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AN ANALYSIS ON THE EXISTENCE OF NONLOCAL HILFER FRACTIONAL NEUTRAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY VIA ALMOST SECTORIAL OPERATORS

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Abstract. We devote this paper to investigate the existence of a solution for a Hilfer fractional differential systems with nonlocal conditions and infinite delay via almost sectorial operators. Firstly, we show the existence of the system using the measure of noncompactness and the Darbo-Sadovskii fixed point theorem. Finally, we extend the results to the existence of a neutral system. An example is provided to demonstrate how the major results might be applied.
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1. Introduction

In 1695, fractional calculus was presented as a major field of mathematics. It happened approximately simultaneously with the development of classical calculus. Researchers have discovered that fractional calculus may accurately portray a range of nonlocal events in the disciplines of architecture and natural science. The most common fields of fractional calculus are rheology, dynamical cycles in self-comparable and porous structures, diffusive transport equivalent to dispersion, liquid stream, optics, viscoelasticity, and others. On the other hand, it has been demonstrated that fractional differential equations are a helpful tool for modeling a variety of occurrences and that fractional-order models are more appropriate than integer-order models in several real-world situations. In [1, 2, 18, 17, 20, 23, 21, 22, 24, 25, 32] are the research papers related to fractional differential systems theory, readers will find several intriguing conclusions related to fractional dynamical systems.

Many researchers have looked at boundary value issues using fractional-order differential equations and inclusions with a variety of boundary conditions, including anti-periodic, classical, multi-point, periodic, fractional-order, nonlocal, and integral boundary conditions. A growing number of researchers are advancing fractional existence for fractional calculus using almost sectorial operators. The researcher devised a new strategy for identifying moderate solutions for the system under study. In addition, fractional calculus, semigroups, multivalued analysis, noncompactness measure, the Laplace transform, Wright-type function, and fixed point theorem, the scholars developed a theory that allowed them to derive various features of related semigroups created by almost sectorial operators. We can refer to [19, 22, 27, 28, 30] for more information.

Hilfer [11] presented another type of fractional derivative, which containing the R-L derivative, as well as the Caputo fractional derivative. Nowadays HF calculus has an important role in researchers. Many scholars have recently expressed a strong interest in this field, which has prompted on the work in [9, 10, 16, 29]. In [12, 14, 4] researchers used Schauder's fixed point theorem to arrive at their conclusions via almost sectorial operators. Researchers recently used the nondense domain, semigroup, cosine families, numerous fixed point approaches, and a measure of noncompactness to build nonlocal fractional differential systems with or without delay. In [15, 26] author established their results via $M\ddot{o}nch$ fixed point theorem with MNC. In [5, 6, 7] the authors developed the ideas of differential systems via Darbo-Sadovskii fixed point method with measure of noncompactness.

Our goals in this article are to examine the existence of HF neutral differential equations via almost sectorial operators, which is motivated by the theory proposed in the preceding articles. In this paper, we shall examine the following topic: HFD equations contain almost sectorial operators with nonlocal condition

$$D_{0^+}^{\eta,\zeta}\mathfrak{u}(\mathfrak{z}) = \mathtt{A}\mathfrak{u}(\mathfrak{z}) + \mathfrak{F}\big(\mathfrak{z},\mathfrak{u}_{\mathfrak{z}}\big), \qquad \mathfrak{z} \in \mathcal{I}' = (0,b], \tag{1.1}$$

$$I_{0^+}^{(1-\eta)(1-\zeta)}\mathfrak{u}(0) + N(\mathfrak{u}_{\mathfrak{z}}) = \xi \in \mathfrak{B}w, \quad \mathfrak{z} \in (-\infty,0], \tag{1.2}$$

where $\mathtt{A}$ denote the almost sectorial operator, which generate an analytic semigroup $\{T(\mathfrak{z}), \mathfrak{z} \geq 0\}$ on Y. $D_{0^+}^{\eta,\zeta}$ denotes the HFD of order $\eta,\ 0<\eta<1$ and type $\zeta,\ 0 \leq \zeta \leq 1$. Let $\mathfrak{u}(\cdot)$ be the state in a Banach space Y with norm $\|\cdot\|$. The histories $\mathfrak{u}_{\mathfrak{z}} : (-\infty,0] \to \mathfrak{B}w,\ \mathfrak{u}_{\mathfrak{z}}(a) = \mathfrak{u}(\mathfrak{z}+a),\ a \leq 0$ associated with phase space $\mathfrak{B}w$. Set $\mathcal{I} = [0,b]$, and let $\mathfrak{F} : \mathcal{I} \times \mathfrak{B}w \to Y$ be the Y-valued function and nonlocal term $N : \mathfrak{B}w \to Y$.

The structure of the article is broken down as follows. The principles of fractional calculus, semigroup, phase spaces, almost sectorial operators, and measure of noncompactness are covered in Section 2. In Section 3, we presented the existence mild solution of system. In Section 4, we continue our study to extend to result to existence of a neutral system with nonlocal condition. We provide an example in Section 5, to demonstrate our main ideas. Finally, some conclusions are offered.

2. Preliminaries

Here, we introduce basic definitions, theorems and lemma that are applied every part of the paper.

Let $\mathtt{C}$ be the collection of all continuous functions from $\mathcal{I}$ to Y, where $\mathcal{I} = [0, b]$ and $\mathcal{I}' = (0, b]$ with $b > 0$. Take

$$\mathcal{X} = \left\{ \mathfrak{u} \in \mathtt{C} : \lim_{\mathfrak{z} \to 0} \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)} \mathfrak{u}(\mathfrak{z}) \text{ exists and finite } \right\},$$

which is the Banach space and its norm on $\| \cdot \|_{\mathcal{X}}$, defined as

$$\|\mathfrak{u}\|_{\mathcal{X}} = \sup_{\mathfrak{z} \in \mathcal{I}'} \left\{ \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)} \|\mathfrak{u}(\mathfrak{z})\| \right\}.$$

Set $\mathfrak{B}_P(\mathcal{I}) = \{u \in \mathtt{C} \text{ such that } \|u\| \leq P\}$. We note that, let

$$\mathfrak{u}(\mathfrak{z}) = \mathfrak{z}^{(1+\eta\vartheta)(\zeta-1)} y(\mathfrak{z}), \ \mathfrak{z} \in (0, b]$$

then, $\mathfrak{u} \in \mathcal{X}$ *iff* $y \in \mathtt{C}$ *and* $\|\mathfrak{u}\|_{\mathcal{X}} = \|y\|$. We introduce $\mathfrak{F}$ with $\|\mathfrak{F}\|_{L^p(\mathcal{I}, \mathbb{R}^+)}$ through $\mathfrak{F} \in L^p(\mathcal{I}, \mathbb{R}^+)$ for some p along with $1 \leq p \leq \infty$. Also $L^p(\mathcal{I}, Y)$ denote the Banach space of functions $\mathfrak{F} : \mathcal{I} \times \mathfrak{B}w \to Y$ which are the Bochner integrable normed by $\|\mathfrak{F}\|_{L^p(\mathcal{I}, Y)}$.

Definition 2.1 [31] The fractional integral of order η for the function $\mathfrak{F} : [b, \infty) \to \mathbb{R}$ having the lower limit b is presented by

$$I^{\eta}_{b^+} \mathfrak{F}(\mathfrak{z}) = \frac{1}{\Gamma(\eta)} \int_b^{\mathfrak{z}} \frac{\mathfrak{F}(\omega)}{(\mathfrak{z} - \omega)^{1-\eta}} d\omega \quad \mathfrak{z} > 0; \ \eta \in \mathbb{R}^+.$$

Definition 2.2 [31] The R-L derivative has order $\eta > 0$, $k - 1 \leq \eta < k, \ k \in \mathbb{N}$, and the function $\mathfrak{F} : [b, +\infty) \to \mathbb{R}$, presented by

$${}^L D^{\eta}_{b+} \mathfrak{F}(\mathfrak{z}) = \frac{1}{\Gamma(k - \eta)} \frac{d^k}{d\mathfrak{z}^k} \int_b^{\mathfrak{z}} \frac{\mathfrak{F}(\omega)}{(\mathfrak{z} - \omega)^{\eta+1-k}} d\omega, \ \mathfrak{z} > b, \omega \in \mathbb{R}^+.$$

Definition 2.3 [31] The Caputo derivative has order $\eta > 0, \ k - 1 \leq \eta < k, \ k \in \mathbb{N}$ for a function $\mathfrak{F} : [b, +\infty) \to \mathbb{R}$, is defined as

$${}^C D^{\eta}_{b+} \mathfrak{F}(\mathfrak{z}) = \frac{1}{\Gamma(k - \eta)} \int_b^{\mathfrak{z}} \frac{\mathfrak{F}^k(\omega)}{(\mathfrak{z} - \omega)^{\eta+1-k}} d\omega = I^{k-\eta}_{b+} \mathfrak{F}^k(\mathfrak{z}), \ \mathfrak{z} > b, \ \omega \in \mathbb{R}^+.$$

Definition 2.4 [11] The HFD of order $0 < \eta < 1$ and type $\zeta \in [0, 1]$ for the function $\mathfrak{F} : [b, +\infty) \to \mathbb{R}$, presented by

$$D^{\eta,\zeta}_{b^+} \mathfrak{F}(\mathfrak{z}) = [I^{(1-\eta)\zeta}_{b^+} D(I^{(1-\eta)(1-\zeta)}_{b^+} \mathfrak{F})](\mathfrak{z}).$$

Remark 2.5

(1) If $\zeta = 0, \ 0 < \eta < 1$, and $b = 0$, then the HFD corresponds to the classical R-L fractional derivative:

$$D^{\eta,0}_{0^+} \mathfrak{F}(\mathfrak{z}) = \frac{d}{d\omega} I^{1-\eta}_{0+} \mathfrak{F}(\mathfrak{z}) = {}^L D^{\eta}_{0+} \mathfrak{F}(\mathfrak{z}).$$

(2) If $\zeta = 1, 0 < \eta < 1$, and $b = 0$, then the HFD corresponds to the classical Caputo fractional derivative:

$$D_{0^+}^{\eta,1}\mathfrak{F}(\mathfrak{z}) = I_{0^+}^{1-\eta}\frac{d}{d\omega}\mathfrak{F}(\mathfrak{z}) =^{C} D_{0^+}^{\eta}\mathfrak{F}(\mathfrak{z}).$$

Now we define the abstract phase space $\mathfrak{B}w$. Let $w : (-\infty, 0] \to (0, +\infty)$ be continuous along $l = \int_{-\infty}^{0} w(\mathfrak{z})d\mathfrak{z} < +\infty$. Now, for every $n > 0$, we have

$$\mathfrak{B} = \big\{\delta : [-n, 0] \to Y \text{ such that } \delta(\mathfrak{z}) \text{ is bounded and measurable}\big\},$$

and set the space $\mathfrak{B}$ with the norm

$$\|\delta\|_{[-n,0]} = \sup_{\tau\in[-n,0]} \|\delta(\tau)\|, \ \textit{for all } \delta \in \mathfrak{B}.$$

