x_{n+1})\leq 1,\ \forall\, n\in\mathbb{N}_0$. By our assumption, $(\mathfrak{X},\mathfrak{m})$ satisfies (α,μ)-regularity property. Thus we have $\alpha(x_n,a^*)\geq 1,\mu(x_n,a^*)\leq 1,\ \forall\, n\in\mathbb{N}_0$ implies $\alpha(x_{2n},a^*)\geq 1,\alpha(x_{2n+1},a^*)\geq 1$ and $\mu(x_{2n},a^*)\leq 1,\mu(x_{2n+1},a^*)\leq 1,\ \forall\, n$. Thus, by inequality 2.1, we have

$$\begin{aligned}
&h(1,\psi(\mathfrak{m}(x_{2n+1},La^*)))\\
&\leq h(\alpha(x_{2n},a^*),\psi(\mathfrak{m}(Kx_{2n},La^*)))\\
&\leq F(\mu(x_{2n},a^*),\theta(\mathcal{M}_{x_{2n},a^*})\eta(\mathcal{M}_{x_{2n},a^*}))\\
&\leq F(1,\theta(\mathcal{M}_{x_{2n},a^*})\eta(\mathcal{M}_{x_{2n},a^*})).
\end{aligned}$$

Since the pair (F,h) is a special upper class of type I, so we have

$$\psi(\mathfrak{m}(x_{2n+1},La^*))\leq\theta(\mathcal{M}_{x_{2n},a^*})\eta(\mathcal{M}_{x_{2n},a^*}),\tag{2.13}$$

where

$$\begin{aligned}
\mathcal{M}_{x_{2n},a^*}=\max\Big\{&\mathfrak{m}(x_{2n},a^*),\mathfrak{m}(x_{2n},x_{2n+1}),\mathfrak{m}(a^*,La^*),\\
&\frac{\mathfrak{m}(x_{2n},x_{2n+1})\mathfrak{m}(a^*,La^*)}{1+\mathfrak{m}(x_{2n},a^*)},\frac{\mathfrak{m}(x_{2n},x_{2n+1})\mathfrak{m}(a^*,La^*)}{1+\mathfrak{m}(x_{2n+1},La^*)}\Big\}.
\end{aligned}$$

Now, $n\to\infty$ in 2.13, and keeping in mind that the super metric $\mathfrak{m}$ satisfies the condition(M) property, we have

$$\psi(\mathfrak{m}(a^*,La^*))\leq\theta(\mathfrak{m}(a^*,La^*))\eta(\mathfrak{m}(a^*,La^*)).\tag{2.14}$$

Now, assume that $\mathfrak{m}(a^*, La^*) > 0$. Hence, from 2.14, we obtain

$$\psi(\mathfrak{m}(a^*, La^*)) \leq \theta(\mathfrak{m}(a^*, La^*))\eta(\mathfrak{m}(a^*, La^*)) < \eta(\mathfrak{m}(a^*, La^*)) \leq \psi(\mathfrak{m}(a^*, La^*)),$$

a contradiction. Thus, we must have $\mathfrak{m}(a^*, La^*) = 0 \Rightarrow a^* = La^*$. In a similar way, by using $\alpha(x_{2n+1}, a^*) \geq 1, \mu(x_{2n+1}, a^*) \leq 1$ and keeping in mind that α, μ both are symmetric and $\mathfrak{m}$ satisfies condition(M), we can show that $Ka^* = a^*$. Thus, we have $a^* = Ka^* = La^*$. Further, by using the technique of Theorem 2.1, we can show that CFP of K and L is unique. □

We now do two important observations.

Observation 1. In [43], the authors have worked on the SMS and they have considered two self mappings T and S. Above equation (26) in their paper, they have used an argument which is " Now suppose that $k, n \in \mathbb{N}$ and $k > n$. If $y_n = y_k$, we have $T^k(y_0) = T^n(y_0)$. " On the other hand, the authors have constructed a sequence $\{y_n\}$ such that $y_{2n+2} = Ty_{2n+1}$ and $y_{2n+1} = Sy_{2n}$, i.e., $y_1 = S_0, y_2 = Ty_1 = TSy_0, y_3 = Sy_2 = STSy_0, y_4 = Ty_3 = TSTSy_0$, and so on. That means any term of that sequence can not be written as a single power of any particular operator T or S, i.e., one can not write $y_n = T^n y_0$ or $y_k = S^k y_0$. Hence, the argument used by S. K. Shah et al. [43] is not correct, but one can overcome such difficulty in the SMS by using our technique given in Theorem 2.1.

Next, we move to our second observation.

Observation 2. In [42], the authors have proved some interpolative fixed point results in the setting of SMSs. We have observed that in Theorem 3.2 below equation (7) or in Theorem 3.7 below equation (14), they are using the fact that $\lim_{n\to\infty} \mathfrak{m}(x_{n+1}, Uu) = \mathfrak{m}(u, Uu)$, which is not correct. Since in general a SMS may not be continuous which we have already shown in Example 1.1. Thus, to rectify those results, authors must mention that either their SMS is continuous or the super metric $\mathfrak{m}$ satisfies condition(M) to overcome such error.

Now, we state two important results as a corollary of our main results.

Corollary 2.1. *Let $(\mathfrak{X}, \mathfrak{m})$ be a CSMS and $K : \mathfrak{X} \to \mathfrak{X}$ be an operator. Let $\psi, \eta : \mathbb{R}^+ \to \mathbb{R}^+$ be two mappings such that $\psi \in \Psi$ and $\eta(x) \leq \psi(x)$, whenever $x > 0$. Also, let $\alpha, \mu : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ be two given mappings such that they are symmetric. Assume that for $\theta \in \Theta$, the mapping K satisfies the following inequality*

$$h(\alpha(x,y), \psi(\mathfrak{m}(Kx, Ky))) \leq F(\mu(x,y), \theta(\mathcal{M}^*_{x,y})\eta(\mathcal{M}^*_{x,y})), \tag{2.15}$$

$\forall\ x, y \in \mathfrak{X}$ with $\alpha(x,y) \geq 1, \mu(x,y) \leq 1$, where the pair (F, h) is a special upper class of type I. Further assume that the mapping K satisfies the following assertions.

(1) *The mapping K is a weakly α-admissible and weakly μ-subadmissible mappings.*
(2) *There exists a point $x_0 \in \mathfrak{X}$ such that $\alpha(x_0, Kx_0) \geq 1$ and $\mu(x_0, Kx_0) \leq 1$.*
(3) *The mapping K is continuous.*

Then we can construct a sequence $\{x_n\}$ in $\mathfrak{X}$ such that $x_n \to a^*(\in \mathfrak{X})$ and that a^* is a fixed point of the mapping K. Moreover, if $a^*, b^* \in Fix(K)$ such that $\mathfrak{m}(a^*, a^*) = \mathfrak{m}(b^*, b^*) = 0$ together with $\alpha(a^*, b^*) \geq 1$ and $\mu(a^*, b^*) \leq 1$. Then the fixed point of the mapping K is unique.

Proof. Proof follows from the proof of Theorem 2.1, by considering $K = L$. $\square$

Next, we state our second corollary

Corollary 2.2. *Let $(\mathfrak{X}, \mathfrak{m})$ be a CSMS together with condition(M). Suppose $K : \mathfrak{X} \to \mathfrak{X}$ be a mapping satisfying all the hypotheses of Corollary 2.1 except assertion 3. Suppose in place of assertion 3, the following assertion is satisfied.*

$$\textit{Assertion } 3^* : \ \textit{The SMS } (\mathfrak{X}, \mathfrak{m}) \textit{ is } (\alpha, \mu) - \textit{regular.}$$

Then we can construct a sequence $\{x_n\}$ in $\mathfrak{X}$ such that $x_n \to a^*(\in \mathfrak{X})$ and that a^* is a fixed point of the mapping K. Moreover, if $a^*, b^* \in Fix(K)$ such that $\mathfrak{m}(a^*, a^*) = \mathfrak{m}(b^*, b^*) = 0$ together with $\alpha(a^*, b^*) \geq 1$ and $\mu(a^*, b^*) \leq 1$. Then the fixed point of the mapping K is unique.

Proof. Proof follows from the proof of Theorem 2.2, by considering $K = L$. $\square$

Before going to our example, first we focus on the following example, which was introduced by Erdal Karapınar and Andreea Fulga (see [28]) as Example 3 in their paper.

Example 2.1. Let $\mathfrak{X} = [0, 1],\ \omega = 1$ and the application $\mathfrak{m} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ be defined as follows:

$$\begin{aligned}
&\mathfrak{m}(x, y) = xy, \text{ for all } x \neq y,\ x, y \in (0, 1);\\
&\mathfrak{m}(x, y) = 0, \text{ for all } x = y,\ x, y \in [0, 1];\\
&\mathfrak{m}(0, y) = \mathfrak{m}(y, 0) = y, \text{ for all } y \in (0, 1];\\
&\mathfrak{m}(1, y) = \mathfrak{m}(y, 1) = 1 - \frac{y}{2}, \text{ for all } y \in [0, 1).
\end{aligned}$$

The authors claimed that this $(\mathfrak{X}, \mathfrak{m})$ is a SMS.

Some authors use example 2.1 to support their main results or they use the above example's main structure in their application (see, for example, [1], [43]). Now, we do the following observation.

Observation 3. Let us consider a sequence $\{x_n\}_{n=0}^{\infty} \subset \mathfrak{X}$, where $x_n = \frac{1}{n+2},\ n \in \{0, 1, 2, 3, \cdots\}$. Let us also consider two distinct real numbers p, q between $(0, 1)$, i.e., $p, q \in (0, 1)$ with $p \neq q$. Now, we have the following

$$\lim_{n\to\infty} \mathfrak{m}(x_n, p) = \lim_{n\to\infty} x_n p = 0 \text{ and } \lim_{n\to\infty} \mathfrak{m}(x_n, q) = \lim_{n\to\infty} x_n q = 0.$$

Now, according to the Definition 1.10, $\lim_{n\to\infty} \mathfrak{m}(x_n, p) = 0$ implies $x_n \to p$ whenever $n \to \infty$. Again, according to the Definition 1.10, $\lim_{n\to\infty} \mathfrak{m}(x_n, q) = 0$ implies $x_n \to q$ whenever $n \to \infty$. That means $\{x_n\}$ converges to both $p, q \in (0, 1)$. In fact, $\{x_n\}$ converges to any real number between $(0, 1)$ whenever $n \to \infty$. But it contradicts Proposition 1.1, since the limit of a convergent sequence is unique, which was proved by Erdal Karapınar and Andreea Fulga in the same paper. Hence, the example, given by Erdal Karapınar and Andreea Fulga, is not correct. Moreover, to show that their constructed SMS is not a JS-metric space, the authors have considered a sequence $\{z_n\}$ such that $\lim_{n\to\infty} z_n = 1$, i.e., $\lim_{n\to\infty} \mathfrak{m}(z_n, 1) = 0$. But, it can be observed that one can not find such type of sequence $\{z_n\}$, which converges to 1 because of their constructed super metric $\mathfrak{m}$ along with their chosen space $\mathfrak{X}$ as $[0,1]$. So, it essential to rectify the example. In our next example, we have corrected the example to support our main result.

