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ULAM STABILITY OF A GENERAL FUNCTIONAL EQUATION IN FUZZY MODULAR SPACES

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Abstract. In this work we improve a fixed point result in fuzzy modular spaces. As application, we prove the Ulam stability of a general functional equation in β-homogeneous fuzzy modular spaces, generalizing some similar stability results published so far for fuzzy modular spaces.
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1. Introduction

In 1922, Banach [3] put the main stone of the fixed point theorem in metric spaces. Since then, many authors have studied this topic and the Banach contraction theorem has been extended to more general spaces (see, for example, [8, 9]). The study of modular spaces as generalizations of normed spaces was pioneered by Nakano [13]. Subsequently, Nourouzi [14] introduced the notion of probabilistic modular spaces, contributing to the development of modular space theory. In 2013, Shen and Chen [17], inspired by the structure of probabilistic modular spaces and the formulation of fuzzy metric spaces by George and Veeramani [7], extended the classical theory by incorporating fuzzy concepts, thereby giving birth to the notion of fuzzy modular spaces. In the same year, Wongkum et al. [21] further advanced this area by establishing fuzzy analogue of the Banach fixed point theorem within the context of fuzzy modular spaces.

On the other hand, it is well known that functional equations are an essential tool in mathematics, as they describe functions through relations between their values at different points. Studying their solutions helps uncover important properties of functions and often leads to new results in areas such as analysis, algebra and number theory. This makes the investigation of functional equations a valuable direction in both pure and applied mathematics. But in many cases, finding the exact solution of an equation is very difficult, and only approximate solutions are available. This raises important questions about how close these approxiamte solutions are to the exact ones, and whether the exact solution can be recovered from them. The theory of Ulam stability deals with such problems. Actually, when we talk about Ulam stability theory, we want to know if there exists an exact solution to an equation which we already know one of its approximate solutions (in some sense). This theory was initiated in 1940 by S. M. Ulam [20], who asked about the stability of the group homomorphism equation. Since then, the theory has been developed and extended by many researchers to cover a wide range of functional equations, e.g., [1, 2, 4, 5, 6, 12, 15, 19, 18, 22, 24]..

In 2016, Wongkum and Kumam [23], employed Khamsi's fixed point technique [11] to establish the generalized Ulam-Hyers-Rassias stability of the following sextic functional equation

$$\begin{aligned} f(ax_1+bx_2,y,z)+f(ax_1-bx_2,y,z) &= 2af(x_1,y,z), \\ f(x,ay_1+by_2,z)+f(x,ay_1-by_2,z) &= 2a^2f(x,y_1,z)+2b^2f(x,y_2,z), \\ f(x,y,az_1+bz_2)+f(x,y,az_1-bz_2) &= ab^2\left(f(x,y,z_1+z_2)+f(x,y,z_1-z_2)\right) \\ &\quad +2a(a^2-b^2)f(x,y,z_1), \end{aligned}$$

where $a,b \in \mathbb{Z}\setminus\{0\}$ with $a \neq \pm 1, \pm b$, and the unknown function f is defined from linear space into fuzzy modular space. The assumptions of lower semi-continuity and β-homogeneity were identified as necessary conditions for their approach. It is worth noting, however, that while the introduction of [23] states an intention to apply the fixed point method of Wongkum et al. [21], the actual proof relies on Khamsi's fixed point theorem.

Next, in 2021, Ramdoss et al. [16] proved the Hyers-Ulam stability of the generalized n-variable mixed-type functional equation

$$\begin{aligned} &\sum_{i=1,j=i+1} f(kx_i+x_j)+f(kx_n+x_1)-k\left(\sum_{i=1,j=i+1}^{n-1} f(x_i+x_j)+f(x_n+x_1)\right) \\ &\qquad = \frac{(1-k)^2}{2}\sum_{i=1}^{n}\left(f(x_i)+f(-x_i)\right)+\frac{1-k}{k^2-k}\sum_{i=1}^{n}\left(k^2f(x_i)-f(kx_i)\right) \end{aligned}$$

for positive integers $n,k \geq 2$, in a fuzzy modular space by using the fixed point technique given in [21].