Now, we define

$$\mathfrak{B}w = \left\{\delta : (-\infty, 0] \to Y \ni \forall n > 0, \delta|_{-n,0} \in \mathfrak{B} \ \& \ \int_{-\infty}^{0} w(\tau)\|\delta\|_{[\tau,0]}d\tau < +\infty\right\}.$$

If $\mathfrak{B}w$ is endowed with $\|\delta\|_{\mathfrak{B}w} = \int_{-\infty}^{0} w(\tau)\|\delta\|_{[\tau,0]}d\tau,$ *for all* $\delta \in \mathfrak{B}w$, therefore $(\mathfrak{B}w, \|\cdot\|)$ is a Banach space. Now, we define the space

$$\mathfrak{B}w' = \big\{\mathfrak{u} : (-\infty, b] \to Y \text{ such that } \mathfrak{u}|_{\mathcal{I}} \in \mathtt{C}, \ \xi \in \mathfrak{B}w\big\}.$$

Consider the seminorm $\|\cdot\|_b$ in $\mathfrak{B}w'$ defined by

$$\|\mathfrak{u}\|_b = \|\xi\|_{\mathfrak{B}w} + \sup\{\|\mathfrak{u}(\tau)\| : \tau \in [0, b]\}, \ \mathfrak{u} \in \mathfrak{B}w'.$$

Lemma 2.6 *Suppose* $\mathfrak{u} \in \mathfrak{B}w'$, *then for* $\mathfrak{z} \in \mathcal{I}, \mathfrak{u}_{\mathfrak{z}} \in \mathfrak{B}w$. *Moreover,*

$$l|\mathfrak{u}(\mathfrak{z})| \leq \|\mathfrak{u}_{\mathfrak{z}}\|_{\mathfrak{B}w} \leq \|\xi\|_{\mathfrak{B}w} + l \sup_{r\in[0,\mathfrak{z}]} |\mathfrak{u}(r)|, \ \textit{where } l = \int_{-\infty}^{0} w(\mathfrak{z})d\mathfrak{z} < \infty.$$

Definition 2.7 [22] For $-1 < \vartheta < 0, \quad 0 < \varphi < \frac{\pi}{2}$, we define the family of closed linear operators $\Theta_{\varphi}^{\vartheta}$, the sector $S_{\varphi} = \{\theta \in \mathbb{C}\backslash\{0\} \text{ with } |arg\ \theta| \leq \varphi\}$ and $\mathtt{A} : D(\mathtt{A}) \subset Y \to Y$ that satisfy

(i) $\sigma(\mathtt{A}) \subseteq S_{\varphi}$;
(ii) $\big\|(\theta - \mathtt{A})^{-1}\big\| \leq \mathcal{K}_{\delta}|\nu|^{\vartheta}$, for all $\varphi < \delta < \pi$ and there exists $\mathcal{K}_{\delta}$ is a constant

then $\mathtt{A} \in \Theta_{\varphi}^{\vartheta}$ is known as almost sectorial operator on Y.

Proposition 2.8 [22] *Suppose* $\mathfrak{u} \in \Theta_{\varphi}^{\vartheta}$ *for* $-1 < \vartheta < 0$ *and* $0 < \varphi < \frac{\pi}{2}$. *Then succeeding are satisfied:*

(a) $T(\mathfrak{z})$ *is analytic and* $\frac{d^k}{d\mathfrak{z}^k}T(\mathfrak{z}) = (-\mathtt{A})^k T(\mathfrak{z}), \qquad \mathfrak{z} \in S_{\frac{\pi}{2}-\varphi}$;
(b) $T(\mathfrak{z} + \omega) = T(\mathfrak{z})T(\omega)$ *for every* $\omega, \mathfrak{z} \in S_{\frac{\pi}{2}-\varphi}$;
(c) $\|T(\mathfrak{z})\|_{L(Y)} \leq \kappa_0 \mathfrak{z}^{-\vartheta-1}$, $\mathfrak{z} > 0$; *where the constant* $\kappa_0 > 0$;
(d) *If* $\Sigma_T = \{\mathfrak{u} \in Y : \lim_{\mathfrak{z}\to 0^+} T(\mathfrak{z})\mathfrak{u} = \mathfrak{u}\}$ *then* $D(\mathtt{A}^{\theta}) \subset \Sigma_T$ *provided* $\theta > 1 + \vartheta$;
(e) $(\nu - \mathtt{A})^{-1} = \int_0^{\infty} e^{-v\omega}T(\omega)d\omega$, $\nu \in \mathbb{C}$ *and* $Re(\nu) > 0$.

Let $\{\eta(\mathfrak{z})\}_{\mathfrak{z}\in S_{\frac{\pi}{2}-\varphi}}, \{\mathcal{Q}_\eta(\mathfrak{z})\}_{\mathfrak{z}\in S_{\frac{\pi}{2}-\varphi}}$ be the operator families defined as follows:

$$\mathcal{S}_\eta(\mathfrak{z}) = \int_0^\infty \mathfrak{W}_\eta(\epsilon)T(\mathfrak{z}^\eta\epsilon)d\epsilon,$$
$$\mathcal{Q}_\eta(\mathfrak{z}) = \int_0^\infty \eta\epsilon\mathfrak{W}_\eta(\epsilon)T(\mathfrak{z}^\eta\epsilon)d\epsilon,$$

where $\mathfrak{W}_\eta(\beta)$ be the Wright-type function:

$$\mathfrak{W}_\eta(\beta) = \sum_{k\in\mathbb{N}} \frac{(-\beta)^{k-1}}{\Gamma(1-\eta k)(k-1)!},\ \beta\in\mathbb{C}. \tag{2.1}$$

Let $-1 < \iota < \infty,\ p > 0$, the given properties are hold.

(a) $\mathfrak{W}_\eta(\theta) \geq 0, \quad \mathfrak{z} > 0$;
(b) $\int_0^\infty \theta^\iota \mathfrak{W}_\eta(\theta)d\theta = \frac{\Gamma(1+\iota)}{\Gamma(1+\eta\iota)}$;
(c) $\int_0^\infty \frac{\eta}{\theta^{(\eta+1)}} e^{-p\theta}\mathfrak{W}_\eta(\frac{1}{\theta^\eta})d\theta = e^{-p^\eta}$.

Theorem 2.9 [31] *For each fixed* $\mathfrak{z}\in S^0_{\frac{\pi}{2}-\varphi}$, $S_\eta(\mathfrak{z})$ *and* $Q_\eta(\mathfrak{z})$ *are the bounded and linear operators on* Y. *Besides*

$$\big\|\mathcal{S}_\eta(\mathfrak{z})\big\| \leq \kappa_s\mathfrak{z}^{-\eta(1+\vartheta)}, \qquad \big\|\mathcal{Q}_\eta(\mathfrak{z})\big\| \leq \kappa_p\mathfrak{z}^{-\eta(1+\vartheta)}, \quad \mathfrak{z}>0,$$

where κ_s *and* κ_p *are constant dependent only on* η *and* ϑ.

Theorem 2.10 [31] $\mathcal{S}_\eta(\mathfrak{z})$ *and* $\mathcal{Q}_\eta(\mathfrak{z})$ *are continuous in the uniform operator topology, for* $\mathfrak{z} > 0$, *for every* $d > 0$, *the continuity is uniform on* $[d,\infty)$.

Lemma 2.11 *System (1.1)-(1.2) is identical to an integral equation presented by*

$$\mathfrak{u}(\mathfrak{z}) = \frac{\xi(0)-N(\mathfrak{u}_\mathfrak{z}))}{\Gamma(\zeta(1-\eta)}\mathfrak{z}^{(1-\eta)(\zeta-1)+\eta)} + \frac{1}{\Gamma(\eta)}\int_0^\mathfrak{z}(\mathfrak{z}-\omega)^{\eta-1}\big[\mathtt{A}\mathfrak{u}_\omega + \mathfrak{F}\big(\omega,\mathfrak{u}_\omega\big)\big]d\omega.$$

Definition 2.12 The mild solution of the Cauchy problem (1.1)-(1.2), is a function $\mathfrak{u}(\mathfrak{z}) \in C(\mathcal{I}', Y)$, that satisfies

$$\mathfrak{u}(\mathfrak{z}) = \mathcal{S}_{\eta,\zeta}(\mathfrak{z})\big[\xi(0)-N(\mathfrak{u}_\mathfrak{z})\big] + \int_0^\mathfrak{z}\mathcal{K}_\eta(\mathfrak{z}-\omega)\mathfrak{F}\big(\omega,\mathfrak{u}_\omega\big)d\omega,\ \mathfrak{z}\in\mathcal{I}, \tag{2.2}$$

where $\mathcal{S}_{\eta,\zeta}(\mathfrak{z}) = I_0^{\zeta(1-\eta)}\mathcal{K}_\eta(\mathfrak{z}),\ \mathcal{K}_\eta(\mathfrak{z}) = \mathfrak{z}^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z})$. i.e.,

$$\mathfrak{u}(\mathfrak{z}) = \mathcal{S}_{\eta,\zeta}(\mathfrak{z})\big[\xi(0)-N(\mathfrak{u}_\mathfrak{z})\big] + \int_0^\mathfrak{z}(\mathfrak{z}-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}-\omega)\mathfrak{F}\big(\omega,\mathfrak{u}_\omega\big)d\omega,\ \mathfrak{z}\in\mathcal{I}.$$

Lemma 2.13 [12]

(1) $\mathcal{K}_\eta(\mathfrak{z})$ *and* $\mathcal{S}_{\eta,\zeta}(\mathfrak{z})$ *are strongly continuous, for* $\mathfrak{z} > 0$.
(2) $\mathcal{K}_\eta(\mathfrak{z})$ *and* $\mathcal{S}_{\eta,\zeta}(\mathfrak{z})$ *are bounded linear operators on* Y, *for every fixed* $\mathfrak{z}\in S_{\frac{\pi}{2}-\varphi}$, *we have*

$$\big\|\mathcal{K}_\eta(\mathfrak{z})\mathfrak{u}\big\| \leq \kappa_p\mathfrak{z}^{-1-\eta\vartheta}\|\mathfrak{u}\|,\ \big\|\mathcal{S}_{\eta,\zeta}(\mathfrak{z})\mathfrak{u}\big\| \leq \frac{\Gamma(-\eta\vartheta)}{\Gamma(\zeta(1-\eta)-\eta\vartheta)}\kappa_p\mathfrak{z}^{\zeta(1-\eta)-\eta\vartheta-1}\|\mathfrak{u}\|. \tag{2.3}$$

Definition 2.14 [13] Suppose E^+ is the positive cone of an order Banach space $(E,\leq)$. Let Φ be the function defined on the set of all bounded subset of the Banach space Y with values in E^+ is known as measure of noncompactness (MNC) on Y iff

$\Phi(conv(\Omega)) = \Phi(\Omega)$ for all bounded subset $\Omega \subset Y$, where $conv(\Omega)$ denoted the closed convex hull of Ω.

Now, we recall some concepts on the Hausdorff measure of noncompactness.