Example 2.2. Let $\mathfrak{X} = \Delta_1 \cup \{0,1,2\} \cup \Delta_2$, where $\Delta_1 = (0,1)$ and $\Delta_2 = \{3,4,5,6,\cdots\}$. Define the application $\mathfrak{m} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ in the following way:

$$\mathfrak{m}(x,y) = x + y, \text{ for all } x \neq y, \ x,y \in \Delta_1 \cup \{2\} \cup \Delta_2;$$
$$\mathfrak{m}(x,y) = 0, \text{ for all } x = y, \ x,y \in \Delta_1 \cup \{0\} \cup \Delta_2;$$
$$\mathfrak{m}(0,y) = \mathfrak{m}(y,0) = y, \text{ for all } y \in \Delta_1 \cup \{2\} \cup \Delta_2;$$
$$\mathfrak{m}(1,y) = \mathfrak{m}(y,1) = (y+1)(| y - 1 |), \text{ for all } y \in \Delta_1 \cup \Delta_2.$$
$$\mathfrak{m}(x,y) = \max\{x,y\}, \text{ if } x,y \in \Delta_2 \text{ and both are even or both are odd;}$$
$$\mathfrak{m}(x,y) = 1, \text{ if } x,y \in \Delta_2 \text{ with one of them is odd and other one is even;}$$
$$\mathfrak{m}(1,1) = \mathfrak{m}(2,2) = 10; \mathfrak{m}(1,2) = \mathfrak{m}(2,1) = 3; \mathfrak{m}(1,0) = \mathfrak{m}(0,1) = 2.$$

Clearly, here conditions $(C_1), (C_2)$ of Definition 1.9 are satisfied. Consider two sequences $\{x_n\}, \{y_n\}$ such that $x_n = \frac{1}{2n}$, $y_n = \frac{1}{2n+1}$, where $n \in \mathbb{N}$ and observe that they are distinct with $\lim_{n\to\infty} \mathfrak{m}(x_n, y_n) = 0$. In fact, for those two sequences, there always exists a "$\omega \geq 1$" such that for every $y \in \mathfrak{X}$, we have

$$\limsup_{n\to\infty} \mathfrak{m}(y_n, y) \leq \omega \limsup_{n\to\infty} \mathfrak{m}(x_n, y).$$

Hence $(\mathfrak{X}, \mathfrak{m})$ is a SMS which is complete also. Take $w_n = 1 - \frac{1}{n}$, where $n \in \mathbb{N}$ and observe that $\lim_{n\to\infty} \mathfrak{m}(w_n, 1) = 0$, i.e., $w_n \to 1$ as $n \to \infty$, i.e., $\{w_n\} \in C(\mathfrak{m}, \mathfrak{X}, 1)$ (see the paper on JS-metric space [26]). Suppose there exists a $C > 0$ such that following inequality holds.

$$(y+1)(| y-1 |) = \mathfrak{m}(1,y) \leq C \limsup_{n\to\infty} \mathfrak{m}(w_n, y) = C \limsup_{n\to\infty} [w_n + y] \leq C[1+y],$$

which implies $| y - 1 | \leq C$, but one can not find any fixed C, since $y \in \Delta_2$. Hence, $(\mathfrak{X}, \mathfrak{m})$ is not a JS-metric space. Moreover, $(\mathfrak{X}, \mathfrak{m})$ is neither a metric space nor a b-metric space nor a b-metric like space. To verify this, we have the following. Let us take three positive integer $2p, 2p+1, 2q \in \Delta_2$ such that $p < q$. Then, we have the following.

$$\mathfrak{m}(2p, 2q) \leq b[\mathfrak{m}(2p, 2p+1) + \mathfrak{m}(2p+1, 2q)] \Rightarrow 2q \leq b[1+1] \Rightarrow q \leq b.$$

Since, q is unbounded so one can not find any fixed b. Now, we define two mappings $K, L : \mathfrak{X} \to \mathfrak{X}$ in the following way

$$K(0) = 0, K(1) = 0, K(2) = 1, L(0) = 0, L(1) = 1, L(2) = 0.$$

$$K(x) = \begin{cases} \frac{x}{x+1}, & \text{if } x \in \Delta_1 \ , \\ x+1, & \text{if } x \in \Delta_2. \end{cases}$$

$$L(x) = \begin{cases} \frac{x}{x+2}, & \text{if } x \in \Delta_1 \ , \\ x+2, & \text{if } x \in \Delta_2. \end{cases}$$

Consider $\Omega = \{(0,0),(1,1),(2,2),(1,0),(0,1)\}$ and define two functions $\alpha, \mu : \mathfrak{X}\times\mathfrak{X} \to \mathbb{R}^+$ in the following way.

$$\alpha(x,y) = \begin{cases} 1.2, & \text{if } (x,y) \in \Omega \ , \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu(x,y) = \begin{cases} .8, & \text{if } (x,y) \in \Omega \ , \\ 2, & \text{otherwise.} \end{cases}$$

Take $h(x,y) = xy$, and $F(s,t) = st$. Define $\psi(x) = \frac{x}{3}, \theta(x) = e^{-\frac{x}{15}}, \eta(x) = \frac{x}{3.2}$. Here the pair (K,L) is a weakly α-admissible mapping. To show that we have the following.

$$\alpha(x,y) \geq 1 \Rightarrow (x,y) \in \Omega.$$

Now,

$$\begin{aligned} \alpha(0,0) \geq 1 \ &\Rightarrow \alpha(K0, LK0) = \alpha(0,0) \geq 1 \text{ and} \\ &\quad\ \alpha(L0, KL0) = \alpha(0,0) \geq 1. \end{aligned}$$

$$\begin{aligned} \alpha(1,1) \geq 1 \ &\Rightarrow \alpha(K1, LK1) = \alpha(0,0) \geq 1 \text{ and} \\ &\quad\ \alpha(L1, KL1) = \alpha(1,0) \geq 1. \end{aligned}$$

$$\begin{aligned} \alpha(2,2) \geq 1 \ &\Rightarrow \alpha(K2, LK2) = \alpha(1,1) \geq 1 \text{ and} \\ &\quad\ \alpha(L2, KL2) = \alpha(0,0) \geq 1. \end{aligned}$$

$$\begin{aligned} \alpha(1,0) \geq 1 \ &\Rightarrow \alpha(K1, LK1) = \alpha(0,0) \geq 1 \text{ and} \\ &\quad\ \alpha(L0, KL0) = \alpha(0,0) \geq 1. \end{aligned}$$

$$\begin{aligned} \alpha(0,1) \geq 1 \ &\Rightarrow \alpha(K0, LK0) = \alpha(0,0) \geq 1 \text{ and} \\ &\quad\ \alpha(L1, KL1) = \alpha(1,0) \geq 1. \end{aligned}$$

In a similar way, it can be shown that the pair (K,L) is a weakly μ-subadmissible mapping. Also, observe that $\alpha(1,K1) \geq 1, \mu(1,K1) \leq 1$. The mappings α, μ both are symmetric. Now, for $\alpha(x,y) \geq 1, \mu(x,y) \leq 1$ we have the following cases.

Case 1. Consider $x = 0, y = 0$. Then

$$\psi(\mathfrak{m}(Kx,Ly)) = \psi(\mathfrak{m}(0,0)) = 0 \Rightarrow \alpha(0,0)\psi(\mathfrak{m}(K0,L0)) \leq \mu(0,0)\theta(\mathcal{M}_{0,0})\eta(\mathcal{M}_{0,0}).$$

Case 2. Consider $x = 1, y = 1$. Then

$$\psi(\mathfrak{m}(Kx,Ly)) = \psi(\mathfrak{m}(0,1)) = \psi(2) = \frac{2}{3}. \text{ Now,}$$

$$\begin{aligned} \mathcal{M}_{x,y} &= \max\{\mathfrak{m}(1,1), \mathfrak{m}(1,0), \mathfrak{m}(1,1), \frac{\mathfrak{m}(1,0)\mathfrak{m}(1,1)}{1+\mathfrak{m}(1,1)}, \frac{\mathfrak{m}(1,0)\mathfrak{m}(1,1)}{1+\mathfrak{m}(0,1)}\} \\ &= \max\{10, 2, 10, \frac{20}{11}, \frac{20}{3}\} = 10. \end{aligned}$$

Consequently, $\theta(\mathcal{M}_{x,y})\eta(\mathcal{M}_{x,y}) = e^{-\frac{2}{3}}\frac{10}{3.2}$.

Hence, $\alpha(1,1)\psi(\mathfrak{m}(K1,L1)) \leq \mu(1,1)\theta(\mathcal{M}_{1,1})\eta(\mathcal{M}_{1,1})$.

Case 3. Consider $x=2, y=2$. Then

$$\psi(\mathfrak{m}(Kx,Ly)) = \psi(\mathfrak{m}(1,0)) = \psi(2) = \frac{2}{3}. \text{ Now,}$$

$$\begin{aligned}\mathcal{M}_{x,y} &= \max\left\{\mathfrak{m}(2,2), \mathfrak{m}(2,1), \mathfrak{m}(2,0), \frac{\mathfrak{m}(2,1)\mathfrak{m}(2,0)}{1+\mathfrak{m}(2,2)}, \frac{\mathfrak{m}(2,1)\mathfrak{m}(2,0)}{1+\mathfrak{m}(1,0)}\right\}\\ &= \max\left\{10, 3, 2, \frac{6}{11}, \frac{6}{3}\right\} = 10.\end{aligned}$$

Consequently, $\theta(\mathcal{M}_{x,y})\eta(\mathcal{M}_{x,y}) = e^{-\frac{2}{3}}\frac{10}{3.2}$.
Hence, $\alpha(2,2)\psi(\mathfrak{m}(K2,L2)) \leq \mu(2,2)\theta(\mathcal{M}_{2,2})\eta(\mathcal{M}_{2,2})$.
Case 4. Consider $x=1, y=0$. Then

$$\psi(\mathfrak{m}(Kx,Ly)) = \psi(\mathfrak{m}(0,0)) = 0 \Rightarrow \alpha(1,0)\psi(\mathfrak{m}(K1,L0)) \leq \mu(1,0)\theta(\mathcal{M}_{1,0})\eta(\mathcal{M}_{1,0}).$$

Case 5. Consider $x=0, y=1$. Then

$$\psi(\mathfrak{m}(K0,L1)) = \psi(\mathfrak{m}(0,1)) = \psi(2) = \frac{2}{3}. \text{ Now,}$$

$$\begin{aligned}\mathcal{M}_{x,y} &= \max\left\{\mathfrak{m}(0,1), \mathfrak{m}(0,0), \mathfrak{m}(1,1), \frac{\mathfrak{m}(0,0)\mathfrak{m}(1,1)}{1+\mathfrak{m}(0,1)}, \frac{\mathfrak{m}(0,0)\mathfrak{m}(1,1)}{1+\mathfrak{m}(0,1)}\right\}\\ &= \max\{2, 0, 10, 0, 0\} = 10.\end{aligned}$$

Consequently, $\theta(\mathcal{M}_{x,y})\eta(\mathcal{M}_{x,y}) = e^{-\frac{2}{3}}\frac{10}{3.2}$.
Hence, $\alpha(0,1)\psi(\mathfrak{m}(K0,L1)) \leq \mu(0,1)\theta(\mathcal{M}_{0,1})\eta(\mathcal{M}_{0,1})$.
In fact, one can check that here all the conditions of Theorem 2.2 are satisfied and 0 is the unique CFP of K and L.