Furthermore in 2024, Jakhar et al. [10] proved the Hyers-Ulam stability of the mixed type cubic and quartic functional equation in fuzzy modular spaces

$$g(u+2v)+g(u-2v)=4\left(g(u+v)+g(u-v)\right)-24g(v)-6g(u)+3g(2v)$$

by using the fixed point method of [21].

Our purpose is to give a fixed point result in fuzzy modular spaces improving a result of Wongkum et al. [21]. In fact, by introducing supplementary assumptions absent from the previous result, we establish a theorem that is structurally analogous to the Banach contraction principle. Next, as application of our result we will prove the Ulam stability of the following general linear functional equation:

$$\sum_{i=1}^{m} K_i\left(F\left(f_i(x_1,\ldots,x_n)\right)\right) = H(x_1,\ldots,x_n), \quad x_1,\ldots,x_n \in X, \tag{1.1}$$

where F is a mapping from a vector space X into a β-homogeneous fuzzy modular space (Y,μ,τ_m), with $\beta \in]0,1]$, and μ lower semi-continuous, m and n are positive integers with $m > 1$ and for each $i \in \mathbb{N}_m := \{1,2,\ldots,m\}$, f_i is a mapping from X^n into X, $(K_i)_{i\in\mathbb{N}_m}$ is a family of non zero commuting endomorphisms of Y and H is a mapping from X^n into Y.

Notice that, since the equation (1.1) generalizes several functional equations studied in the literature, many known results concerning the stability of such equations in fuzzy modular spaces can be derived as special cases of our findings. Finally, we conclude the paper with some corollaries, a remark and an example to illustrate our results.

This paper is organized as follows: In Section 2, we recall some basic definitions and properties of a fuzzy modular space. In Section 3, we show a Banach contraction type fixed point theorem in such spaces, improving a result of [21]. In Section 4, using this fixed point theorem, we present a result on Ulam stability of the equation (1.1). We end with some corollaries and an example to illustrate our results.

2. Preliminary

In this section, we recall some basic definitions and properties of a fuzzy modular space. These notions are standard in the literature and are presented here for the convenience of the reader.

Definition 2.1. A real function ρ on an arbitrary vector space E is said to be a modular if it satisfies the following conditions:

- $\rho(x) = 0$ if and only if $x = 0$,
- $\rho(x) = \rho(-x)$,
- $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $x, y \in E$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A modular space E_ρ is defined by corresponding modular ρ, that is

$$E_\rho := \{x \in E : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Definition 2.2. A binary operation $\tau : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

- τ is commutative and associative,
- τ is continuous,
- $a\tau 1 = a$ for every $a \in [0,1]$,
- $a\tau b \leq c\tau d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.3. A fuzzy set μ in E is a function with domain E and value in $[0,1]$.

Definition 2.4. The triple (E, μ, τ) is said to be a fuzzy modular space (FM-space) if E is a real or complex vector space, τ is a continuous t-norm, and μ is a fuzzy set on $E \times R_+$ satisfying the following conditions, for all $x, y \in E$, $s, t > 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$:

- $\mu(x, t) > 0$,
- $\mu(x, t) = 1$ for all $t > 0$ if and only if $x = 0$,
- $\mu(x, t) = \mu(-x, t)$,
- $\mu(\alpha x + \beta y, s + t) \geq \mu(x, s)\tau\mu(y, t)$,
- the mapping $t \to \mu(x, t)$ is continuous at each fixed $x \in E$.

Example 2.1. Let E be a real complex vector space, ρ be a modular on E, and let τ be a continuous t-norm. For every $t > 0$, define

$$\mu(x, t) = \frac{t}{t + \rho(x)}, \quad x \in E.$$

Then (E, μ, τ) is a FM-space.