Definition 2.15 [13] For a bounded set B in a Banach space Y, the Hausdorff measure of noncompactness μ is defined as

$$\mu(B) = \inf\{\epsilon > 0 : B \text{ can be covered by a finite number of balls with radii } \epsilon\}. \quad (2.4)$$

Lemma 2.16 [13] *Suppose Y is a Banach space and $\mathfrak{B}, C \subseteq Y$ are bounded. Then, the succeeding properties satisfy.*

(i) *$\mathfrak{B}$ is precompact iff $\mu(\mathfrak{B}) = 0$;*
(ii) *$\mu(\mathfrak{B}) = \mu(\overline{\mathfrak{B}}) = \mu(conv(\mathfrak{B}))$, where $\overline{\mathfrak{B}}$ and $conv(\mathfrak{B})$ are denote the closure and convex hull of $\mathfrak{B}$, respectively;*
(iii) *If $\mathfrak{B} \subseteq C$ then $\mu(\mathfrak{B}) \leq \mu(C)$;*
(iv) *$\mu(\mathfrak{B} + C) \leq \mu(\mathfrak{B}) + \mu(C)$, such that $B + C = \{a_1 + a_2 : a_1 \in B, a_2 \in C\}$;*
(v) *$\mu(\mathfrak{B} \cup C) \leq \max\{\mu(\mathfrak{B}), \mu(C)\}$;*
(vi) *$\mu(\lambda\mathfrak{B}) = |\lambda|\mu(\mathfrak{B})$ for every $\lambda \in \mathbb{R}$, when Y be a real Banach space;*
(vii) *if the operator $\Psi : D(\Psi) \subseteq Y \to Y_1$ is Lipschitz continuous with constant κ_1 then we know $\rho(\Psi(\mathfrak{B})) \leq \kappa_1\mu(\mathfrak{B})$ for any bounded subset $\mathfrak{B} \subset D(\Psi)$, where Y_1 is the another Banach space and ρ represent the Hausdorff measure of noncompactness in Y_1.*

Definition 2.17 [5] The operator $\Psi : D(\Psi) \subseteq Y \to Y$ is said to be a $\mu-contraction$ if there exists a positive constant $\kappa_1 < 1$ such that $\mu(\Psi(B)) \leq \kappa_1\mu(B)$ for any bounded closed subset $B \subseteq D(\Psi)$.

Theorem 2.18 [13] *if $\{\mathfrak{u}_k\}_{k=1}^{\infty}$ is a sequence of Bochner integrable functions from $\mathcal{I}$ into Y with the estimation $\|\mathfrak{u}_k(\mathfrak{z})\| \leq \mu_1(\mathfrak{z})$ for almost all $\mathfrak{z} \in \mathcal{I}$ and every $k \geq 1$, where $\mu_1 \in L^1(\mathcal{I}, \mathbb{R})$, then the function $\varphi(\mathfrak{z}) = \mu\big(\{\mathfrak{u}_k(\mathfrak{z}) : k \geq 1\}\big)$ be in $L^1(\mathcal{I}, \mathbb{R})$ and satisfies*

$$\mu\left(\left\{\int_0^{\mathfrak{z}} \mathfrak{u}_k(\omega)d\omega : \ k \geq 1\right\}\right) \leq 2\int_0^{\mathfrak{z}} \varphi(\omega)d\omega.$$

Lemma 2.19 [5] *Let $B \subset Y$ be a bounded set, then there exists a countable set $B_0 \subset B$ such that $\mu(B) \leq 2\mu(B_0)$.* We denote by μ_c the Hausdorff measure of noncompactness in the space $\mathtt{C}$.

Lemma 2.20 [5] *Let $B \subset \mathtt{C}$ be bounded and equicontinuous, then*

(1) *$\mu(B(\mathfrak{z}))$ is continuous on $[0, b]$,*
(2) *$\mu_c(B) = \max_{\mathfrak{z} \in [0,b]} \big(\mu(B(\mathfrak{z}))\big)$.*

Lemma 2.21 [31](Darbo-Sadovskii theorem) *Let $B \subset Y$ be a bounded, closed, and convex set. If $\Psi : D(\Psi) \subseteq Y \to Y$ is a continuous and $\mu -$ contraction operator. Then, Ψ has at least one fixed point in B.*

3. Existence

We require the succeeding hypotheses:

(H_1) Let $\mathtt{A}$ be the almost sectorial operator of the analytic semigroup $T(\mathfrak{z}),\ \mathfrak{z} > 0$ in Y such that $\|T(\mathfrak{z})\| \leq \mathcal{K}_1$ where $\mathcal{K}_1 \geq 0$ be the constant.

(H_2) The function $\mathfrak{F} : \mathcal{I} \times \mathfrak{B}w \to Y$ satisfies:

(a) Caratheodory condition: $\mathfrak{F}(\cdot, \mathfrak{u})$ is strongly measurable forevery $\mathfrak{u} \in \mathfrak{B}w$ and $\mathfrak{F}(\mathfrak{z}, \cdot)$ is continuous for a.e. $\mathfrak{z} \in \mathcal{I},\ \mathfrak{F}(\cdot, \cdot) : [0, S] \to Y$ is strongly measurable;

(b) there exists a constant $0 < \eta_1 < \eta$ and $m \in L^{\frac{1}{\eta_1}}(\mathcal{I}, \mathbb{R}^+)$ and non-decreasing continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\left\|\mathfrak{F}(\mathfrak{z}, \mathfrak{u})\right\| \leq m(\mathfrak{z}) f(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)} \|\mathfrak{u}\|),\ \mathfrak{u} \in Y,\ \mathfrak{z} \in \mathcal{I}$ where f satisfies $\lim \inf_{k\to\infty} \frac{f(k)}{k} = 0$;

(c) there exists a constant $0 < \eta_2 < \eta$ and $h \in L^{\frac{1}{\eta_2}}(\mathcal{I}, \mathbb{R}^+)$ such that, for all bounded subset $M \subset Y,\ \mu(\mathfrak{F}(\mathfrak{z}, M)) \leq h(\mathfrak{z})\beta(M)$ for a.e. $\mathfrak{z} \in \mathcal{I}$.

(H_3) The function $N : C(\mathcal{I}, Y) \to Y$ is continuous, compact operator and there exists $L_1 > 0$ be the value such that $\|N(\mathfrak{u}_1) - N(\mathfrak{u}_2)\| \leq L_1 \|\mathfrak{u}_1 - \mathfrak{u}_2\|$.

Theorem 3.1 *Suppose $(H_1) - (H_3)$ holds, then the HF system (1.1)-(1.2) has a solution on $\mathcal{I}$ provided, $\xi(0) \in D(\mathtt{A}^\theta)$ with $\theta > 1 + \vartheta$.*

Proof. Consider the operator $\Psi : \mathfrak{B}w' \to \mathfrak{B}w'$, defined

$$\Psi(\mathfrak{u}(\mathfrak{z})) = \begin{cases} \Psi_1(\mathfrak{z}), \mathfrak{z} \in (-\infty, 0], \\ \mathcal{S}_{\eta,\zeta}(\mathfrak{z})\big[\xi(0) - N(\mathfrak{u}_\mathfrak{z})\big] + \int_0^\mathfrak{z} (\mathfrak{z} - \omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z} - \omega)\mathfrak{F}\big(\omega, \mathfrak{u}_\omega\big) d\omega,\ \mathfrak{z} \in \mathcal{I}. \end{cases} \tag{3.1}$$

For $\Psi_1 \in \mathfrak{B}w$, we define $\widehat{\Psi}$ by

$$\widehat{\Psi}(\mathfrak{z}) = \begin{cases} \Psi_1(\mathfrak{z}), & \mathfrak{z} \in (-\infty, 0], \\ \mathcal{S}_{\eta,\zeta}(\mathfrak{z})\xi(0), & \mathfrak{z} \in \mathcal{I}, \end{cases}$$

then $\widehat{\Psi} \in \mathfrak{B}w'$. Let $\mathfrak{u}(\mathfrak{z}) = \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v(\mathfrak{z}) + \widehat{\Psi}(\mathfrak{z})],\ -\infty < \mathfrak{z} \leq b$. It can be easily shown that $\mathfrak{u}$ satisfies from Equation (2.2) iff v satisfies v_0 and

$$\begin{aligned} v(\mathfrak{z}) = & -\mathcal{S}_{\eta,\zeta}(\mathfrak{z}) N\big(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v_\mathfrak{z} + \widehat{\Psi}_\mathfrak{z}]\big) \\ & + \int_0^\mathfrak{z} (\mathfrak{z} - \omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z} - \omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega. \end{aligned}$$

Let $\mathfrak{B}w'' = \{v \in \mathfrak{B}w' : v_0 \in \mathfrak{B}w\}$. For any $v \in \mathfrak{B}w'$,

$$\begin{aligned} \|v\|_b = & \|v_0\|_{\mathfrak{B}w} + \sup\{\|v(\omega)\| : 0 \leq \omega \leq b\} \\ = & \sup\{\|v(\omega)\| : 0 \leq \omega \leq b\}. \end{aligned}$$

Hence, $(\mathfrak{B}w'', \|\cdot\|)$ is a Banach space.

For $P > 0$, choose $\mathfrak{B}_P = \{v \in \mathfrak{B}w'' : \|v\|_b \leq P\}$, then $\mathfrak{B}_P \subset \mathfrak{B}w''$ is uniformly bounded, and for $v \in \mathfrak{B}_P$, from Lemma 2,

$$\begin{aligned} \|v_\mathfrak{z} + \widehat{\Psi}_\mathfrak{z}\|_{\mathfrak{B}w} & \leq \|v_\mathfrak{z}\|_{\mathfrak{B}w} + \|\widehat{\Psi}\|_{\mathfrak{B}w} \\ & \leq l\left(P + \frac{\Gamma(-\eta\vartheta)}{\Gamma(\zeta(1-\eta) - \eta\vartheta)} \kappa_p \mathfrak{z}^{\zeta(1-\eta)-\eta\vartheta-1}\right) + \|\Psi_1\|_{\mathfrak{B}w} \\ & = P'. \end{aligned}$$

Let us introduce an operator $\Phi : \mathfrak{B}w'' \to \mathfrak{B}w''$, defined by

$$\Phi v(\mathfrak{z}) = \begin{cases} 0, \ \mathfrak{z} \in (-\infty, 0], \\ -\mathcal{S}_{\eta,\zeta}(\mathfrak{z}) N\big(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}} + \widehat{\Psi}_{\mathfrak{z}}]\big) + \int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}-\omega) \\ \qquad \times \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega, \qquad \mathfrak{z} \in \mathcal{I}. \end{cases}$$

Next to show that Φ has a fixed point.