Note. In the above example, one can not apply Theorem 2.1 due to the following reason.
Consider a sequence $\{x_n\}$, defined by $x_n = 1 - \frac{1}{n+1}$, where $n \in \mathbb{N}$. Now,

$$\begin{aligned}\lim_{n\to\infty}\mathfrak{m}(x_n,1) &= \lim_{n\to\infty}\left[\left(1-\frac{1}{n+1}+1\right)\left(\mid 1-\frac{1}{n+1}-1\mid\right)\right]\\ &= \lim_{n\to\infty}\left[\left(2-\frac{1}{n+1}\right)\left(\mid -\frac{1}{n+1}\mid\right)\right] = 0,\end{aligned}$$

which means $x_n \to 1$, whenever $n \to \infty$. Now,

$$Kx_n = K\Big[1-\frac{1}{n+1}\Big] = \frac{1-\frac{1}{n+1}}{2-\frac{1}{n+1}}, K_1 = 0,\ Lx_n = L\Big[1-\frac{1}{n+1}\Big] = \frac{1-\frac{1}{n+1}}{3-\frac{1}{n+1}},\ L_1 = 1.$$

Next, we check the following.

$$\lim_{n\to\infty}\mathfrak{m}(Kx_n,K1) = \lim_{n\to\infty}\mathfrak{m}\left(\frac{1-\frac{1}{n+1}}{2-\frac{1}{n+1}}, 0\right) = \lim_{n\to\infty}\left[\frac{1-\frac{1}{n+1}}{2-\frac{1}{n+1}}\right] = \frac{1}{2} \neq 0,$$

and

$$\lim_{n\to\infty}\mathfrak{m}(Lx_n,L1) = \lim_{n\to\infty}\mathfrak{m}\left(\frac{1-\frac{1}{n+1}}{3-\frac{1}{n+1}}, 1\right)$$

$$= \lim_{n\to\infty}\left[\left(\frac{1-\frac{1}{n+1}}{3-\frac{1}{n+1}}+1\right)\left(\mid \frac{1-\frac{1}{n+1}}{3-\frac{1}{n+1}}-1\mid\right)\right]=\frac{8}{9}\neq 0.$$

Hence, applying Definition 1.12, we can conclude that both K and L are not continuous, i.e., Theorem 2.1 is not applicable in example 2.2.
Like Geraghty contraction, there is another important generalization of the BCP given by Popescu [38] as we mentioned it earlier. The author introduced a new type of contraction, named as P-contraction which greatly improves the famous BCP. It should be noted that in a SMS, a mapping that satisfies Banach contraction always satisfies P-contraction whereas the converse is not true. Before going to our second main result, we first introduce the following definition.

Definition 2.3. Let K be a self mapping defined on a SMS $(\mathfrak{X},\mathfrak{m})$ with $\mathfrak{m}(x,x)=0,\ \forall\, x\in\mathfrak{X}$ and $\alpha:\mathfrak{X}\times\mathfrak{X}\to\mathbb{R}^+$ be another function. Then, the mapping K is called an almost generalized $(\alpha,\mathcal{F}_{(\psi,\varphi)},\phi)$ type P-contraction if K satisfies the following inequality.

$$\psi(\mathfrak{m}(Kx,Ky))\leq\mathcal{F}(\psi(A(x,y)),\varphi(A(x,y)))+\phi(B(x,y))\psi(B(x,y)),\qquad(2.16)$$

for all $x,y\in\mathfrak{X}$, with $\alpha(x,y)\geq 1$, where $\mathcal{F}\in\mathcal{C}$, $\psi,\varphi\in\Psi$, $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is a function with $\overline{\lim_{x\to y^+}}\phi(x)$ exists for all $y\in\mathbb{R}^+$ and

$$A(x,y)=\mathfrak{m}(x,y)+\mid\mathfrak{m}(x,Kx)-\mathfrak{m}(y,Ky)\mid,$$

$$B(x,y)=\min\{\mathfrak{m}(x,Kx),\mathfrak{m}(y,Ky),\mathfrak{m}(y,Kx)),\mathfrak{m}(x,Ky)\}.$$

We write $\Lambda_P(\mathfrak{X},\alpha,\mathcal{F}_{(\psi,\varphi)},\phi)_{\mathfrak{m}}$ to denote the collection of all almost generalized $(\alpha,\mathcal{F}_{(\psi,\varphi)},\phi)$ type P-contraction in the setting of the SMS $(\mathfrak{X},\mathfrak{m})$.
Note. The last part of the contraction proposed in the Definition 2.3 is inspired by the famous Berinde type contraction (see [13]).
Now, we prove our second main result.

Theorem 2.3. *Let $(\mathfrak{X},\mathfrak{m})$ be a CSMS with $\mathfrak{m}(x,x)=0,\ \forall\, x\in\mathfrak{X}$. Let $K:\mathfrak{X}\to\mathfrak{X},\alpha:\mathfrak{X}\times\mathfrak{X}\to\mathbb{R}^+$ be two given mappings. Suppose that the following assertions hold.*

1. *K is an α-admissible mapping;*
2. *$K\in\Lambda_P(\mathfrak{X},\alpha,\mathcal{F}_{(\psi,\varphi)},\phi)_{\mathfrak{m}}$;*
3. *there exists a $x_0\in\mathfrak{X}$ such that $\alpha(x_0,Kx_0)\geq 1$;*
4. *either K is continuous or $(\mathfrak{X},\mathfrak{m})$ is α-regular satisfying condition(M).*

Then K has a fixed point in $\mathfrak{X}$.

Proof. From assumption 3, there exists a point $x_0\in\mathfrak{X}$ such that $\alpha(x_0,Kx_0)\geq 1$. Clearly, we can construct a Picard iterative sequence $\{x_n\}$ in $\mathfrak{X}$ such that $x_{n+1}=Kx_n,\ \forall\, n\in\mathbb{N}_0$. If there exists a $\tilde{n}\in\mathbb{N}_0$ for which $x_{\tilde{n}}=x_{\tilde{n}+1}$ holds, then $x_{\tilde{n}}$ is the fixed point of the mapping K and consequently, our proof will be completed. Thus, from now we will assume that $x_n\neq x_{n+1}$ for each $n\in\mathbb{N}_0$, i.e., $\mathfrak{m}(x_n,x_{n+1})>0,\ \forall\, n\in\mathbb{N}_0$. Our goal is to show that $\lim_{n\to\infty}\mathfrak{m}(x_n,x_{n+1})=0$. Since K is an α-admissible mapping and $\alpha(x_0,x_1)=\alpha(x_0,Kx_0)\geq 1$, we obtain $\alpha(x_1,x_2)=\alpha(Kx_0,Kx_1)\geq 1$. Continuing in this way, we get $\alpha(x_n, x_{n+1}) \geq 1,\ \forall\ n \in \mathbb{N}_0$. Now, $\mathrm{K} \in \Lambda_P(\mathfrak{X}, \alpha, \mathcal{F}_{(\psi,\varphi)}, \phi)_{\mathfrak{m}}$, consequently, by putting $x = x_n, y = x_{n+1}$ in 2.16, we have

$$\psi(\mathfrak{m}(Kx_n, Kx_{n+1})) \leq \mathcal{F}(\psi(A(x_n, x_{n+1})), \varphi(A(x_n, x_{n+1}))) + \phi(B(x_n, x_{n+1}))\psi(B(x_n, x_{n+1})). \tag{2.17}$$

Clearly, $B(x_n, x_{n+1}) = 0$, since $\mathfrak{m}(x_{n+1}, Kx_n) = 0$. Thus, inequality 2.17 becomes

$$\psi(\mathfrak{m}(x_{n+1}, x_{n+2})) = \psi(\mathfrak{m}(Kx_n, Kx_{n+1})) \leq \mathcal{F}(\psi(A(x_n, x_{n+1})), \varphi(A(x_n, x_{n+1}))) \leq \psi(A(x_n, x_{n+1})), \tag{2.18}$$

where $A(x_n, x_{n+1}) = \mathfrak{m}(x_n, x_{n+1}) + \mid \mathfrak{m}(x_n, x_{n+1}) - \mathfrak{m}(x_{n+1}, x_{n+2}) \mid$. Suppose there exists a $\hat{n} \in \mathbb{N}_0$ for which $\mathfrak{m}(x_{\hat{n}}, x_{\hat{n}+1}) \leq \mathfrak{m}(x_{\hat{n}+1}, x_{\hat{n}+2})$. Then, from 2.18, we obtain

$$\psi(\mathfrak{m}(x_{\hat{n}+1}, x_{\hat{n}+2})) \leq \mathcal{F}(\psi(A(x_{\hat{n}}, x_{\hat{n}+1})), \varphi(A(x_{\hat{n}}, x_{\hat{n}+1}))) \leq \psi(A(x_{\hat{n}}, x_{\hat{n}+1}))$$

implies

$$\psi(\mathfrak{m}(x_{\hat{n}+1}, x_{\hat{n}+2})) \leq \mathcal{F}(\psi(\mathfrak{m}(x_{\hat{n}+1}, x_{\hat{n}+2})), \varphi(\mathfrak{m}(x_{\hat{n}+1}, x_{\hat{n}+2})) \leq \psi(\mathfrak{m}(x_{\hat{n}+1}, x_{\hat{n}+2}))$$

implies

$$\mathcal{F}(\psi(\mathfrak{m}(x_{\hat{n}+1}, x_{\hat{n}+2})), \varphi(\mathfrak{m}(x_{\hat{n}+1}, x_{\hat{n}+2})) = \psi(\mathfrak{m}(x_{\hat{n}+1}, x_{\hat{n}+2})).$$