Remark 2.1. If (E, μ, τ) is a FM-space, then the mapping $\mu(x, .)$ is a non-decreasing function for all $x \in E$.

Definition 2.5. If the fuzzy modular μ has this property: for every $x \in E$ and a non zero real λ, the equality

$$\mu(\lambda x, t) = \mu(x, \frac{t}{|\lambda|^\beta})$$

holds for some fixed $\beta \in]0, 1]$. Then (E, μ, τ) is said to be a β-homogeneous fuzzy modular space.

Notice that the β-homogeneity plays a fundamental role in fuzzy modular spaces, as it establishes a scaling law, that is, a precise principle governing the behavior of the fuzzy modular under scalar multiplication. This property guarantees the coherent interaction between vector scaling and the modular parameter, thereby ensuring the well-posedness of notions such as convergence, continuity, and the induced topology. Moreover, β-homogeneity is indispensable in the development of fixed point theory and stability results, serving as the analogue of the homogeneity property in normed spaces, where scalar multiplication of a vector results in a predictable rescaling of its norm.

Definition 2.6. We say that the fuzzy modular space (E, μ, τ) satisfies the lower semi-continuity if, for any sequence (x_n) of E and μ-converging to a point $x \in E$,

$$\mu(x, t) \leq \lim_{n \to +\infty} \inf \mu(x_n, t)$$

for all $t > 0$.

Notice that, the lower semi-continuity condition in fuzzy modular spaces is a regularity assumption that guarantees the coherence of modular convergence, supports the induced topology, and provides the technical foundation for applying limit-based arguments in fixed point and stability theory.

3. A fixed point result in fuzzy modular spaces

We assume some familiarity with certain definitions, notions and properties of fuzzy modular spaces, which can be found in [17].

Notice that in [21], the authors proved the following result:

Theorem 3.1 (Theorem 6, [21])**.** *Let E be a real vector space equipped with a β-homogeneous fuzzy modular μ and the strongest triangular norm τ_m such that (E, μ, τ_m) is μ-complete. Suppose that at each $x \in E$, $\mu(x, t) \to 1$ as $t \to +\infty$. If $f : E \to E$ is an inner contraction with constant $q \in]0, 1[$, then f has a unique fixed point.*

The following theorem is our main result. It improves Theorem 3.1, and it is structurally analogous to the Banach contraction principle.

Theorem 3.2. *Let (E, μ, τ_m) be a μ-complete β-homogeneous fuzzy modular space where τ_m is the minimum t-norm, such that $\lim_{t\to+\infty} \mu(\xi, t) = 1$ for each $\xi \in E$. If $J : E \to E$ is a μ-inner contraction with constant $q \in]0, 1[$, i.e:*

$$\mu(J\xi - J\xi^{'}, qt) \geq \mu(\xi - \xi^{'}, t), \quad \xi, \xi^{'} \in E, \quad t > 0,$$

then there exists a unique fixed point $\xi^\star$ of J such that for all $\xi \in E$, $(J^n\xi)_n$ μ-converges to $\xi^\star$.
If in addition $q < 2^{-\beta}$, then for every $\xi \in X$ and $t > 0$, we have

$$\mu\left(J^n\xi - \xi, t\right) \geq \mu\left(J\xi - \xi, \frac{(1-2^\beta q)t}{2^\beta}\right), \qquad n \geq 2, \tag{3.1}$$

$$\mu(\xi^\star - \xi, t) \geq \mu\left(J\xi - \xi, \frac{(1-2^\beta q)t}{2^{2\beta+1}}\right). \tag{3.2}$$

Moreover, if μ is lower semi-continuous, then for every $\xi \in X$ and $t > 0$

$$\mu\left(J^n\xi - \xi^\star, t\right) \geq \mu\left(J\xi - \xi, \frac{(1-2^\beta q)t}{2^\beta q^n}\right), \qquad n \in \mathbb{N}. \tag{3.3}$$

Proof. The existence and uniqueness of $\xi^\star$ the fixed point of J is shown in the real case in [21], the same proof works also in the complex case.