Step 1: We have to prove there exists a positive value P such that $\Phi(\mathfrak{B}_P(\mathcal{I})) \subseteq \mathfrak{B}_P(\mathcal{I})$. Assume the statement is false i.e., there for each $P > 0$, there exits $v^P \in \mathfrak{B}_P(\mathcal{I})$, but $\Phi(v^P)$ not in $\mathfrak{B}_P(\mathcal{I})$, that is,

$$\begin{aligned} \|v^P\| \leq P &< \sup \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)} \big\| \big(\Phi v^P(\mathfrak{z})\big) \big\| \\ &\leq b^{(1+\eta\vartheta)(1-\zeta)} \big\| -\mathcal{S}_{\eta,\zeta}(\mathfrak{z}) N\big(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v^P_{\mathfrak{z}} + \widehat{\Psi}_{\mathfrak{z}}]\big) \big\| \\ &\quad + b^{(1+\eta\vartheta(1-\zeta))} \int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\eta-1} \Big\| \mathcal{Q}_\eta(\mathfrak{z}-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v^P_\omega + \widehat{\Psi}_\omega]\big) \Big\| d\omega \\ &\leq b^{(1+\eta\vartheta)(1-\zeta)} \Big[\|\mathcal{S}_{\eta,\zeta}(\mathfrak{z})\| \big[L_1 \|v^P_{\mathfrak{z}} + \widehat{\Psi}_{\mathfrak{z}}\| + \|N(0)\|\big] \\ &\quad + \int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{(\eta-1)} \|\mathcal{Q}_\eta(\mathfrak{z}-\omega)\| m(b) f(P') d\omega \Big] \\ &\leq b^{(1+\eta\vartheta)(1-\zeta)} \Big[\frac{\Gamma(-\eta\vartheta)}{\Gamma(\zeta(1-\eta)-\eta\vartheta)} \kappa_p b^{\zeta(1-\eta)-\eta\vartheta-1} \big[L_1 P' + \|N(0)\|\big] \\ &\quad + \frac{b^{-\eta\vartheta}}{\eta\vartheta} \kappa_p m(b) f(P') \Big] \\ &\leq \kappa_p b^{(1+\eta\vartheta)(1-\zeta)} M^*, \end{aligned}$$

where $M^* = \frac{\Gamma(-\eta\vartheta)}{\Gamma(\zeta(1-\eta)-\eta\vartheta)} b^{\zeta(1-\eta)-\eta\vartheta-1}\big[L_1 P' + \|N(0)\|\big] + \frac{b^{-\eta\vartheta}}{\eta\vartheta} m(b) f(P')$.

The above inequality is divided by P and applying the limit as $P \to \infty$, we obtain $1 \leq 0$, which is the contradiction. Therefore, $\Phi(\mathfrak{B}_P(\mathcal{I})) \subseteq \mathfrak{B}_P(\mathcal{I})$.

Step 2: In this section, we are proving, the operator Φ is continuous on $\mathfrak{B}_P(\mathcal{I})$. For Φ maps, $\mathfrak{B}_P(\mathcal{I})$ into $\mathfrak{B}_P(\mathcal{I})$. For any $v^k, \ v \in \mathfrak{B}_P(\mathcal{I}), \ k = 0, 1, 2, ...$ such that $\lim\limits_{k\to\infty} v^k = v$, then we have

$$\lim_{k\to\infty} v^k(\mathfrak{z}) = v(\mathfrak{z}), \text{ and } \lim_{k\to\infty} \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)} v^k(\mathfrak{z}) = \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)} v(\mathfrak{z}).$$

By using the hypothesis (H_2),

$$\begin{aligned} \mathfrak{F}(\mathfrak{z}, \mathfrak{u}_k(\mathfrak{z})) &= \mathfrak{F}\big(\mathfrak{z}, \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta))}[v^k(\mathfrak{z}) + \widehat{\Psi}(\mathfrak{z})]\big) \to \mathfrak{F}\big(\mathfrak{z}, \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta))}[v(\mathfrak{z}) + \widehat{\Psi}(\mathfrak{z})]\big) \\ &= \mathfrak{F}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z})) \ as \ k \to \infty. \end{aligned}$$

Next, let us consider the values:

$$F_k(\omega) = \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta))}[v^k_\omega + \widehat{\Psi}_\omega]\big) \ and \ F(\omega) = \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta))}[v_\omega + \widehat{\Psi}_\omega]\big).$$

Then, from hypotheses (H_2) and Lebesgue's dominated convergence theorem, we have obtained

$$\int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{(\eta-1)} \|F_k(\omega)-F(\omega)\| d\omega \to 0 \ as \ k \to \infty, \ \mathfrak{z} \in I. \tag{3.2}$$

Take $\mathcal{N}_k(\mathfrak{z}) = N\big(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v^k_{\mathfrak{z}} + \widehat{\Psi}_{\mathfrak{z}}]\big)$ and $\mathcal{N}(\mathfrak{z}) = N\big(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}} + \widehat{\Psi}_{\mathfrak{z}}]\big)$, from (H_4), we have

$$\|\mathcal{N}_k(\mathfrak{z}) - \mathcal{N}(\mathfrak{z})\| \to 0 \ as \ k \to \infty. \tag{3.3}$$

From the above equation, we can written as:

$$\begin{aligned}\big\|\Phi v^k - \Phi v\big\|_b \le{}& \frac{\Gamma(-\eta\vartheta)}{\Gamma(\zeta(1-\eta)-\eta\vartheta)}\kappa_p b^{\zeta(1-\eta)-\eta\vartheta-1}\big\|\mathcal{N}_k(\mathfrak{z}) - \mathcal{N}(\mathfrak{z})\big\| \\ &+ \kappa_p \frac{b^{-\eta\vartheta}}{\eta\vartheta}\big\|F_k(\omega)-F(\omega)\big\| ds. \end{aligned} \tag{3.4}$$

From above equations, we obtain $\big\|\Phi v^k - \Phi v\big\|_b \to 0 \ as \ k \to \infty$.Therefore, Φ is continuous on $\mathfrak{B}_P$.

Step 3: Next, we have to prove Φ is equicontinuous. For $\mathfrak{u} \in \mathfrak{B}_P(\mathcal{I})$, and $0 \le \mathfrak{z}_1 < \mathfrak{z}_2 \le b$. For our convenience, we can take $\big\|\Phi\mathfrak{u}(\mathfrak{z}_2) - \Phi\mathfrak{u}(\mathfrak{z}_1)\big\| = \Xi$, we have

$$\begin{aligned}\Xi ={}& \Bigg\| -\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\Bigg(\mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2)N\big(\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_2} + \widehat{\Psi}_{\mathfrak{z}_2}]\big) \\
&+ \int_0^{\mathfrak{z}_2} (\mathfrak{z}_2-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega)\mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big)d\omega\Bigg) \\
&- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}\Bigg(-\mathcal{S}_{\eta,\zeta}(\mathfrak{z}_1)N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big) \\
&+ \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}_1-\omega)\mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big)d\omega\Bigg)\Bigg\| \\
\le{}& \Big\|\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2)N\big(\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_2} + \widehat{\Psi}_{\mathfrak{z}_2}]\big) \\
&- \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2)N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big)\Big\| \\
&+ \Big\|\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2)N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big) \\
&- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}\mathcal{S}_{\eta,\zeta}(\mathfrak{z}_1)N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big)\Big\| \\
&+ \Big\|\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\int_0^{\mathfrak{z}_1} (\mathfrak{z}_2-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega)\mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big)d\omega \\
&- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}\int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega)\mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big)d\omega\Big\| \\
&+ \Big\|\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}\int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega)\mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big)d\omega
\end{aligned}$$

$$
\begin{aligned}
&- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_1-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
&+ \Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} (\mathfrak{z}_2-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_2-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
\leq & \sum_{i=1}^{5} I_i.
\end{aligned}
$$

Therefore,

$$
\big\| \Phi\mathfrak{u}(\mathfrak{z}_2) - \Phi\mathfrak{u}(\mathfrak{z}_1) \big\| = \sum_{i=1}^{5} I_i.
$$

$$
\begin{aligned}
I_1 = & \Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2) N\big(\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_2} + \widehat{\Psi}_{\mathfrak{z}_2}]\big) \\
& - \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2) N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big) \Big\|.
\end{aligned}
$$

From hypotheses (H_3) and Equation 3.3, we have obtained I_1 become to zero as $\mathfrak{z}_2 \to \mathfrak{z}_1$.

$$
\begin{aligned}
I_2 = & \Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2) N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big) \\
& - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_1) N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big) \Big\| \\
\leq & \Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2) - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_1) \Big\| \Big\| N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big) \Big\|.
\end{aligned}
$$

By the strong continuity of $\mathcal{S}_{\eta,\zeta}(\mathfrak{z})$ and hypothesis (H_3), we have obtained $I_2 \to 0$ as $\mathfrak{z}_2 \to \mathfrak{z}_1$.

$$
\begin{aligned}
I_3 = & \Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_2-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_2-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \\
& - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_2-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
\leq & \Big\| \Big(\int_0^{\mathfrak{z}_1} \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_2-\omega)^{\eta-1} - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_1-\omega)^{\eta-1} \Big) \\
& \times \mathcal{Q}_\eta(\mathfrak{z}_2-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
\leq & \kappa_p \Big\| \Big(\int_0^{\mathfrak{z}_1} \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_2-\omega)^{\eta-1} - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_1-\omega)^{\eta-1} \Big) \\
& \times (\mathfrak{z}_2-\omega)^{-\eta(1+\vartheta)} m(b) f(P') d\omega \Big\|.
\end{aligned}
$$

Implies, we have gotten $I_3 \to 0$ as $\mathfrak{z}_2 \to \mathfrak{z}_1$.

$$
I_4 = \Big\| \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_2-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega
$$

$$
\begin{aligned}
& - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1 - \omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_1 - \omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
\leq & \Big\| \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1 - \omega)^{\eta-1} \big[\mathcal{Q}_\eta(\mathfrak{z}_2 - \omega) - \mathcal{Q}_\eta(\mathfrak{z}_1 - \omega)\big] \\
& \times \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
\leq & \Big\| \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1 - \omega)^{\eta-1} \big[\mathcal{Q}_\eta(\mathfrak{z}_2 - \omega) - \mathcal{Q}_\eta(\mathfrak{z}_1 - \omega)\big] m(b) f(P') d\omega \Big\|.
\end{aligned}
$$

Since $\mathcal{Q}_\eta(\mathfrak{z})$ is uniformly continuous in operator norm topology, we obtain $I_4 \to 0$ as $\mathfrak{z}_2 \to \mathfrak{z}_1$.

$$
\begin{aligned}
I_5 &= \Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} (\mathfrak{z}_2 - \omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_2 - \omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
&\leq \kappa_p \Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} (\mathfrak{z}_2 - \omega)^{-\eta\vartheta-1} m(b) f(P') d\omega \Big\|.
\end{aligned}
$$

Integrating and $\mathfrak{z}_2 \to \mathfrak{z}_1 \Longrightarrow I_5 = 0$. Therefore, Φ is equicontinuous on $\mathcal{I}$.