Now, by using second property of C-class function, we have either $\psi(\mathfrak{m}(x_{\hat{n}+1}, x_{\hat{n}+2})) = 0$ or $\varphi(\mathfrak{m}(x_{\hat{n}+1}, x_{\hat{n}+2})) = 0$ implies $\mathfrak{m}(x_{\hat{n}+1}, x_{\hat{n}+2}) = 0$ implies $x_{\hat{n}+1} = x_{\hat{n}+2}$, a contradiction. Hence we have $\mathfrak{m}(x_{n+1}, x_{n+2}) \leq \mathfrak{m}(x_n, x_{n+1}),\ \forall\ n \in \mathbb{N}_0$, i.e., $\{\mathfrak{m}(x_n, x_{n+1})\}_{n=0}^{\infty}$ is a decreasing sequence which is bounded below by 0. Thus we have $\lim_{n\to\infty} \mathfrak{m}(x_n, x_{n+1}) = b$, where $b \geq 0$. Now, we wish to show that $b = 0$, suppose on the contrary, i.e., $b > 0$. Taking limit as $n \to \infty$ and using the fact that $\mathcal{F}$ is a continuous mapping, from 2.18, we get

$$\psi(b) \leq \mathcal{F}(\psi(b), \varphi(b)) \leq \psi(b)$$

implies

$$\mathcal{F}(\psi(b), \varphi(b)) = \psi(b)$$

implies either $\psi(b) = 0$ or $\varphi(b) = 0$ implies $b = 0$, a contradiction. Thus we must have

$$\lim_{n\to\infty} \mathfrak{m}(x_n, x_{n+1}) = 0. \tag{2.19}$$

Now, suppose that $n, p \in \mathbb{N}$ with $p > n$. If $x_n = x_p$, then we have $K^p(x_0) = K^n(x_0)$. Thus, we obtain $K^{p-n}(K^n x_0) = K^n(x_0)$. Hence, $K^n x_0$ is a fixed point of K^{p-n}. Again, we have

$$K(K^{p-n}(K^n x_0)) = K^{p-n}(K(K^n x_0)) = K(K^n x_0),$$

which means $K(K^n x_0)$ is a fixed point of K^{p-n}. Thus, we obtain $K(K^n x_0) = K^n x_0$. Consequently, $K^n x_0$ is a fixed point of K. So without loss of generality, we may suppose that $x_n \neq x_p$. Therefore

$$\overline{\lim_{n\to\infty}} \mathfrak{m}(x_n, x_{n+2}) \leq \omega \overline{\lim_{n\to\infty}} \mathfrak{m}(x_{n+1}, x_{n+2}).$$

By using 2.19, we have $\overline{\lim_{n\to\infty}} \mathfrak{m}(x_n, x_{n+2}) = 0$. Again, since $\overline{\lim_{n\to\infty}} \mathfrak{m}(x_n, x_{n+2}) = 0$, we have

$$\overline{\lim_{n\to\infty}} \mathfrak{m}(x_n, x_{n+3}) \leq \omega \overline{\lim_{n\to\infty}} \mathfrak{m}(x_{n+2}, x_{n+3}).$$

Consequently, by applying mathematical induction, we conclude that

$$\overline{\lim_{n\to\infty}}\{\mathfrak{m}(x_n, x_p) : p > n\} = 0,$$

which implies that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since $(\mathfrak{X}, \mathfrak{m})$ is a CSMS, hence the sequence $\{x_n\}$ will converge to a point in $\mathfrak{X}$, say a^*. Now, we discuss the following two cases.

Case 1. Suppose that K is a continuous mapping, then we have

$$a^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} K(x_n) = K(\lim_{n\to\infty} x_n) = Ka^*.$$

Suppose K is not continuous. Hence, we discuss the second case.

Case 2. Here $x_n \to a^*$ as $n \to \infty$ with $\alpha(x_n, x_{n+1}) \geq 1$. Since $(\mathfrak{X}, \mathfrak{m})$ is α-regular so $\alpha(x_n, a^*) \geq 1$. Thus, we have

$$\psi(\mathfrak{m}(Kx_n, Ka^*)) \leq \mathcal{F}(\psi(A(x_n, a^*)), \varphi(A(x_n, a^*))) + \phi(B(x_n, a^*))\psi(B(x_n, a^*)). \tag{2.20}$$

Now, considering lim sup in both sides of 2.20 and using 2.19 together with condition(M), we have

$$\psi(\mathfrak{m}(a^*, Ka^*)) \leq \mathcal{F}(\psi(\mathfrak{m}(a^*, Ka^*)), \varphi(\mathfrak{m}(a^*, Ka^*))) \leq \psi(\mathfrak{m}(a^*, Ka^*))$$

implies

$$\mathcal{F}(\psi(\mathfrak{m}(a^*, Ka^*)), \varphi(\mathfrak{m}(a^*, Ka^*))) = \psi(\mathfrak{m}(a^*, Ka^*))$$

implies either $\psi(\mathfrak{m}(a^*, Ka^*)) = 0$ or $\varphi(\mathfrak{m}(a^*, Ka^*)) = 0$ implies $\mathfrak{m}(a^*, Ka^*) = 0$ implies $a^* = Ka^*$. □

Next, we state and prove a theorem on uniqueness of fixed points of the operator K.

Theorem 2.4. *In addition to the hypotheses of Theorem 2.3, suppose that for every $a_1^*, a_2^* \in$ Fix(K), we have $\alpha(a_1^*, a_2^*) \geq 1$. Then K has a unique fixed point in $\mathfrak{X}$.*

Proof. We use method of contradiction to obtain our desired result. Suppose there are two fixed point of K, i.e., $a_1^*, a_2^* \in$ Fix(K) with $Ka_1^* = a_1^* \neq a_2^* = Ka_2^*$. Clearly, from our assumption, we have $\alpha(a_1^*, a_2^*) \geq 1$, which implies

$$\psi(\mathfrak{m}(Ka_1^*, Ka_2^*)) \leq \mathcal{F}(\psi(A(a_1^*, a_2^*)), \varphi(A(a_1^*, a_2^*))) + \phi(B(a_1^*, a_2^*))\psi(B(a_1^*, a_2^*)),$$

where

$$A(a_1^*, a_2^*) = \mathfrak{m}(a_1^*, a_2^*) + \mid \mathfrak{m}(a_1^*, Ka_1^*) - \mathfrak{m}(a_2^*, Ka_2^*) \mid,$$

$$B(a_1^*, a_2^*) = \min\{\mathfrak{m}(a_1^*, Ka_1^*), \mathfrak{m}(a_2^*, Ka_2^*), \mathfrak{m}(a_2^*, Ka_1^*)), \mathfrak{m}(a_1^*, Ka_2^*)\}.$$

Clearly, from the above inequality, we have

$$\mathcal{F}(\psi(\mathfrak{m}(a_1^*, a_2^*)), \varphi(\mathfrak{m}(a_1^*, a_2^*))) = \psi(\mathfrak{m}(a_1^*, a_2^*))$$

implies either $\psi(\mathfrak{m}(a_1^*, a_2^*)) = 0$ or $\varphi(\mathfrak{m}(a_1^*, a_2^*)) = 0 \Rightarrow \mathfrak{m}(a_1^*, a_2^*) = 0 \Rightarrow a_1^* = a_2^*$, which is our desired result. □

We now give an example to support our result.

Example 2.3. Let $\mathfrak{X} = [0,2] \cup [3,\infty)$ and $\mathfrak{m} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ be a mapping defined as

$$\mathfrak{m}(x,y) = (x-y)^2;\ \forall\ x,y\ \in \mathfrak{X} \setminus \{1\},$$
$$\mathfrak{m}(1,x) = \mathfrak{m}(x,1) = (1-x^3)^2,\ \forall\ x \in \mathfrak{X}.$$

Clearly, $(\mathfrak{X},\mathfrak{m})$ is a CSMS with $\mathfrak{m}(x,x) = 0,\ \forall\ x \in \mathfrak{X}$. We now define two mappings $K : \mathfrak{X} \to \mathfrak{X}$ and $\alpha : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ in the following way.

$$K(x) = \begin{cases} 1, & \text{if } x \in [0,2], \\ 0, & \text{if } x \in [3,\infty). \end{cases}$$

$$\alpha(x,y) = \begin{cases} 1.5, & \text{if } x,y \in [0,2] \cup [3,5], \\ 0, & \text{if } x,y \in (5,\infty). \end{cases}$$

Now, for $\alpha(x,y) \geq 1 \Rightarrow\ x,y \in [0,2] \cup [3,5]$. Hence, we do the following cases.
Case 1. $x,y \in [0,2]\ \Rightarrow \mathfrak{m}(Kx,Ky) = \mathfrak{m}(1,1) = 0$.
Case 2. $x,y \in [3,5]\ \Rightarrow \mathfrak{m}(Kx,Ky) = \mathfrak{m}(0,0) = 0$.
Case 3. If $x \in [0,2]$ and $y \in [3,5]\ \Rightarrow \mathfrak{m}(Kx,Ky) = \mathfrak{m}(1,0) = 1$. In this case, we calculate the following quantity.

$$\mathfrak{m}(x,y)+ \mid \mathfrak{m}(x,Kx) - \mathfrak{m}(y,Ky) \mid$$
$$= \begin{cases} (x-y)^2+ \mid \mathfrak{m}(x,1) - \mathfrak{m}(y,0) \mid, & \text{if x} \neq 1, \\ (1-y^3)^2+ \mid \mathfrak{m}(1,1) - \mathfrak{m}(y,0) \mid, & \text{if x} = 1. \end{cases}$$
$$= \begin{cases} (y-x)^2+ \mid (1-x^3)^2 - y^2 \mid, & \text{if x} \neq 1, \\ (y^3-1)^2 + y^2, & \text{if x} = 1. \end{cases}$$
$$= \begin{cases} (y-x)^2+ \mid y^2 - (x^3-1)^2 \mid & > 1.5, \\ (y^3-1)^2 + y^2 & > 1.5. \end{cases}$$

By using MATLAB, we have checked that when $x \neq 1$ then the minimum value of $\mathfrak{m}(x,y)+ \mid \mathfrak{m}(x,Kx) - \mathfrak{m}(y,Ky) \mid$ is 2.018371 and for $x = 1$ it is trivial. A 3D view of $x \neq 1$ case is given in figure-1.

Thus, from any case, we can write the following

$$\mathfrak{m}(Kx,Ky) = 1 = \Big(\frac{2}{3}\Big)\Big(\frac{3}{2}\Big) \leq \frac{2}{3}\Big[\mathfrak{m}(x,y)+ \mid \mathfrak{m}(x,Kx) - \mathfrak{m}(y,Ky) \mid \Big] + B(x,y).$$

Here $\alpha(5,K5) \geq 1$, K is an α-admissible mapping as well as continuous function. Moreover, consider $\mathcal{F}(s,t) = s-t, \psi(x) = x, \varphi(x) = \frac{x}{3}, \phi(x) = 1$. Hence, $K \in \Lambda_P(\mathfrak{X},\alpha,\mathcal{F}_{(\psi,\varphi)},\phi)_{\mathfrak{m}}$. Thus, by applying Theorem 2.3, we can ensure the existence of a fixed point of the operator K. Here 1 is that fixed point, which is unique. Next, we mention an interesting note.