Now, assume that $q < 2^{-\beta}$ and fix $\xi \in X$ and $t > 0$, we claim that for every $n \geq 2$, we have

$$\mu\bigg(J^n(\xi) - \xi, \big((2^\beta q)^{n-1} + 2^\beta \sum_{i=1}^{n-1} (2^\beta q)^{i-1}\big)t\bigg) \geq \mu(J(\xi) - \xi, t). \tag{3.4}$$

The case $n = 2$ is easy. Let $n \geq 2$ and assume that (3.4) is satisfied. Then

$$\begin{aligned}\mu\bigg(& J^{n+1}(\xi) - \xi, \big((2^\beta q)^n + 2^\beta \sum_{i=1}^{n} (2^\beta q)^{i-1}\big)\ t\bigg) \\ &= \mu\bigg(\frac{1}{2}\big(J^{n+1}(\xi) - J(\xi)\big) + \frac{1}{2}(J(\xi) - \xi), \big(2^{\beta(n-1)} q^n + \sum_{i=2}^{n} (2^\beta q)^{i-1} + 1\big)\ t\bigg)\end{aligned}$$

$$\geq \mu\bigg(J^{n+1}(\xi) - J(\xi), \big(2^{\beta(n-1)} q^n + \sum_{i=2}^{n} (2^\beta q)^{i-1}\big)\ t\bigg) \tau_m \mu(J(\xi)-\xi, t).$$

But, the contractivity of J gives

$$\begin{aligned}\mu\bigg(J^{n+1}(\xi) - J(\xi), &\big(2^{\beta(n-1)} q^n + \sum_{i=2}^{n} (2^\beta q)^{i-1}\big) t\bigg) \\ &\geq \mu\bigg(J^n(\xi) - \xi, \big((2^\beta q)^{n-1} + 2^\beta \sum_{j=1}^{n-1} (2^\beta q)^{j-1}\big) t\bigg).\end{aligned}$$

Using the assumption (3.4) on n, we get

$$\mu\bigg(J^{n+1}(\xi) - J(\xi), \big(2^{\beta(n-1)} q^n + \sum_{i=2}^{n} (2^\beta q)^{i-1}\big) t\bigg) \geq \mu(J(\xi) - x, t).$$

It follows that (3.4) holds for every $n \geq 2$. Next, since $2^\beta q < 1$, one gets

$$(2^\beta q)^{n-1} + 2^\beta \sum_{i=1}^{n-1} (2^\beta q)^{i-1} \leq \frac{2^\beta}{1-2^\beta q}.$$

Since $t \mapsto \mu(\xi, t)$ is non-decreasing, for every $n \geq 2$, we have

$$\begin{aligned}\mu\bigg(J^n(\xi) - \xi, \frac{2^\beta t}{(1-2^\beta) q}\bigg) &\geq \mu\bigg(J^n(\xi) - \xi, \big((2^\beta q)^{n-1} + 2^\beta \sum_{i=1}^{n-1} (2^\beta q)^{i-1}\big)\ t\bigg) \\ &\geq \mu(J(\xi) - \xi, t).\end{aligned}$$

Therefore (3.1) holds true. Furthermore,

$$\begin{aligned}\mu(\xi^\star - \xi, t) &= \mu\left(\frac{1}{2}(\xi^\star - J^n(\xi)) + \frac{1}{2}(J^n(\xi) - \xi), \frac{t}{2^{\beta+1}} + \frac{t}{2^{\beta+1}}\right) \\ &\geq \min\left(\mu\left(\xi^\star - J^n(\xi), \frac{t}{2^{\beta+1}}\right), \mu\left(J(\xi) - \xi, \frac{(1-2^\beta q)t}{2^{2\beta+1}}\right)\right).\end{aligned}$$

Letting n tend to infinity, we get (3.2), since $(J^n(\xi))_n$ μ-converges to $\xi^\star$.