Step 4: Prove that $\Phi : \mathfrak{B}_P(\mathcal{I}) \to \mathfrak{B}_P(\mathcal{I})$ is a μ_c-contraction operator. Let $B \subset \mathfrak{B}_P$, then by Lemma 2.19 there exists a countable set B_0 such that $B_0 = \{\mathfrak{u}_k\} \subset B$ and $\Phi(B_0)$ becomes a countable set of $\Phi(B)$. Thus, $\mu_c(\Phi(B)) \leq 2\mu_c(\Phi(B_0))$. Since $\Phi(B_0)$ is bounded and equicontinuous, then from Lemma 2.20, we get

$$
\mu_c(\Phi(B_0)) = \max_{\mathfrak{z}\in[0,b]} \big(\mu(\Phi(B_0))\big). \tag{3.5}
$$

Then,

$$
\begin{aligned}
& \leq 2\mu_c(\Phi(B_0)) \\
& = 2 \max_{\mathfrak{z}\in[0,b]} \big(\mu(\Phi(B_0)(\mathfrak{z}))\big) \\
& = 2 \max_{\mathfrak{z}\in[0,b]} \Bigg(\mu\bigg(- \mathcal{S}_{\eta,\zeta}(\mathfrak{z}) N(B_0) + \int_0^{\mathfrak{z}} (\mathfrak{z} - \omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z} - \omega) \mathfrak{F}\big(\omega, B_0(\omega)\big) d\omega\bigg)\Bigg) \\
& \leq 2 \max_{\mathfrak{z}\in[0,b]} \Bigg(\mu\bigg(- \mathcal{S}_{\eta,\zeta}(\mathfrak{z}) N(B_0)\bigg) + \mu\bigg(\int_0^{\mathfrak{z}} (\mathfrak{z} - \omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z} - \omega) \mathfrak{F}\big(\omega, B_0(\omega)\big) d\omega\bigg)\Bigg).
\end{aligned}
$$

Since N is compact, then $\mathcal{S}_{\eta,\zeta}(\mathfrak{z}) N(B_0)$ is relatively compact, we get

$$
\mu_c(\Phi(B)) \leq 2 \max_{\mathfrak{z}\in[0,b]} \Bigg(\mu\bigg(\int_0^{\mathfrak{z}} (\mathfrak{z} - \omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z} - \omega) \mathfrak{F}\big(\omega, B_0(\omega)\big) d\omega\bigg)\Bigg).
$$

From Theorem 2.18, we write

$$
\begin{aligned}
\mu_c(\Phi(B)) & \leq 4 \max_{\mathfrak{z}\in[0,b]} \Bigg(\int_0^{\mathfrak{z}} \mu\bigg((\mathfrak{z} - \omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z} - \omega) \mathfrak{F}\big(\omega, B_0(\omega)\big)\bigg) d\omega\Bigg) \\
& \leq 4 \max_{\mathfrak{z}\in[0,b]} \Bigg(\int_0^{\mathfrak{z}} (\mathfrak{z} - \omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z} - \omega) \mu\bigg(\mathfrak{F}\big(\omega, B_0(\omega)\big)\bigg) d\omega\Bigg) \\
& \leq 4\kappa_p \max_{\mathfrak{z}\in[0,b]} \Bigg(\int_0^{\mathfrak{z}} (\mathfrak{z} - \omega)^{-\eta\vartheta-1} h(\omega) \mu(B) d\omega\Bigg)
\end{aligned}
$$

$$\leq 4\kappa_p \max_{\mathfrak{z}\in[0,b]} \left(\int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\frac{-\eta\vartheta-1}{1-q}} d\omega \right)^{1-q} \left(\int_0^{\mathfrak{z}} \left\| h(\omega) \right\|^q d\omega \right)^{\frac{1}{q}} \mu(B)$$
$$\leq 4\kappa_p K_\eta \|h\|_{L^{\frac{1}{\eta_1}}(\mathcal{I},\mathbb{R}^+)} \times \mu(B),$$

$\mu_c(\Phi(B)) \leq \kappa_1 \mu_c(B)$, where, $K_\eta = \left[\left(\frac{1-\eta_1}{-\eta\vartheta-1}\right) b^{\left(\frac{-\eta\vartheta-1}{1-\eta_1}\right)} \right]$, $\kappa_1 = 4\kappa_p K_\eta \|h\|_{L^{\frac{1}{\eta_1}}(\mathcal{I},\mathbb{R}^+)}$. Therefore, from Definition 2.17 Φ is a μ_c-contraction operator. Hence, by Lemma 2.21, Φ has atleast one fixed point, which is the mild solution.

4. Neutral Systems

Neutral systems have gotten increased attention in the present generation due to important applications in several domains of applied mathematics. Several partial neutral systems that emerge in challenges related to heat flow in materials, wave propagation, visco-elasticity, and other neutral advancements benefit from neutral systems with or without delay. For a very instructive treatment of neutral systems that are involved in differential equations.

Consider the HF neutral evolution equations have the forms,

$$D_{0^+}^{\eta,\zeta}\left[\mathfrak{u}(\mathfrak{z}) - H(\mathfrak{z},\mathfrak{u}_{\mathfrak{z}})\right] = \mathtt{A}\mathfrak{u}(\mathfrak{z}) + \mathfrak{F}(\mathfrak{z},\mathfrak{u}_{\mathfrak{z}}), \qquad \mathfrak{z}\in\mathcal{I}' = (0,b], \tag{4.1}$$
$$I_{0^+}^{(1-\eta)(1-\zeta)}\mathfrak{u}(0) + N(\mathfrak{u}_{\mathfrak{z}}) = \xi \in \mathfrak{B}w, \quad \mathfrak{z}\in(-\infty,0], \tag{4.2}$$

where $H : \mathcal{I}\times\mathfrak{B}w \to Y$ is an appropriate function.

Proposition 4.1 *Suppose* $0 < \eta < 1,\ 0 < q \leq 1$ *and for all* $\mathfrak{u}\in D(\mathtt{A})$*, then there exists a* $\kappa_q > 0$ *such that*

$$\|\mathtt{A}^q \mathcal{Q}_\eta(\mathfrak{z})\mathfrak{u}\| \leq \frac{\eta\kappa_q\Gamma(2-q)}{\mathfrak{z}^{\eta q}\Gamma(1+\eta(1-q))}\|\mathfrak{u}\|,\ 0<\mathfrak{z}<d.$$

Consider the following hypotheses:

(H_4) The function $H : \mathcal{I}\times\mathfrak{B}w \to Y$ is continuous and there exists $q > 0,\ 0 < q < 1$ such that $H \in D(\mathtt{A}^q)$ for any $\mathfrak{u}\in Y,\ \mathfrak{z}\in\mathcal{I},\ \mathtt{A}^q H(\cdot,\mathfrak{u})$ is strongly measurable, then there exists $M_w > 0,\ M_w' > 0$ such that $\gamma_1,\gamma_2\in Y$ and $\mathtt{A}^q H(\mathfrak{z},\cdot)$ satisfy the following:

$$\left\|\mathtt{A}^q H(\mathfrak{z},\gamma_1(\mathfrak{z})) - \mathtt{A}^q H(\mathfrak{z},\gamma_2(\mathfrak{z}))\right\| \leq M_w \mathfrak{z}^{(1+\eta)(1-\eta\vartheta)}\|\gamma_1(\mathfrak{z})-\gamma_2(\mathfrak{z})\|_{\mathfrak{B}w},$$
$$\left\|\mathtt{A}^q H(\mathfrak{z},\mathfrak{u}(\mathfrak{z}))\right\| \leq M_w'\left(1+\mathfrak{z}^{(1+\eta)(1-\eta\vartheta)}\|\mathfrak{u}\|_{\mathfrak{B}w}\right).$$

Take $\|\mathtt{A}^{-q}\| = M_0$.

Definition 4.2 A continuous function $\mathfrak{u} : (-\infty,b] \to Y$ is said to be mild solution of (4.1)-(4.2), that satisfied

$$\begin{aligned}\mathfrak{u}(\mathfrak{z}) &= \mathcal{S}_{\eta,\zeta}(\mathfrak{z})\left[\xi(0) - N(\mathfrak{u}_{\mathfrak{z}}) - H(0,\xi(0))\right] + H(\mathfrak{z},\mathfrak{u}_{\mathfrak{z}}) \\ &\quad + \int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\eta-1}\mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}-\omega)H(\omega,\mathfrak{u}_\omega)d\omega \\ &\quad + \int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{(\eta-1)}\mathcal{Q}_\eta(\mathfrak{z}-\omega)\mathfrak{F}(\omega,\mathfrak{u}_\omega)d\omega.\end{aligned}$$

Theorem 4.3 *Suppose* $(H_1)-(H_4)$ *holds, then the HF system (4.1)-(4.2) has a solution on* $\mathcal{I}$ *provided,* $\xi(0)\in D(\mathtt{A}^{\theta})$ *with* $\theta>1+\vartheta$.
Proof. Consider the operator $\Psi:\mathfrak{B}w'\to\mathfrak{B}w'$, defined

$$\Psi(\mathfrak{u}(\mathfrak{z}))=\begin{cases}\Psi_1(\mathfrak{z}),\ \mathfrak{z}\in(-\infty,0],\\ \mathcal{S}_{\eta,\zeta}(\mathfrak{z})\big[\xi(0)-N(\mathfrak{u}_{\mathfrak{z}})-H(0,\xi(0))\big]+H(\mathfrak{z},\mathfrak{u}_{\mathfrak{z}})\\ +\int_0^{\mathfrak{z}}(\mathfrak{z}-\omega)^{\eta-1}\mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}-\omega)H(\omega,\mathfrak{u}_\omega)d\omega\\ +\int_0^{\mathfrak{z}}(\mathfrak{z}-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}-\omega)\mathfrak{F}(\omega,\mathfrak{u}_\omega)d\omega,\ \mathfrak{z}\in\mathcal{I}.\end{cases}\tag{4.3}$$

For $\Psi_1\in\mathfrak{B}w$, we define $\widehat{\Psi}$ by

$$\widehat{\Psi}(\mathfrak{z})=\begin{cases}\Psi_1(\mathfrak{z}),\quad \mathfrak{z}\in(-\infty,0],\\ \mathcal{S}_{\eta,\zeta}(\mathfrak{z})\xi(0),\quad \mathfrak{z}\in\mathcal{I},\end{cases}$$

then $\widehat{\Psi}\in\mathfrak{B}w'$. Let $\mathfrak{u}(\mathfrak{z})=\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v(\mathfrak{z})+\widehat{\Psi}(\mathfrak{z})],\ -\infty<\mathfrak{z}\le b$. It can be easily shown that $\mathfrak{u}$ satisfies from (4) iff v satisfies v_0 and

$$\begin{aligned}v(\mathfrak{z})=&-\mathcal{S}_{\eta,\zeta}(\mathfrak{z})\big[N\big(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}}+\widehat{\Psi}_{\mathfrak{z}}]\big)+H(0,\xi(0))\big]\\ &+H\big(\mathfrak{z},\big(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}}+\widehat{\Psi}_{\mathfrak{z}}]\big)\big)\\ &+\int_0^{\mathfrak{z}}(\mathfrak{z}-\omega)^{\eta-1}\mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}-\omega)H\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega\\ &+\int_0^{\mathfrak{z}}(\mathfrak{z}-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}-\omega)\mathfrak{F}\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega.\end{aligned}$$

Let $\mathfrak{B}w''=\{v\in\mathfrak{B}w':v_0\in\mathfrak{B}w\}$. For any $v\in\mathfrak{B}w'$,

$$\begin{aligned}\|v\|_b=&\|v_0\|_{\mathfrak{B}w}+\sup\{\|v(\omega)\|:0\le\omega\le b\}\\ =&\sup\{\|v(\omega)\|:0\le\omega\le b\}.\end{aligned}$$