Note. Erdal Karapınar and Farshid Khojasteh introduced the concept of a SMS and they proved BCP in the context of a SMS in Theorem 2.6 in their paper (see [30]). Consider $x = 2, y = 3$, then from Example 2.3, we have $\mathfrak{m}(K2,K3) = \mathfrak{m}(1,0) = 1$. Hence, there does not exists any $\tilde{a} \in (0,1)$ for which the following inequality

$$\mathfrak{m}(K2,K3) \leq \tilde{a}\mathfrak{m}(2,3),$$

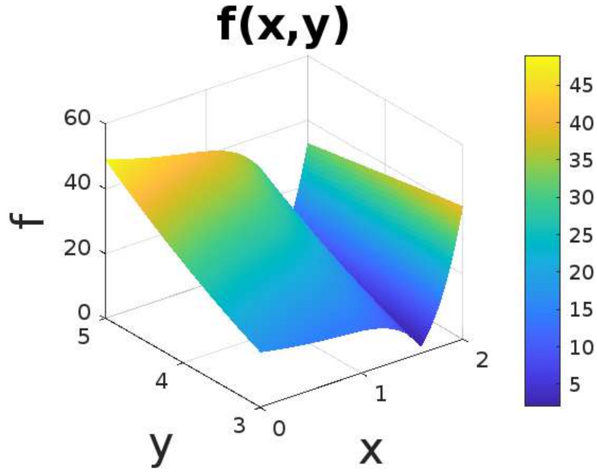

FIGURE 1. 3D view of $x \neq 1$ case.

holds. Hence, Theorem 2.6 (i.e. BCP in the setting of a SMS) of Karapınar and Khojasteh is not applicable in Example 2.3.

Next, we investigate one of the classical problem of metric fixed point theory in the setting of a SMS. In [39], Billy E. Rhoades introduced the following open question.

“ What are the contractive conditions which are strong enough to generate a fixed point but which do not force the map to be continuous at fixed point? ”

Pant attained first solution to this open problem (see [37]). Motivated by the work of Pant, many interesting fixed point results in this direction have been appeared in the literature (see, for example, [12], [14], [15], [20]). Next inspired by the famous Dass and Gupta contraction ([17]), we state our another main result on Rhoades open question in the setting of a SMS.

Theorem 2.5. *Let $(\mathfrak{X}, \mathfrak{m})$ be a CSMS such that the super metric $\mathfrak{m}$ is continuous together with $\mathfrak{m}(x, x) = 0$, $\forall\, x \in \mathfrak{X}$. Let $K, L : \mathfrak{X} \to \mathfrak{X}$ be two mappings such that for any $x, y \in \mathfrak{X}$ the following assertions hold.*

(1) *There exists a function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lambda(s) < s$ for $s > 0$ and*

$$\mathfrak{m}(Kx, Ly) \leq \lambda(D_c(x, y)), \tag{2.21}$$

where

$$D_c(x, y) = \max\Big\{\mathfrak{m}(x, y), \mathfrak{m}(x, Kx), \mathfrak{m}(y, Ly), a_1 \frac{\mathfrak{m}(x, Kx)[1 + \mathfrak{m}(y, Ly)]}{1 + \mathfrak{m}(x, y)}, a_2 \frac{\mathfrak{m}(y, Ly)[1 + \mathfrak{m}(x, Kx)]}{1 + \mathfrak{m}(x, y)}\Big\},$$

$$0 < a_1, a_2 < 1.$$

(2) *For a given $\epsilon > 0$, there exists a $\delta(\epsilon)$ such that*

$$\epsilon < D_c(x, y) < \epsilon + \delta \text{ implies } \mathfrak{m}(Kx, Ly) < \epsilon.$$

Then K and L have a unique CFP $a^* \in \mathfrak{X}$ and the sequence $x_n \to a^*$ for any $x_0 \in \mathfrak{X}$, where $x_{2n+1} = Kx_{2n}, x_{2n+2} = Lx_{2n+1},\ n \in \mathbb{N}_0$. Also, atleast one of K and L is discontinuous at a^* iff $D_c(x_n, a^*) \nrightarrow 0$ whenever $x_n \to a^*$ or $D_c(a^*, y_n) \nrightarrow 0$ whenever $y_n \to a^*$.

Proof. Suppose x_0 be any point in $\mathfrak{X}$ with $x_0 \neq Kx_0,\ x_0 \neq Lx_0$. Let us define a sequence $\{x_n\}$ in $\mathfrak{X}$ in the following way $x_{2n+1} = Kx_{2n}, x_{2n+2} = Lx_{2n+1},\ n \in \mathbb{N}_0$. Suppose there exists a $c \in \mathbb{N}$ such that $\mathfrak{m}(x_{2c}, x_{2c+1}) = 0$. Then $x_{2c} = x_{2c+1} = Kx_{2c}$, i.e., x_{2c} is a fixed point of K. Now to show that x_{2c} is a fixed point of L, it is sufficient to show that $x_{2c} = x_{2c+1} = x_{2c+2}$. Consider $\mathfrak{m}(x_{2c+1}, x_{2c+2}) > 0$. Then putting $x = x_{2c}$ and $y = x_{2c+1}$ in 2.21.

$$\mathfrak{m}(x_{2c+1}, x_{2c+2}) = \mathfrak{m}(Kx_{2c}, Lx_{2c+1}) \leq \lambda(D_c(x_{2c}, x_{2c+1})),$$

where

$$\begin{aligned} D_c(x_{2c}, x_{2c+1}) = \max\Big\{&\mathfrak{m}(x_{2c}, x_{2c+1}), \mathfrak{m}(x_{2c}, x_{2c+1}), \mathfrak{m}(x_{2c+1}, x_{2c+2}), \\ &a_1 \frac{\mathfrak{m}(x_{2c}, x_{2c+1})[1 + \mathfrak{m}(x_{2c+1}, x_{2c+2})]}{1 + \mathfrak{m}(x_{2c}, x_{2c+1})}, \\ &a_2 \frac{\mathfrak{m}(x_{2c+1}, x_{2c+2})[1 + \mathfrak{m}(x_{2c}, x_{2c+1})]}{1 + \mathfrak{m}(x_{2c}, x_{2c+1})}\Big\} \\ &= \mathfrak{m}(x_{2c+1}, x_{2c+2}). \end{aligned}$$

Now, by property of λ, we get

$$\mathfrak{m}(x_{2c+1}, x_{2c+2}) \leq \lambda(\mathfrak{m}(x_{2c+1}, x_{2c+2})) < \mathfrak{m}(x_{2c+1}, x_{2c+2}),$$

a contradiction. Thus $\mathfrak{m}(x_{2c+1}, x_{2c+2}) = 0 \Rightarrow x_{2c+1} = x_{2c+2}$. Hence, x_{2c} is a CFP of K and L. Now, if $\mathfrak{m}(x_{2c-1}, x_{2c}) = 0$ then $x_{2c-1} = x_{2c} = Lx_{2c-1}$. In a similar way, we can show that x_{2c-1} is a fixed point of K. Hence, we assume that $\mathfrak{m}(x_n, x_{n+1}) \neq 0$, $\forall\ n \in \mathbb{N}_0$. Suppose that $b_n = \mathfrak{m}(x_n, x_{n+1}),\ \forall\ n \in \mathbb{N}_0$. By using inequality 2.21, we have

$$b_1 = \mathfrak{m}(x_1, x_2) = \mathfrak{m}(Kx_0, Lx_1) \leq \lambda(D_c(x_0, x_1)), \text{ where}$$

$$\begin{aligned} D_c(x_0, x_1) = \max\Big\{&\mathfrak{m}(x_0, x_1), \mathfrak{m}(x_0, Kx_0), \mathfrak{m}(x_1, Lx_1), \\ &a_1 \frac{\mathfrak{m}(x_0, Kx_0)[1 + \mathfrak{m}(x_1, Lx_1)]}{1 + \mathfrak{m}(x_0, x_1)}, a_2 \frac{\mathfrak{m}(x_1, Lx_1)[1 + \mathfrak{m}(x_0, Kx_0)]}{1 + \mathfrak{m}(x_0, x_1)}\Big\} \\ = \max\Big\{&\mathfrak{m}(x_0, x_1), \mathfrak{m}(x_1, x_2), a_1 \frac{\mathfrak{m}(x_0, x_1)[1 + \mathfrak{m}(x_1, x_2)]}{1 + \mathfrak{m}(x_0, x_1)}\Big\} = A. \end{aligned}$$

We now consider the following three cases.

Case 1. If $A = \mathfrak{m}(x_1, x_2)$, then $\mathfrak{m}(x_1, x_2) \leq \lambda(\mathfrak{m}(x_1, x_2)) < \mathfrak{m}(x_1, x_2)$, a contradiction.

Case 2. If $A = \mathfrak{m}(x_0, x_1)$, then $\mathfrak{m}(x_1, x_2) \leq \lambda(\mathfrak{m}(x_0, x_1)) < \mathfrak{m}(x_0, x_1)$.

Case 3. Suppose

$$A = a_1 \frac{\mathfrak{m}(x_0, x_1)[1 + \mathfrak{m}(x_1, x_2)]}{1 + \mathfrak{m}(x_0, x_1)},$$

which means

$$\max\{\mathfrak{m}(x_0, x_1), \mathfrak{m}(x_1, x_2)\} \leq a_1 \frac{\mathfrak{m}(x_0, x_1)[1 + \mathfrak{m}(x_1, x_2)]}{1 + \mathfrak{m}(x_0, x_1)}. \tag{2.22}$$

We now consider the following two sub-cases.
Sub-case 3a. Suppose $\max\{\mathfrak{m}(x_0, x_1), \mathfrak{m}(x_1, x_2)\} = \mathfrak{m}(x_1, x_2)$, i.e,

$$\mathfrak{m}(x_0, x_1) < \mathfrak{m}(x_1, x_2). \tag{2.23}$$

On the other hand,

$$\mathfrak{m}(x_1, x_2) < a_1 \frac{\mathfrak{m}(x_0, x_1)[1 + \mathfrak{m}(x_1, x_2)]}{1 + \mathfrak{m}(x_0, x_1)} < \frac{\mathfrak{m}(x_0, x_1)[1 + \mathfrak{m}(x_1, x_2)]}{1 + \mathfrak{m}(x_0, x_1)}$$

implies $\mathfrak{m}(x_1, x_2) < \mathfrak{m}(x_0, x_1)$, a contradiction to 2.23.
Sub-case 3b. Suppose $\max\{\mathfrak{m}(x_0, x_1), \mathfrak{m}(x_1, x_2)\} = \mathfrak{m}(x_0, x_1)$, i.e,

$$\mathfrak{m}(x_1, x_2) < \mathfrak{m}(x_0, x_1). \tag{2.24}$$

Moreover, from 2.22, we obtain

$$\mathfrak{m}(x_0, x_1) \leq a_1 \frac{\mathfrak{m}(x_0, x_1)[1 + \mathfrak{m}(x_1, x_2)]}{1 + \mathfrak{m}(x_0, x_1)} < \frac{\mathfrak{m}(x_0, x_1)[1 + \mathfrak{m}(x_1, x_2)]}{1 + \mathfrak{m}(x_0, x_1)}.$$

implies $\mathfrak{m}(x_0, x_1) < \mathfrak{m}(x_1, x_2)$, a contradiction to 2.24. Thus, from Case 2, we have

$$\mathfrak{m}(x_1, x_2) < \mathfrak{m}(x_0, x_1).$$

Next, we do the following.

$$b_2 = \mathfrak{m}(x_2, x_3) = \mathfrak{m}(Kx_2, Lx_1) \leq \lambda(D_c(x_2, x_1)),$$

where

$$\begin{aligned} D_c(x_2, x_1) = &\max\Big\{\mathfrak{m}(x_2, x_1), \mathfrak{m}(x_2, Kx_2), \mathfrak{m}(x_1, Lx_1), \\ &a_1 \frac{\mathfrak{m}(x_2, Kx_2)[1 + \mathfrak{m}(x_1, Lx_1)]}{1 + \mathfrak{m}(x_2, x_1)}, a_2 \frac{\mathfrak{m}(x_1, Lx_1)[1 + \mathfrak{m}(x_2, Kx_2)]}{1 + \mathfrak{m}(x_2, x_1)}\Big\} \\ = &\max\Big\{\mathfrak{m}(x_1, x_2), \mathfrak{m}(x_2, x_3), a_2 \frac{\mathfrak{m}(x_1, x_2)[1 + \mathfrak{m}(x_2, x_3)]}{1 + \mathfrak{m}(x_1, x_2)}\Big\}. \end{aligned}$$

Again, by using the the previous technique (i.e. used in b_1), we can show that

$$\mathfrak{m}(x_2, x_3) < \mathfrak{m}(x_1, x_2).$$

Thus, in general, we can write $\mathfrak{m}(x_n, x_{n+1}) < \mathfrak{m}(x_{n-1}, x_n)\ \forall\ n \in \mathbb{N}$. Since $\{b_n\}$ is a strictly decreasing sequence in $\mathbb{R}^+$, hence it will converge to a non-negative real number, say $b^* \geq 0$. We consider $b^* > 0$. Then there exists a positive integer l such that $l \leq n$ implies

$$b^* < b_n < b^* + \delta(\epsilon).$$

Now, from assertion 2 and $b_n < b_{n-1}$, we have

$$b_n < b^*\ (n \geq l),$$

which is a contradiction. Thus, we must have $b^* = 0$, i.e.,

$$\lim_{n\to\infty} \mathfrak{m}(x_n, x_{n+1}) = 0. \tag{2.25}$$

Now, by using the technique given in Theorem 2.1, we can show that $\{x_n\}$ is a Cauchy sequence. Since $(\mathfrak{X}, \mathfrak{m})$ is a CSMS so it will converge to a point in $\mathfrak{X}$, say a^*, which means $x_{2n+1} \to a^*$ and $x_{2n+2} \to a^*$ as $n \to \infty$. Now, we show that a^* is a CFP of K and L, i.e., $a^* = Ka^* = La^*$. Suppose, $a^* \neq Ka^*$, i.e., $\mathfrak{m}(a^*, Ka^*) > 0$. Then, putting $x = a^*$ and $y = x_{2n+1}$ in 2.21, we have

$$\mathfrak{m}(Ka^*, x_{2n+2}) = \mathfrak{m}(Ka^*, Lx_{2n+1}) \leq \lambda(D_c(a^*, x_{2n+1})). \tag{2.26}$$

Now, taking limit as $n \to \infty$ in both sides of 2.26 together with 2.25 and $\mathfrak{m}$ is continuous, we have

$$\mathfrak{m}(a^*, Ka^*) \leq \lambda(\mathfrak{m}(a^*, Ka^*)) < \mathfrak{m}(a^*, Ka^*),$$

a contradiction. Thus, we obtain $\mathfrak{m}(a^*, Ka^*) = 0 \Rightarrow a^* = Ka^*$. Next, we suppose that $a^* \neq La^*$. Then, by using 2.21 with $x = x_{2n}, y = a^*$, we have

$$\mathfrak{m}(x_{2n+1}, La^*) = \mathfrak{m}(Kx_{2n}, La^*) \leq \lambda(D_c(x_{2n}, a^*)). \tag{2.27}$$

Now, taking limit as $n \to \infty$ in both sides of 2.27 together with 2.25 and $\mathfrak{m}$ is continuous, we have

$$\mathfrak{m}(a^*, La^*) \leq \lambda(\mathfrak{m}(a^*, La^*)) < \mathfrak{m}(a^*, La^*),$$

a contradiction. Thus, we obtain $\mathfrak{m}(a^*, La^*) = 0 \Rightarrow a^* = La^*$. Consequently, we get $a^* = Ka^* = La^*$. Our next intention is to show that a^* is a unique CFP of K and L. Suppose on the contrary, i.e., take a^{**} be another CFP of K and L. Since $a^* \neq a^{**} \Rightarrow \mathfrak{m}(a^*, a^{**}) > 0$. By using 2.21, we have

$$\mathfrak{m}(a^*, a^{**}) = \mathfrak{m}(Ka^*, La^{**}) \leq \lambda(D_c(a^*, a^{**})) = \lambda(\mathfrak{m}(a^*, a^{**})) < \mathfrak{m}(a^*, a^{**}),$$

a contradiction. Thus $\mathfrak{m}(a^*, a^{**}) = 0 \Rightarrow a^* = a^{**}$, i.e., a^* is a unique CFP of K and L. Finally, our aim is to prove that atleast one of the self mapping K and L is discontinuous at a^* iff $D_c(x_n, a^*) \nrightarrow 0$ whenever $x_n \to a^*$ or $D_c(a^*, y_n) \nrightarrow 0$ whenever $y_n \to a^*$. First, we are assuming $D_c(x_n, a^*) \to 0$ whenever $x_n \to a^*$ and $D_c(a^*, y_n) \to 0$ whenever $y_n \to a^*$. Now, we will show that whenever $z_n \to a^*$ as $n \to \infty$ implies $Kz_n \to Ka^*$ and $Lz_n \to La^*$. Here, we are not considering the trivial case, i.e., except a few finite terms, all terms of $\{z_n\}$ are equal to a^*. Since, in that case it is obvious. So, we are considering that each term of the sequence $\{z_n\}$ is distinct. Since $z_n \to a^*$, consequently by our assumption, we have $D_c(z_n, a^*) \to 0$ and $D_c(a^*, z_n) \to 0$ whenever $n \to \infty$. Now,

$$\begin{gathered}
D_c(z_n, a^*) \to 0 \\
\Rightarrow \max\Big\{\mathfrak{m}(z_n, a^*), \mathfrak{m}(z_n, Kz_n), \mathfrak{m}(a^*, La^*), \\
a_1\frac{\mathfrak{m}(z_n, Kz_n)[1 + \mathfrak{m}(a^*, La^*)]}{1 + \mathfrak{m}(z_n, a^*)}, a_2\frac{\mathfrak{m}(a^*, La^*)[1 + \mathfrak{m}(z_n, Kz_n)]}{1 + \mathfrak{m}(z_n, a^*)}\Big\} \to 0 \\
\Rightarrow \mathfrak{m}(z_n, Kz_n) \to 0,
\end{gathered}$$

and

$$\begin{gathered}
D_c(a^*, z_n) \to 0 \\
\Rightarrow \max\Big\{\mathfrak{m}(a^*, z_n), \mathfrak{m}(a^*, Ka^*), \mathfrak{m}(z_n, Lz_n), \\
a_1\frac{\mathfrak{m}(a^*, Ka^*)[1+\mathfrak{m}(z_n, Lz_n)]}{1+\mathfrak{m}(a^*, z_n)}, a_2\frac{\mathfrak{m}(z_n, Lz_n)[1+\mathfrak{m}(a^*, Ka^*)]}{1+\mathfrak{m}(a^*, z_n)}\Big\} \to 0 \\
\Rightarrow \mathfrak{m}(z_n, Lz_n) \to 0.
\end{gathered}$$

Now observe that the sequences $\{z_n\}$ and $\{Kz_n\}$ are two distinct sequences such that $\lim_{z_n \to a^*} \mathfrak{m}(z_n, Kz_n) = 0$. Suppose they are not distinct. Hence there exists a $n^* \in \mathbb{N}_0$ such that $z_{n^*} = Kz_{n^*}$, i.e., $\mathfrak{m}(z_{n^*}, Kz_{n^*}) = 0$. Putting $x = z_{n^*}$ and $y = z_{n^*}$ in 2.21, we have

$$\mathfrak{m}(z_{n^*}, Lz_{n^*}) = \mathfrak{m}(Kz_{n^*}, Lz_{n^*}) \le \lambda(D_c(z_{n^*}, z_{n^*})), \text{ where} \tag{2.28}$$

$$\begin{aligned}
&D_c(z_{n^*}, z_{n^*}) \\
&= \max\Big\{\mathfrak{m}(z_{n^*}, z_{n^*}), \mathfrak{m}(z_{n^*}, Kz_{n^*}), \mathfrak{m}(z_{n^*}, Lz_{n^*}), a_1\frac{\mathfrak{m}(z_{n^*}, Kz_{n^*})[1+\mathfrak{m}(z_{n^*}, Lz_{n^*})]}{1+\mathfrak{m}(z_{n^*}, z_{n^*})}, \\
&a_2\frac{\mathfrak{m}(z_{n^*}, Lz_{n^*})[1+\mathfrak{m}(z_{n^*}, Kz_{n^*})]}{1+\mathfrak{m}(z_{n^*}, z_{n^*})}\Big\} \\
&= \mathfrak{m}(z_{n^*}, Lz_{n^*}).
\end{aligned}$$