Hence, $(\mathfrak{B}w'',\|\cdot\|)$ is a Banach space. For $P>0$, choose $\mathfrak{B}_P=\{v\in\mathfrak{B}w'':\|v\|_b\le P\}$, then $\mathfrak{B}_P\subset\mathfrak{B}w''$ is uniformly bounded, and for $v\in\mathfrak{B}_P$, from Lemma 2.6,

$$\begin{aligned}\|v_{\mathfrak{z}}+\widehat{\Psi}_{\mathfrak{z}}\|_{\mathfrak{B}w}&\le\|v_{\mathfrak{z}}\|_{\mathfrak{B}w}+\|\widehat{\Psi}\|_{\mathfrak{B}w}\\ &\le l\left(P+\frac{\Gamma(-\eta\vartheta)}{\Gamma(\zeta(1-\eta)-\eta\vartheta)}\kappa_p\mathfrak{z}^{\zeta(1-\eta)-\eta\vartheta-1}\right)+\|\Psi_1\|_{\mathfrak{B}w}\\ &=P'.\end{aligned}$$

Let us introduce an operator $\Phi:\mathfrak{B}w''\to\mathfrak{B}w''$, defined by

$$\Phi v(\mathfrak{z})=\begin{cases}0,\ \mathfrak{z}\in(-\infty,0],\\ -\mathcal{S}_{\eta,\zeta}(\mathfrak{z})\big[N\big(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}}+\widehat{\Psi}_{\mathfrak{z}}]\big)+H(0,\xi(0))\big]\\ \quad+H\big(\mathfrak{z},\big(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}}+\widehat{\Psi}_{\mathfrak{z}}]\big)\big)\\ \quad+\int_0^{\mathfrak{z}}(\mathfrak{z}-\omega)^{\eta-1}\mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}-\omega)H\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega\\ \quad+\int_0^{\mathfrak{z}}(\mathfrak{z}-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}-\omega)\mathfrak{F}\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega,\qquad \mathfrak{z}\in\mathcal{I}.\end{cases}$$

Next to show that Φ has a fixed point.

Step 1: We have to prove there exists a positive value P such that $\Phi(\mathfrak{B}_P(\mathcal{I})) \subseteq \mathfrak{B}_P(\mathcal{I})$. Assume the statement is false i.e., there for each $P > 0$, there exits $v^P \in \mathfrak{B}_P(\mathcal{I})$, but $\Phi(v^P)$ not in $\mathfrak{B}_P(\mathcal{I})$, that is,

$$\begin{aligned}
\|v^P\| &\leq P < \sup \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)} \|(\Phi v^P(\mathfrak{z}))\| \\
&\leq b^{(1+\eta\vartheta)(1-\zeta)} \Bigg[\| - \mathcal{S}_{\eta,\zeta}(\mathfrak{z}) \big[N\big(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v^P_{\mathfrak{z}} + \widehat{\Psi}_{\mathfrak{z}}]\big) + H(0,\xi(0))\big] \| \\
&\quad + \big\|H\big(\mathfrak{z}, \mathfrak{z}^{(1+\eta)(1-\eta\vartheta)}[v^P_{\mathfrak{z}} + \widehat{\Psi}_{\mathfrak{z}}]\big)\big\| \\
&\quad + \int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\eta-1} \Big\| \mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}-\omega) H\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v^P_\omega + \widehat{\Psi}_\omega]\big) \Big\| d\omega \\
&\quad + \int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\eta-1} \Big\| \mathcal{Q}_\eta(\mathfrak{z}-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v^P_\omega + \widehat{\Psi}_\omega]\big) \Big\| d\omega \Bigg] \\
&\leq b^{(1+\eta\vartheta)(1-\zeta)} \Bigg[\|\mathcal{S}_{\eta,\zeta}(\mathfrak{z})\| \big[L_1 \|v^P_{\mathfrak{z}} + \widehat{\Psi}_{\mathfrak{z}}\| + \|N(0)\| + M'_w \|\xi\|\big] \\
&\quad + \int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{(\eta-1)} \Big\| \mathtt{A}^{1-q} \mathcal{Q}_\eta(\mathfrak{z}-\omega) \Big\| \Big\| \mathtt{A}^q H\big(\omega, \omega^{(1+\eta)(1-\eta\vartheta)}[v^P_\omega + \widehat{\Psi}_\omega]\big) \Big\| d\omega \\
&\quad + M_0 M_w (1 + b^{(1+\eta)(1-\eta\vartheta)}) P' + \int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{(\eta-1)} \|\mathcal{Q}_\eta(\mathfrak{z}-\omega)\| m(b) f(P') d\omega \Bigg] \\
&\leq b^{(1+\eta\vartheta)(1-\zeta)} \Bigg[\frac{\Gamma(-\eta\vartheta)}{\Gamma(\zeta(1-\eta)-\eta\vartheta)} \kappa_p b^{\zeta(1-\eta)-\eta\vartheta-1} \big[L_1 P' + \|N(0)\| + M'_w \|\xi\|\big] \\
&\quad + M_0 M_w \big(1 + b^{(1+\eta\vartheta)(1-\zeta)}\big) P' + \frac{M'_w \kappa_{1-q} \Gamma(1+q)}{q\Gamma(1+\eta q)} \big(1 + b^{(1+\eta\vartheta)(1-\zeta)} P'\big) b^{\eta q} \\
&\quad + \frac{b^{-\eta\vartheta}}{\eta\vartheta} \kappa_p m(b) f(P') \Bigg] \\
&\leq \kappa_p b^{(1+\eta\vartheta)(1-\zeta)} \big(M^* + M^{**}\big),
\end{aligned}$$

where

$$M^{**} = \kappa_p^{-1} M_0 M_w \big(1 + b^{(1+\eta\vartheta)(1-\zeta)}\big) P' + \frac{M_0 \kappa_{1-q} \Gamma(1+q)}{q\Gamma(1+\eta q)} \big(1 + b^{(1+\eta\vartheta)(1-\zeta)}\big) b^{\eta q}.$$

The above inequality is divided by P and applying the limit as $P \to \infty$, we obtain $1 \leq 0$, which is the contradiction. Therefore, $\Phi(\mathfrak{B}_P(\mathcal{I})) \subseteq \mathfrak{B}_P(\mathcal{I})$.

Step 2: The operator Φ is continuous on $\mathfrak{B}_P(\mathcal{I})$. For Φ maps, $\mathfrak{B}_P(\mathcal{I})$ into $\mathfrak{B}_P(\mathcal{I})$. For any $v^k,\ v \in \mathfrak{B}_P(\mathcal{I}),\ k = 0, 1, 2, ...$ such that $\lim_{k\to\infty} v^k = v$, then we have

$$\lim_{k\to\infty} v^k(\mathfrak{z}) = v(\mathfrak{z}) \text{ and } \lim_{k\to\infty} \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)} v^k(\mathfrak{z}) = \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)} v(\mathfrak{z}).$$

By (H_2),

$$\begin{aligned}
\mathfrak{F}(\mathfrak{z}, \mathfrak{u}_k(\mathfrak{z})) = \mathfrak{F}\big(\mathfrak{z}, \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta))}[v^k(\mathfrak{z}) + \widehat{\Psi}(\mathfrak{z})]\big) &\to \mathfrak{F}\big(\mathfrak{z}, \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta))}[v(\mathfrak{z}) + \widehat{\Psi}(\mathfrak{z})]\big) \\
&= \mathfrak{F}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z})) \text{ as } k \to \infty.
\end{aligned}$$

Take

$$F_k(\omega) = \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta))}[v^k_\omega + \widehat{\Psi}_\omega]\big) \text{ and } F(\omega) = \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta))}[v_\omega + \widehat{\Psi}_\omega]\big).$$

Then, from hypotheses (H_2) and Lebesgue's dominated convergence theorem, we can obtain

$$\int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{(\eta-1)} \|F_k(\omega) - F(\omega)\| d\omega \to 0 \ as \ k \to \infty, \mathfrak{z} \in I. \tag{4.4}$$

Take $\mathcal{N}_k(\mathfrak{z}) = N\big(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}}^k + \widehat{\Psi}_{\mathfrak{z}}]\big)$ and $\mathcal{N}(\mathfrak{z}) = N\big(\mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}} + \widehat{\Psi}_{\mathfrak{z}}]\big)$, from (H_4), we have

$$\|\mathcal{N}_k(\mathfrak{z}) - \mathcal{N}(\mathfrak{z})\| \to 0 \ as \ k \to \infty. \tag{4.5}$$

Next,

$$H_k(\mathfrak{z}, \mathfrak{u}_k(\mathfrak{z})) = H\big(\mathfrak{z}, \mathfrak{z}^{(1+\eta)(1-\eta\vartheta)}[v_{\mathfrak{z}}^k + \widehat{\Psi}_{\mathfrak{z}}]\big), \ H(\mathfrak{z}, \mathfrak{u}(\mathfrak{z})) = H\big(\mathfrak{z}, \mathfrak{z}^{(1+\eta)(1-\eta\vartheta)}[v_{\mathfrak{z}} + \widehat{\Psi}_{\mathfrak{z}}]\big).$$

From hypotheses (H_5), we have obtained

$$H(\mathfrak{z}, \mathfrak{u}_k(\mathfrak{z})) \to H(\mathfrak{z}, \mathfrak{u}(\mathfrak{z})) \ as \ k \to \infty. \tag{4.6}$$

Now,

$$\begin{aligned}\big\|\Phi v^k - \Phi v\big\|_b \leq& \frac{\Gamma(-\eta\vartheta)}{\Gamma(\zeta(1-\eta)-\eta\vartheta)} \kappa_p b^{\zeta(1-\eta)-\eta\vartheta-1} \big\|\mathcal{N}_k(\mathfrak{z}) - \mathcal{N}(\mathfrak{z})\big\| \\ &+ \|H_k(\mathfrak{z}, \mathfrak{u}_k(\mathfrak{z})) - H(\mathfrak{z}, \mathfrak{u}_{\mathfrak{z}})\| + \kappa_p \frac{b^{-\eta\vartheta}}{\eta\vartheta} \big\|F_k(\omega) - F(\omega)\big\| ds. \end{aligned} \tag{4.7}$$

Using (4.4), (4.5) and (4.6) we obtain $\big\|\Phi v^k - \Phi v\big\|_b \to 0 \ as \ k \to \infty$. Therefore, Φ is continuous on $\mathfrak{B}_P$.