Clearly, from 2.28, we arrive at a contradiction, if we consider $\mathfrak{m}(z_{n^*}, Lz_{n^*}) > 0$. Thus, we must have $\mathfrak{m}(z_{n^*}, Lz_{n^*}) = 0 \Rightarrow z_{n^*} = Lz_{n^*}$, i.e., z_{n^*} is a CFP of K and L, which can not be possible since a^* is the unique CFP of K and L. In a similar way, one can show that the sequences $\{z_n\}$ and $\{Lz_n\}$ are two distinct sequences such that $\lim_{z_n \to a^*} \mathfrak{m}(z_n, Lz_n) = 0$. Now, by using third condition of Definition 1.9, we have

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\, \mathfrak{m}(Kz_n, a^*) \le \omega \overline{\lim_{n\to\infty}}\, \mathfrak{m}(z_n, a^*) \\
&\Rightarrow \lim_{n\to\infty} \mathfrak{m}(Kz_n, a^*) = 0 \text{ [since } \mathfrak{m}(x,y) \ge 0] \\
&\Rightarrow \lim_{n\to\infty} Kz_n = a^* = Ka^*,
\end{aligned}$$

and

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\, \mathfrak{m}(Lz_n, a^*) \le \omega \overline{\lim_{n\to\infty}}\, \mathfrak{m}(z_n, a^*) \\
&\Rightarrow \lim_{n\to\infty} \mathfrak{m}(Lz_n, a^*) = 0 \text{ [since } \mathfrak{m}(x,y) \ge 0] \\
&\Rightarrow \lim_{n\to\infty} Lz_n = a^* = La^*.
\end{aligned}$$

Since the sequence $\{z_n\}$ is arbitrary which converges to a^*, so the self mappings K and L both are continuous at a^*. For the converse part suppose that K and L both are continuous at a^*. Let $\{x_n\}$ and $\{y_n\}$ be any two sequences such that $x_n \to a^*$ and $y_n \to a^*$ whenever $n \to \infty$. Then by continuity of K and L we have

$$\lim_{n\to\infty} \mathfrak{m}(Kx_n, Ka^*) = 0 \text{ and } \lim_{n\to\infty} \mathfrak{m}(Ly_n, La^*) = 0.$$

Now,

$$D_c(x_n, a^*) = \max\Big\{\mathfrak{m}(x_n, a^*), \mathfrak{m}(x_n, Kx_n), \mathfrak{m}(a^*, La^*),$$
$$a_1\frac{\mathfrak{m}(x_n, Kx_n)[1+\mathfrak{m}(a^*, La^*)]}{1+\mathfrak{m}(x_n, a^*)}, a_2\frac{\mathfrak{m}(a^*, La^*)[1+\mathfrak{m}(x_n, Kx_n)]}{1+\mathfrak{m}(x_n, a^*)}\Big\} \to 0$$

(using the continuity of the super metric $\mathfrak{m}$), and

$$D_c(a^*, y_n) = \max\Big\{\mathfrak{m}(a^*, y_n), \mathfrak{m}(a^*, Ka^*), \mathfrak{m}(y_n, Ly_n),$$
$$a_1\frac{\mathfrak{m}(a^*, Ka^*)[1+\mathfrak{m}(y_n, Ly_n)]}{1+\mathfrak{m}(a^*, y_n)}, a_2\frac{\mathfrak{m}(y_n, Ly_n)[1+\mathfrak{m}(a^*, Ka^*)]}{1+\mathfrak{m}(a^*, y_n)}\Big\} \to 0$$

(using the continuity of the super metric $\mathfrak{m}$). ☐

Next, we prove a power contraction, where we assume the commutativity of the self mappings K and L, i.e., $KLx = LKx$, for all $x \in \mathfrak{X}$.

Corollary 2.3. *Suppose that all the hypotheses of Theorem 2.5 are satisfied by two mappings K^p and L^q, where $p, q \in \mathbb{N}$ and K, L are two self mappings from $\mathfrak{X}$ into $\mathfrak{X}$ satisfying commutativity property. Then a^* is the unique CFP of K and L.*

Proof. By the proof of Theorem 2.5, we obtain a^* is a unique CFP of K^p and L^q, i.e., $a^* = K^pa^* = La^*$. Then $Ka^* = K(K^pa^*) = K^p(Ka^*)$ and by using commutativity of property $Ka^* = K(L^qa^*) = L^q(Ka^*)$. Thus $Ka^* = K^p(Ka^*) = L^q(Ka^*)$, i.e., Ka^* is a CFP of K^p and L^q. But by the uniqueness of CFP of K^p and L^q, we have $a^* = Ka^*$. Again, $La^* = L(L^qa^*) = L^q(La^*)$ and by using commutativity of property $La^* = L(K^pa^*) = K^p(La^*)$. Hence, La^* is a CFP of K^p and L^q. From the uniqueness of CFP, we have $a^* = Ka^* = La^*$, a^* is a CFP of K and L. Next, we show that the uniqueness of CFP of the operators K and L. Let a^{**} be another CFP of K and L, i.e., $a^{**} = Ka^{**} = La^{**}$ implies $a^{**} = K^pa^{**} = L^qa^{**}$. By the uniqueness of CFP of K^p and L^q, we obtain $a^* = a^{**}$. Hence, a^* is the unique CFP of K and L. ☐

Example 2.4. Let $\mathfrak{X} = \mathbb{R}^+$ and $\mathfrak{m} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ be an application defined as

$$\mathfrak{m}(x, y) = (x - y)^2, \ \forall\, x, y \in \mathfrak{X}.$$

It is easy to verify that $(\mathfrak{X}, \mathfrak{m})$ is a SMS. Let us define two self mappings K and L in the following way

$$K(x) = \begin{cases} 1, & \text{if } x \in [0,1]\ , \\ 0.7, & \text{if } x \in (1, \infty). \end{cases}$$

$$L(x) = \begin{cases} 1, & \text{if } x \in [0,1]\ , \\ 0.75, & \text{if } x \in (1, \infty). \end{cases}$$

One can check that here 1 is the CFP of K and L, which is unique. Observe that both of the self mappings are discontinuous at $x = 1$. Now, we have the following.

$$\text{For } x, y \in [0,1] \Rightarrow \mathfrak{m}(Kx, Ky) = 0 \text{ and } 0 \le D_c(x, y) \le 1. \tag{2.29}$$

$$\text{For } x, y \in (1,\infty) \Rightarrow \mathfrak{m}(Kx, Ky) = 0.0025 \text{ and } 0 < D_c(x, y) < \infty. \tag{2.30}$$

$$\text{For } x \in [0,1] \text{ and } y \in (1,\infty) \Rightarrow \mathfrak{m}(Kx, Ky) = 0.0625 \text{ and } 0 < D_c(x, y) < \infty. \tag{2.31}$$

$$\text{For } x \in (1,\infty) \text{ and } y \in [0,1] \Rightarrow \mathfrak{m}(Kx,Ky) = 0.09 \text{ and } 0 < D_c(x,y) < \infty. \tag{2.32}$$

The self mappings K and L satisfy the assertion 1 of Theorem 2.5 with

$$\lambda(s) = \begin{cases} 0.09, & \text{if } 0.09 < s \\ 0.0625, & \text{if } 0.0625 < s \le 0.09 \\ \frac{4s}{5}, & \text{if } 0 < s \le 0.0625 \end{cases}$$

and both the self mappings satisfy the assertion 2 of Theorem 2.5 with

$$\delta(\epsilon) = \begin{cases} 1, & \text{if } 0.09 \le \epsilon \\ \frac{0.09-\epsilon}{2}, & \text{if } 0.0625 \le \epsilon < 0.09 \\ \frac{0.0625-\epsilon}{2}, & \text{if } 0.0025 \le \epsilon < 0.09 \\ \frac{0.0025-\epsilon}{2}, & \text{if } 0 < \epsilon < 0.0025 \end{cases}$$

for any $a_1, a_2 \in (0,1)$. Now for $x, y \in (1,\infty)$, we have $Kx = Ky = 0.7, Lx = Ly = 0.75$ and

$$D_c(x,y) = \max\Big\{(x-y)^2, (x-0.7)^2, (y-0.75)^2, \\ a_1\frac{(x-0.7)^2[1+(y-0.75)^2]}{1+(x-y)^2}, a_2\frac{(y-0.75)^2[1+(x-0.7)^2]}{1+(x-y)^2}\Big\} > 0.09.$$

From definition of λ, we have $\lambda(s) < s$ for $s > 0$. Thus, for $x, y \in (1,\infty)$, we have

$$\mathfrak{m}(Kx, Ly) = 0.0025 < \lambda(D_c(x,y)).$$

In fact, it can be checked that if we consider the other cases given by 2.29, 2.31, 2.32, then also the assertion 1 of Theorem 2.5 holds. Moreover, one can also check that our constructed $\delta(\epsilon)$ satisfies assertion 2 of Theorem 2.5. Here, $(\mathfrak{X}, \mathfrak{m})$ is a CSMS which is continuous satisfying $\mathfrak{m}(x,x) = 0,\ \forall\ x \in \mathfrak{X}$. Consider two sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ in $\mathfrak{X}$ and defined them as $x_n = 1 + \frac{1}{2n+1}$, $y_n = 1 + \frac{1}{2n+2}$, $n \in \mathbb{N}_0$. Then one can check that $x_n \to 1$ and $y_n \to 1$ but $D_c(x_n, 1) \nrightarrow 0$ whenever $x_n \to 1$ and $D_c(1, y_n) \nrightarrow 0$ whenever $y_n \to 1$.

3. Application

First, we recall some basic definitions of fractional calculus (see [33], [41]). For a continuous function $g : \mathbb{R}^+ \to \mathbb{R}$, the Caputo derivative of fractional order β is defined as

$${}^C D^{\beta} g(p) = \frac{1}{\Gamma(n-\beta)} \int_0^p (p-q)^{n-\beta-1} g^{(n)}(q)dq \ (n-1 < \beta < n, n = [\beta]+1),$$

where $[\beta]$ denotes the integer part of the positive real number β and Γ is the gamma function. Let us denote $\mathfrak{X} = C([0,1],\mathbb{R})$, where $C([0,1],\mathbb{R})$ is set of all continuous functions from $[0,1]$ into $\mathbb{R}$. Next, we define a mapping $\mathfrak{m} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ as

$$\mathfrak{m}(x,y) = \sup_{p \in [0,1]} (\mid x(p) \mid + \mid y(p) \mid).$$

Clearly, here $(\mathfrak{X}, \mathfrak{m})$ is a CSMS satisfying condition(M). In this section, we present an application of Corollary 2.2 to find a solution of the following fractional thermostat model inspired by [36].

$$^{C}D^{\beta}x(p) = -f(p, x(p)) \ (0 \leq p \leq 1, 1 < \beta \leq 2) \tag{3.1}$$

with boundary conditions $x'(0) = 0, \sigma^{C}D^{\beta}x(p) + x(t) = x'(0)$, where $\sigma > 0, 0 < t \leq 1$ are given constants and $x(p) \in \mathfrak{X}$, $^{C}D^{\beta}$ is the Caputo derivative of fractional order β and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. According to [36], any function $x(p) \in \mathfrak{X}$ is a solution of 3.1 iff

$$x(p) = \int_0^1 G(p,q) f(q, x(q)) dq, \tag{3.2}$$

where $G(p,q)$ is the Green's function (based on β), which is presented by

$$G(p,q) = \sigma + R_t(q) - R_p(q)$$

and $R_p : [0,1] \to \mathbb{R}$ is characterized by

$$R_p(q) = \begin{cases} \frac{(p-q)^{\beta-1}}{\Gamma(\beta)}, & \text{if } q \leq p \\ 0, & \text{if } q > p. \end{cases}$$

Let us write Ξ to denote all functions $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(1) $\xi(s) \leq s, \ \forall \ s \in \mathbb{R}^+$,
(2) ξ is non-decreasing.