Step 3: Next, we have to prove Φ is equicontinuous. For $\mathfrak{u} \in \mathfrak{B}_P(\mathcal{I})$, and $0 \leq \mathfrak{z}_1 < \mathfrak{z}_2 \leq b$, we have

$$\begin{aligned}\big\|\Phi\mathfrak{u}(\mathfrak{z}_2) - \Phi\mathfrak{u}(\mathfrak{z}_1)\big\| =& \bigg\|\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\bigg(- \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2)\big[N\big(\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_2} + \widehat{\Psi}_{\mathfrak{z}_2}]\big) \\ &+ H(0,\xi(0))\big] + H\big(\mathfrak{z}_2, \big(\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_2} + \widehat{\Psi}_{\mathfrak{z}_2}]\big)\big) \\ &+ \int_0^{\mathfrak{z}_2} (\mathfrak{z}-\omega)^{\eta-1} \mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega) H\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \\ &+ \int_0^{\mathfrak{z}_2} (\mathfrak{z}_2-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_2-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega\bigg) \\ &- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}\bigg(- \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_1)\big[N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big) + H(0,\xi(0))\big] \\ &+ H\big(\mathfrak{z}_1, \big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big)\big) \\ &+ \int_0^{\mathfrak{z}_1} (\mathfrak{z}-\omega)^{\eta-1} \mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}_1-\omega) H\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \\ &+ \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_1-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega\bigg)\bigg\| \\ \leq& \bigg\|\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2)\big[N\big(\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_2} + \widehat{\Psi}_{\mathfrak{z}_2}]\big) + H(0,\xi(0))\big] \end{aligned}$$

$$
\begin{aligned}
&- \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2)\big[N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1}+\widehat{\Psi}_{\mathfrak{z}_1}]\big)+H(0,\xi(0))\big]\Big\| \\
&+ \Big\|\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2)\big[N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1}+\widehat{\Psi}_{\mathfrak{z}_1}]\big)+H(0,\xi(0))\big]
\end{aligned}
$$

$$
\begin{aligned}
&- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}\mathcal{S}_{\eta,\zeta}(\mathfrak{z}_1)\big[N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1}+\widehat{\Psi}_{\mathfrak{z}_1}]\big)+H(0,\xi(0))\big]\Big\| \\
&+ \Big\|\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}H\big(\mathfrak{z}_2,\big(\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_2}+\widehat{\Psi}_{\mathfrak{z}_2}]\big)\big) \\
&- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}H\big(\mathfrak{z}_1,\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1}+\widehat{\Psi}_{\mathfrak{z}_1}]\big)\big)\Big\| \\
&+ \Big\|\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\int_0^{\mathfrak{z}_1}(\mathfrak{z}_2-\omega)^{\eta-1}\mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega)H\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega \\
&- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}\int_0^{\mathfrak{z}_1}(\mathfrak{z}_1-\omega)^{\eta-1}\mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega)H\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega\Big\| \\
&+ \Big\|\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}\int_0^{\mathfrak{z}_1}(\mathfrak{z}_1-\omega)^{\eta-1}\mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega)H\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega \\
&- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}\int_0^{\mathfrak{z}_1}(\mathfrak{z}_1-\omega)^{\eta-1}\mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}_1-\omega)H\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega\Big\| \\
&+ \Big\|\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}(\mathfrak{z}_2-\omega)^{\eta-1}\mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega)H\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega\Big\| \\
&+ \Big\|\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\int_0^{\mathfrak{z}_1}(\mathfrak{z}_2-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega)\mathfrak{F}\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega \\
&- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}\int_0^{\mathfrak{z}_1}(\mathfrak{z}_1-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega)\mathfrak{F}\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega\Big\| \\
&+ \Big\|\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}\int_0^{\mathfrak{z}_1}(\mathfrak{z}_1-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega)\mathfrak{F}\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega \\
&- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}\int_0^{\mathfrak{z}_1}(\mathfrak{z}_1-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}_1-\omega)\mathfrak{F}\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega\Big\| \\
&+ \Big\|\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}(\mathfrak{z}_2-\omega)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega)\mathfrak{F}\big(\omega,\omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega+\widehat{\Psi}_\omega]\big)d\omega\Big\| \\
&\le \sum_{i=1}^{9} I_i.
\end{aligned}
$$

Now, we consider the following terms:

$$
\begin{aligned}
I_1 = \Big\|&\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2)\big[N\big(\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_2}+\widehat{\Psi}_{\mathfrak{z}_2}]\big)+H(0,\xi(0))\big] \\
&- \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}\mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2)\big[N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1}+\widehat{\Psi}_{\mathfrak{z}_1}]\big)+H(0,\xi(0))\big]\Big\|
\end{aligned}
$$

$$\leq \Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2) \Big(N\big(\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_2} + \widehat{\Psi}_{\mathfrak{z}_2}]\big) - N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big)\Big)\Big\|.$$

From hypotheses (H_3) and obtained derivation, we get I_1 tends to zero as $\mathfrak{z}_2 \to \mathfrak{z}_1$.

$$\begin{aligned} I_2 = &\Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2) \big[N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big) + H(0,\xi(0))\big] \\ &- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_1) \big[N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big) + H(0,\xi(0))\big] \Big\| \\ \leq &\Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_2) - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \mathcal{S}_{\eta,\zeta}(\mathfrak{z}_1) \Big\| \\ &\times \Big\| N\big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big) H(0,\xi(0)) \Big\|. \end{aligned}$$

By the strong continuity of $\mathcal{S}_{\eta,\zeta}(\mathfrak{z})$ and (H_3), we get $I_2 \to 0$ *as* $\mathfrak{z}_2 \to \mathfrak{z}_1$.

$$\begin{aligned} I_3 = &\big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} H\big(\mathfrak{z}_2, \big(\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_2} + \widehat{\Psi}_{\mathfrak{z}_2}]\big)\big) \\ &- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} H\big(\mathfrak{z}_1, \big(\mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)}[v_{\mathfrak{z}_1} + \widehat{\Psi}_{\mathfrak{z}_1}]\big)\big) \big\| \\ \leq &M_0 M'_w (1+P') \big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \big\|. \end{aligned}$$

From hypotheses (H_4), we obtain $I_3 \to 0$ *as* $\mathfrak{z}_2 \to \mathfrak{z}_1$.

$$\begin{aligned} I_4 = &\Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_2 - \omega)^{\eta-1} \mathtt{A} \mathcal{Q}_\eta(\mathfrak{z}_2 - \omega) H\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \\ &- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1 - \omega)^{\eta-1} \mathtt{A} \mathcal{Q}_\eta(\mathfrak{z}_2 - \omega) H\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\ \leq &\Big\| \Big(\int_0^{\mathfrak{z}_1} \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_2 - \omega)^{\eta-1} - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_1 - \omega)^{\eta-1} \Big) \\ &\times \mathtt{A} \mathcal{Q}_\eta(\mathfrak{z}_2 - \omega) H\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\ \leq &\Big\| \Big(\int_0^{\mathfrak{z}_1} \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_2 - \omega)^{\eta-1} - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_1 - \omega)^{\eta-1} \Big) \Big\| \\ &\times \Big\| \mathtt{A}^{1-q} \mathcal{Q}_\eta(\mathfrak{z}_2 - \omega) \Big\| \Big\| \mathtt{A}^q H\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) \Big\| d\omega \\ \leq &\frac{M'_w \kappa_{1-q} \eta \Gamma(1+q)}{q\Gamma(1+\eta q)} \big(1 + b^{(1+\eta\vartheta)(1-\zeta)} P'\big) \\ &\times \Big\| \Big(\int_0^{\mathfrak{z}_1} (\mathfrak{z}_2 - \omega)^{\eta(q-1)} \big[\mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_2 - \omega)^{\eta-1} - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_1 - \omega)^{\eta-1} \big] \Big) d\omega \Big\|. \end{aligned}$$

Implies, $I_4 \to 0 \ as \ \mathfrak{z}_2 \to \mathfrak{z}_1$.

$$\begin{aligned}
I_5 = \Big\| & \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega) H\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \\
& - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}_1-\omega) H\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
\leq & \Big\| \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \big[\mathcal{Q}_\eta(\mathfrak{z}_2-\omega) - \mathcal{Q}_\eta(\mathfrak{z}_1-\omega)\big] \\
& \times \mathtt{A} H\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
\leq & M_0 M'_w (1 + b^{(1+\eta\vartheta)(1-\zeta)} P') \\
& \times \Big\| \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \big[\mathcal{Q}_\eta(\mathfrak{z}_2-\omega) - \mathcal{Q}_\eta(\mathfrak{z}_1-\omega)\big] d\omega \Big\|.
\end{aligned}$$

Since $\mathcal{Q}_\eta(\mathfrak{z})$ is uniformly continuous in operator norm topology, $I_5 \to 0 \ as \ \mathfrak{z}_2 \to \mathfrak{z}_1$.

$$\begin{aligned}
I_6 = & \Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} (\mathfrak{z}_2-\omega)^{\eta-1} \mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}_2-\omega) H\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
\leq & b^{(1+\eta\vartheta)(1-\zeta)} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} (\mathfrak{z}_2-\omega)^{\eta-1} \Big\| \mathtt{A}^{1-q} \mathcal{Q}_\eta(\mathfrak{z}_2-\omega) \Big\| \\
& \times \Big\| \mathtt{A}^q H\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) \Big\| d\omega \\
\leq & \frac{M'_w \kappa_{1-q} \eta \Gamma(1+q)}{q\Gamma(1+\eta q)} \big(1 + b^{(1+\eta\vartheta)(1-\zeta)} P'\big) b^{(1+\eta\vartheta)(1-\zeta)} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} (\mathfrak{z}_2-\omega)^{\eta q-1} d\omega.
\end{aligned}$$

Integrating and $\mathfrak{z}_2 \to \mathfrak{z}_1 \Longrightarrow I_6 = 0$.

$$\begin{aligned}
I_7 = & \Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_2-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_2-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \\
& - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_2-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
\leq & \Big\| \Big(\int_0^{\mathfrak{z}_1} \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_2-\omega)^{\eta-1} - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_1-\omega)^{\eta-1} \Big) \\
& \times \mathcal{Q}_\eta(\mathfrak{z}_2-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
\leq & \kappa_p \Big\| \Big(\int_0^{\mathfrak{z}_1} \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_2-\omega)^{\eta-1} - \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} (\mathfrak{z}_1-\omega)^{\eta-1} \Big) \\
& \times (\mathfrak{z}_2-\omega)^{-\eta(1+\vartheta)} m(b) f(P') d\omega \Big\|.
\end{aligned}$$

Implies $I_7 \to 0$ as $\mathfrak{z}_2 \to \mathfrak{z}_1$.

$$I_8 = \Big\| \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_2-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega$$

$$
\begin{aligned}
&- \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_1-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
\leq & \Big\| \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \\
& \times \big[\mathcal{Q}_\eta(\mathfrak{z}_2-\omega) - \mathcal{Q}_\eta(\mathfrak{z}_1-\omega)\big] \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
\leq & \Big\| \mathfrak{z}_1^{(1+\eta\vartheta)(1-\zeta)} \int_0^{\mathfrak{z}_1} (\mathfrak{z}_1-\omega)^{\eta-1} \big[\mathcal{Q}_\eta(\mathfrak{z}_2-\omega) - \mathcal{Q}_\eta(\mathfrak{z}_1-\omega)\big] m(b) f(P') d\omega \Big\|.
\end{aligned}
$$

Since $\mathcal{Q}_\eta(\mathfrak{z})$ is uniformly continuous in operator norm topology, we obtain $I_8 \to 0$ as $\mathfrak{z}_2 \to \mathfrak{z}_1$.

$$
\begin{aligned}
I_9 &= \Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} (\mathfrak{z}_2-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}_2-\omega) \mathfrak{F}\big(\omega, \omega^{(1+\eta\vartheta)(1-\zeta)}[v_\omega + \widehat{\Psi}_\omega]\big) d\omega \Big\| \\
&\leq \kappa_p \Big\| \mathfrak{z}_2^{(1+\eta\vartheta)(1-\zeta)} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} (\mathfrak{z}_2-\omega)^{-\eta\vartheta-1} m(b) f(P') d\omega \Big\|.
\end{aligned}
$$

Integrating and $\mathfrak{z}_2 \to \mathfrak{z}_1 \Longrightarrow I_9 = 0$. Therefore, Φ is equicontinuous on $\mathcal{I}$.