Now we prove our existence theorem.

Theorem 3.1. *Consider the fractional thermostat model given by 3.1. Let $\zeta_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (for i = 1, 2) be any two given functions. Assume that the following assertions hold.*

(1) *Suppose there exists a $x_0 \in \mathfrak{X}$ such that $\zeta_1(x_0(p), Kx_0(p)) \geq 0$, $\zeta_2(x_0(p), Kx_0(p)) \leq 0$ for all $p \in [0,1]$, where $K : \mathfrak{X} \to \mathfrak{X}$ is a function that is characterized by*

$$Kx(p) = \int_0^1 G(p,q) f(q, x(q)) dq;$$

(2) *there exists a constant Q and a $\xi \in \Xi$ such that*

$$| f(p, x(p)) | + | f(p, y(p)) | \leq Q\xi\Big(\sqrt{\frac{\log[(| x(p) | + | y(p) |)^2 + 1]}{1 + \mathcal{M}^*_{x,y}}}\Big),$$

where $x, y \in \mathfrak{X}$, $\zeta_1(x(p), y(p)) \geq 0, \zeta_2(x(p), y(p)) \leq 0$ for all $p \in [0,1]$ together with

$$Q(\sigma + \frac{2}{\Gamma(\beta+1)}) \leq 1;$$

(3) *for any $p \in [0,1]$ and $x, y \in \mathfrak{X}$, $\zeta_1(x(p), y(p)) \geq 0$ implies $\zeta_1(Kx(p), K^2x(p)) \geq 0$, $\zeta_1(Ky(p), K^2y(p)) \geq 0$ and $\zeta_2(x(p), y(p)) \leq 0$ implies $\zeta_2(Kx(p), K^2x(p)) \leq 0$, $\zeta_2(Ky(p), K^2y(p)) \geq 0$. Also, suppose that $\zeta_1(a,b) \geq 0 \Leftrightarrow \zeta_1(b,a) \geq 0$ and $\zeta_2(a,b) \leq 0 \Leftrightarrow \zeta_2(b,a) \leq 0$;*

(4) *for any* $p \in [0,1]$, *if* $\{x_n\}$ *is a sequence in* $\mathfrak{X}$ *such that* $x_n \to x^*$ *in* $\mathfrak{X}$ *and* $\zeta_1(x_n(p), x_{n+1}(p)) \geq 0$, $\zeta_2(x_n(p), x_{n+1}(p)) \leq 0$, $\forall\ n \in \mathbb{N}$, *then* $\zeta_1(x_n(p), x^*(p)) \geq 0$, $\zeta_2(x_n(p), x^*(p)) \leq 0$ *for each* $n \in \mathbb{N}$.

Then, the problem 3.1 has at least one solution in $\mathfrak{X}$.

Proof. In 3.2, we have already mentioned that to find a solution of 3.1 is equivalent to find a solution of 3.2. Hence, to find a solution of 3.1 is similar to find a fixed point of the operator K defined in assertion 1. Now, let $x, y \in \mathfrak{X}$ such that $\zeta_1(x(p), y(p)) \geq 0, \zeta_2(x(p), y(p)) \leq 0$ for all $p \in [0,1]$. By assertion 2, we get the following

$$
\begin{aligned}
&\mid Kx(p) \mid + \mid Ky(p) \mid \\
&=\mid \int_0^1 G(p,q) f(q,x(q)) dq \mid + \mid \int_0^1 G(p,q) f(q,y(q)) dq \mid \\
&=\mid \sigma \int_0^1 f(q,x(q)) dq + \int_0^t \frac{(t-q)^{\beta-1}}{\Gamma(\beta)} f(q,x(q)) dq - \int_0^p \frac{(p-q)^{\beta-1}}{\Gamma(\beta)} f(q,x(q)) dq \mid \\
&+ \mid \sigma \int_0^1 f(q,y(q)) dq + \int_0^t \frac{(t-q)^{\beta-1}}{\Gamma(\beta)} f(q,y(q)) dq - \int_0^p \frac{(p-q)^{\beta-1}}{\Gamma(\beta)} f(q,y(q)) dq \mid \\
&\leq \sigma \int_0^1 \mid f(q,x(q)) \mid dq + \int_0^t \frac{(t-q)^{\beta-1}}{\Gamma(\beta)} \mid f(q,x(q)) \mid dq \\
&+ \int_0^p \frac{(p-q)^{\beta-1}}{\Gamma(\beta)} \mid f(q,x(q)) \mid dq \\
&+ \sigma \int_0^1 \mid f(q,y(q)) \mid dq + \int_0^t \frac{(t-q)^{\beta-1}}{\Gamma(\beta)} \mid f(q,y(q)) \mid dq \\
&+ \int_0^p \frac{(p-q)^{\beta-1}}{\Gamma(\beta)} \mid f(q,y(q)) \mid dq \\
&= \sigma \int_0^1 \Big[\mid f(q,x(q)) \mid + \mid f(q,y(q)) \mid \Big] dq \\
&+ \int_0^t \frac{(t-q)^{\beta-1}}{\Gamma(\beta)} \Big[\mid f(q,x(q)) \mid + \mid f(q,y(q)) \mid \Big] dq \\
&+ \int_0^p \frac{(p-q)^{\beta-1}}{\Gamma(\beta)} \Big[\mid f(q,x(q)) \mid + \mid f(q,y(q)) \mid \Big] dq \\
&\leq \sigma \int_0^1 Q\xi \Big(\sqrt{\frac{\log[(\mid x(p) \mid + \mid y(p) \mid)^2 + 1]}{1 + \mathcal{M}^*_{x,y}}} \Big) dq \\
&+ \int_0^t \frac{(t-q)^{\beta-1}}{\Gamma(\beta)} Q\xi \Big(\sqrt{\frac{\log[(\mid x(p) \mid + \mid y(p) \mid)^2 + 1]}{1 + \mathcal{M}^*_{x,y}}} \Big) dq \\
&+ \int_0^p \frac{(p-q)^{\beta-1}}{\Gamma(\beta)} Q\xi \Big(\sqrt{\frac{\log[(\mid x(p) \mid + \mid y(p) \mid)^2 + 1]}{1 + \mathcal{M}^*_{x,y}}} \Big) dq
\end{aligned}
$$

$$\leq \sigma \int_0^1 Q\xi\Big(\sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}\Big)dq + \int_0^t \frac{(t-q)^{\beta-1}}{\Gamma(\beta)} Q\xi\Big(\sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}\Big)dq$$

$$+ \int_0^p \frac{(p-q)^{\beta-1}}{\Gamma(\beta)} Q\xi\Big(\sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}\Big)dq$$

$$\leq \sigma Q \int_0^1 \sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}dq + Q\int_0^t \frac{(t-q)^{\beta-1}}{\Gamma(\beta)} \sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}dq$$

$$+ Q\int_0^p \frac{(p-q)^{\beta-1}}{\Gamma(\beta)} \sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}dq$$

$$= \sigma Q\sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}\int_0^1 dq + Q\sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}\int_0^t \frac{(t-q)^{\beta-1}}{\Gamma(\beta)}dq$$

$$+ Q\sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}\int_0^p \frac{(p-q)^{\beta-1}}{\Gamma(\beta)}dq$$

$$\leq \sigma Q\sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}} + \frac{2Q}{\Gamma(\beta+1)}\sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}$$

$$\leq \sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}Q\Big[\sigma + \frac{2}{\Gamma(\beta+1)}\Big]$$

$$\leq \sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}.$$

Hence, for every $p \in [0,1]$, we have the following

$$\mid Kx(p) \mid + \mid Ky(p) \mid \leq \sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}.$$

In LHS of the above inequality, we consider the supremum and we have the following

$$\mathfrak{m}(Kx,Ky) \leq \sqrt{\frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}}}.$$

Taking square on both sides of the above inequality, we obtain

$$\mathfrak{m}(Kx,Ky)^2 \leq \frac{\log[\mathfrak{m}(x,y)^2+1]}{1+\mathcal{M}^*_{x,y}} \leq \frac{\log[(\mathcal{M}^*_{x,y})^2+1]}{1+\mathcal{M}^*_{x,y}}.$$

In Corollary 2.2, we consider $\psi(s) = s^2$;

$$\theta(s) = \begin{cases} \frac{1}{1+s}, & \text{if } s > 0; \\ \frac{1}{2}, & \text{if } s = 0; \end{cases}$$

$\eta(s) = \log[1+s^2]$ together with $h(x,y) = y$ and $F(s,t) = t$. Let us define two functions $\alpha, \mu : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ in the following way.

$$\alpha(x,y) = \begin{cases} 2, & \text{if } \zeta_1(x(p), y(p)) \geq 0, \ \forall\, p \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu(x,y) = \begin{cases} 0, & \text{if } \zeta_2(x(p), y(p)) \leq 0, \ \forall\, p \in [0,1] \\ 2, & \text{otherwise.} \end{cases}$$

Clearly, then by assertion 1, the mapping K is weakly α-admissible and weakly μ-subadmissible mapping. Further from assertion 4, for any sequence $\{x_n\}$ in $\mathfrak{X}$ such that $x_n \to x^*$ with $\alpha(x_n, x_{n+1}) \geq 1, \mu(x_n, x_{n+1}) \leq 1$ implies $\alpha(x_n, x^*) \geq 1, \mu(x_n, x^*) \leq 1$ for each $n \in \mathbb{N}$, i.e., the SMS $(\mathfrak{X}, \mathfrak{m})$ is (α, μ)-regular. Hence, by applying Corollary 2.2, we conclude that there exists a point a^* in $\mathfrak{X}$ such that $Ka^* = a^*$, i.e., a^* is a solution of the equation 3.1. Thus, our proof is completed. $\square$

4. Conclusion

In this paper, we have studied some famous fixed point results in the context of SMSs. Our results are supported by suitable examples. Moreover, we have rectified some errors in some of the published papers in the framework of SMSs. Lastly, we have given an application in fractional thermostat model to show the applicability of our obtained new findings.

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