Step 4: Prove that $\Phi : \mathfrak{B}_P(\mathcal{I}) \to \mathfrak{B}_P(\mathcal{I})$ is a μ_c-*contraction* operator. Let $B \subset \mathfrak{B}_P$, then by Lemma 2.19 there exists a countable set B_0 such that

$$B_0 = \{\mathfrak{u}_k = \mathfrak{z}^{(1+\eta\vartheta)(1-\zeta)}[v_k(\mathfrak{z}) + \widehat{\Psi}_k(\mathfrak{z})]\} \subset B$$

and $\Phi(B_0)$ becomes a countable set of $\Phi(B)$. Thus, $\mu_c(\Phi(B)) \leq 2\mu_c(\Phi(B_0))$. Since $\Phi(B_0)$ is bounded and equicontinuous, then from Lemma 2.20, we get

$$\mu_c(\Phi(B_0)) = \max_{\mathfrak{z}\in[0,b]} \big(\mu(\Phi(B_0))\big).$$

Then, it become

$$
\begin{aligned}
\mu_c(\Phi(B)) &\leq 2\mu_c(\Phi(B_0)) \\
&= 2 \max_{\mathfrak{z}\in[0,b]} \big(\mu(\Phi(B_0)(\mathfrak{z}))\big) \\
&= 2 \max_{\mathfrak{z}\in[0,b]} \Bigg(\mu\bigg(- \mathcal{S}_{\eta,\zeta}(\mathfrak{z})\big[N(B_0) + H(0,\xi(0))\big] + H(\mathfrak{z}, B_0(\mathfrak{z})) \\
&\quad + \int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\eta-1} \mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}-\omega) H\big(\omega, B_0(\omega)\big) d\omega \\
&\quad + \int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}-\omega) \mathfrak{F}\big(\omega, B_0(\omega)\big) d\omega \bigg)\Bigg) \\
&\leq 2 \max_{\mathfrak{z}\in[0,b]} \Bigg[\mu\bigg(- \mathcal{S}_{\eta,\zeta}(\mathfrak{z})\big[N(B_0) + H(0,\xi(0))\bigg) + \mu\big(H(\mathfrak{z}, B_0(\mathfrak{z}))\big) \\
&\quad + \mu\bigg(\int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\eta-1} \mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}-\omega) H\big(\omega, B_0(\omega)\big) d\omega \bigg)\bigg) \\
&\quad + \mu\bigg(\int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}-\omega) \mathfrak{F}\big(\omega, B_0(\omega)\big) d\omega \bigg)\Bigg] \leq 2 \sum_{j=1}^{4} J_j.
\end{aligned}
$$

Consider the following:

$$J_1 = \max_{\mathfrak{z}\in[0,b]} \mu\bigg(- \mathcal{S}_{\eta,\zeta}(\mathfrak{z})\big[N(B_0) + H(0, \xi(0))\big]\bigg).$$

Since N is compact, then $\mathcal{S}_{\eta,\zeta}(\mathfrak{z})\big[N(B_0) + H(0, \xi(0))\big]$ is relatively compact, we get J_1 becomes zero. Next consider,

$$J_2 = \max_{\mathfrak{z}\in[0,b]} \mu\big(H(\mathfrak{z}, B_0(\mathfrak{z}))\big)$$

$$J_3 = \max_{\mathfrak{z}\in[0,b]} \mu\bigg(\int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\eta-1} \mathtt{A}\mathcal{Q}_\eta(\mathfrak{z}-\omega) H\big(\omega, B_0(\omega)\big) d\omega \bigg)\bigg).$$

From hypotheses (H_4) and properties of function H *and* $\mathtt{A}\mathcal{Q}_\eta$, we obtain, that terms are relatively compact. So J_2 and J_3 become zero. Therefore,

$$\mu_c(\Phi(B)) \le 2 \max_{\mathfrak{z}\in[0,b]} \bigg(\mu\bigg(\int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}-\omega) \mathfrak{F}\big(\omega, B_0(\omega)\big) d\omega \bigg)\bigg).$$

From Theorem 2.18, we write

$$\begin{aligned}
\mu_c(\Phi(B)) &\le 4 \max_{\mathfrak{z}\in[0,b]} \bigg(\int_0^{\mathfrak{z}} \mu\bigg((\mathfrak{z}-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}-\omega) \mathfrak{F}\big(\omega, B_0(\omega)\big)\bigg) d\omega \bigg) \\
&\le 4 \max_{\mathfrak{z}\in[0,b]} \bigg(\int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{z}-\omega) \mu\bigg(\mathfrak{F}\big(\omega, B_0(\omega)\big)\bigg) d\omega \bigg) \\
&\le 4\kappa_p \max_{\mathfrak{z}\in[0,b]} \bigg(\int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{-\eta\vartheta-1} h(\omega) \mu(B) d\omega \bigg) \\
&\le 4\kappa_p \max_{\mathfrak{z}\in[0,b]} \bigg(\int_0^{\mathfrak{z}} (\mathfrak{z}-\omega)^{\frac{-\eta\vartheta-1}{1-q}} d\omega \bigg)^{1-q} \bigg(\int_0^{\mathfrak{z}} \big\|h(\omega)\big\|^q d\omega \bigg)^{\frac{1}{q}} \mu(B) \\
&\le 4\kappa_p K_\eta \|h\|_{L^{\frac{1}{\eta_1}}(\mathcal{I},\mathbb{R}^+)} \times \mu(B)
\end{aligned}$$

i.e., $\mu_c(\Phi(B)) \le \kappa_1 \mu_c(B)$,

where, $K_\eta = \bigg[\big(\frac{1-\eta_1}{-\eta\vartheta-1}\big) b^{\left(\frac{-\eta\vartheta-1}{1-\eta_1}\right)}\bigg]$, $\kappa_1 = 4\kappa_p K_\eta \|h\|_{L^{\frac{1}{\eta_1}}(\mathcal{I},\mathbb{R}^+)}$.

Therefore, from Definition 2.20 Φ is a μ_c-*contraction* operator. Hence, by Lemma 2.21 Φ has atleast one fixed point, which is the mild solution.

5. Example

Consider the HF neutral differential system with non-local condition of the form

$$\begin{cases}
D_{0^+}^{\frac{2}{3},\zeta}\big[\mathfrak{u}(\mathfrak{z},\tau) + \int_0^\pi \rho(\alpha,\tau)\mathfrak{u}(\mathfrak{z},\tau)d\alpha\big] = \mathfrak{u}_{\tau\tau}(\mathfrak{z},\tau) + \chi\bigg(\mathfrak{z}, \int_\infty^{\mathfrak{z}} \chi_1(\omega-\mathfrak{z})\mathfrak{u}(\mathfrak{z},\tau)d\omega\bigg), \\
\mathfrak{u}(\mathfrak{z},0) = \mathfrak{u}(\mathfrak{z},\pi) = 0, \ \mathfrak{z}\in\mathcal{I}, \\
I_{0^+}^{(1-\frac{2}{3})(1-\zeta)}\mathfrak{u}(0,\tau) + \int_0^\pi \mathcal{N}(\alpha,\tau)\mathfrak{u}(\mathfrak{z},\tau)d\alpha = \mathfrak{u}(0,\tau), \ \tau\in[0,\pi], \ \mathfrak{z}\in(-\infty,0),
\end{cases} \tag{5.1}$$

where $D_{0^+}^{\frac{2}{3},\zeta}$ denoted the HFD of order $\eta = 2/3$, type ζ and $\chi,\ \chi_1,\ \rho$ and $\mathcal{N}$ are the required functions. To change this system into an abstract structure, let $Y = L^2[0,\pi]$ and $\mathtt{A} : D(\mathtt{A}) \subset Y \to Y$ is defined as $\mathtt{A}x = x'$ with

$$D(A) = \left\{x \in Y : x, x' \textit{ are absolutely continuous},\ x'' \in Y,\ x(0) = x(\pi) = 0\right\}$$

and

$$\mathtt{A}x = \sum_{k=1}^{\infty} k^2 \langle x, \varrho_k \rangle \varrho_k,\ \ \varrho \in D(\mathtt{A}),$$

where $\varrho_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx),\ k \in \mathbb{N}$ is the orthogonal set of eigen vectors of $\mathtt{A}$.
We know that $\mathtt{A}$ is the almost sectorial operator of the analytic semigroup $\{T(\mathfrak{z}), t \geq 0\}$ in Y, $T(\mathfrak{z})$ is noncompact semigroup on Y with $\mu(T(\mathfrak{z})B) \leq \mu(B)$, where μ denoted the Hausdorff measure of noncompactness and there exists a constant $\mathcal{K}_1 \geq 1$, satisfy $\sup\limits_{\mathfrak{z}\in\mathcal{I}} \|T(\mathfrak{z})\| \leq \mathcal{K}_1$.
Define, $\mathfrak{F} : \mathcal{I} \times \mathfrak{B}w \to Y,\ H : \mathcal{I} \times \mathfrak{B}w \to Y \textit{ and } N : \mathfrak{B}w \to Y$ are the appropriate functions, which satisfy the hypotheses $(H_1) - (H_4)$,

$$\begin{aligned}
\mathfrak{F}(\mathfrak{z}, \mathfrak{u}_{\mathfrak{z}})(\tau), &= \chi\left(\mathfrak{z}, \int_{\infty}^{\mathfrak{z}} \chi_1(\omega - \mathfrak{z})\mathfrak{u}(\mathfrak{z}, \tau)d\omega\right), \\
H(\mathfrak{z}, \mathfrak{u}_{\mathfrak{z}})(\tau) &= \int_0^{\pi} \rho(\alpha, \tau)\mathfrak{u}(\mathfrak{z}, \tau)d\alpha \\
N(\mathfrak{u}_{\mathfrak{z}})(\tau) &= \int_0^{\pi} \mathcal{N}(\alpha, \tau)\mathfrak{u}(\mathfrak{z}, \tau)d\alpha.
\end{aligned}$$

We also provide some acceptable conditions for the above-mentioned functions in order to validate all of the Theorem 4.3 assumptions, and we conclude that the Hilfer fractional system (4.1)-(4.2) has a mild solution.

6. Conclusion

We concentrated on the existence of mild solution of the Hilfer fractional neutral differential equations with infinite delay via almost sectorial operator in this study. The major conclusions are established by applying the results and ideas belonging to almost sectorial operators, fractional calculus, measure of noncompactness and fixed point method. Finally, to explain the principle, we offered an example. In the future, we will study the exact controllability of the Hilfer fractional differential system with infinite delay via almost sectorial operators using fixed point method.

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