Now, whenever $m - n \geq 2$, we have:

$$\mu\left(J^n(\xi) - J^m(\xi), t\right) \geq \mu\left(\xi - J^{m-n}(\xi), \frac{t}{q^n}\right).$$

Using (3.1) we get

$$\mu\left(J^n(\xi) - J^m(\xi), t\right) \geq \mu\left(J(\xi) - \xi, \frac{(1-2^\beta q)t}{2^\beta q^n}\right).$$

If μ happens to be lower semi-continuous, then we get

$$\begin{aligned}\mu(J^n(\xi) - \xi^\star, t) &\geq \limsup_{m\to+\infty} \mu(J^n(\xi) - J^m(\xi), t) \\ &\geq \mu\bigg(J(\xi) - \xi, \frac{(1-2^\beta q)t}{2^\beta q^n}\bigg).\end{aligned}$$

Therefore (3.3) holds true, and this finishes the proof. $\square$

Remark 3.1. Theorem 3.1 is given in real modular space framework, but our result holds in both real and complex situations. We even go beyond that and obtain the approximation inequalities (3.1), (3.2) and (3.3) showing how the iterations of J behave with respect to the fixed point of J. So our result is more similar to the Banach contraction principle than that of [21].

4. Application of Theorem 3.2 in Ulam stability

In this section, using Theorem 3.2, to prove the stability of (1.1) in a β-homogeneous fuzzy modular spaces.

Let X and Y be vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. On Y we consider a complete β-homogeneous FM-space structure (Y, μ, τ_m), with $\beta \in]0,1]$ and μ lower semi-continuous. Fix positive integers m and n, with $m > 1$, and for each $i \in \mathbb{N}_m := \{1, 2, \ldots, m\}$, $f_i : X^n \to X$. Let $(K_i)_{i \in \mathbb{N}_m}$ be a family of non zero commuting endomorphisms of Y, each K_i being continuous with respect to the topology of the FM-space (Y, μ, τ_m) as in Theorem 14 of [17]. For every nonempty subset I of $\mathbb{N}_m$, we will write K_I for $\sum_{i \in I} K_i$. In case $I = \{1, 2, \ldots, m\}$, we will write K instead of $K_{\{1,2,\ldots,m\}}$.

We will denote an element $(x_1, \ldots, x_n)$ of X^n by $\overline{x}$.

Theorem 4.1. *Let $H : X^n \to Y$ and $\chi : X \to X^n$ so that*

$$\sum_{i=1}^{m} K_i \left(H(\chi \circ f_i(\overline{x})) \right) = \sum_{i=1}^{m} K_i \left(H(f_i \circ \chi(x_1), \ldots, f_i \circ \chi(x_n)) \right) \tag{4.1}$$

for every $\overline{x} \in X^n$. Suppose there exist $\theta : X^n \times\ ]0, +\infty[\to [0, 1]$, a proper subset I of $\mathbb{N}_m$ and positive numbers ω_i and non-zero real numbers α_i, for $i \notin I$, so that $t \mapsto \theta(\overline{x}, t)$ is non-decreasing for every $\overline{x} \in X^n$, K_I possesses an eigenvalue $M \neq 0$, whose corresponding eigenspace Y_M contains $F(X)$ and $H(X^n)$, and for every $j \in I$, $i \notin I$, $p \in \mathbb{N}_m$, $\overline{x} \in X^n$, $y \in Y$, $x \in X$ and $t > 0$, we have

$$\lim_{t \to +\infty} \theta(\overline{x}, t) = 1, \tag{4.2}$$

$$f_j \circ \chi = Id_X, \tag{4.3}$$

$$\theta(f_i \circ \chi(x_1), \ldots, f_i \circ \chi(x_n), t) \geq \theta(\overline{x}, \omega_i t), \tag{4.4}$$

$$f_p(f_i \circ \chi(x_1), \ldots, f_i \circ \chi(x_n)) = f_i \circ \chi \circ f_p(\overline{x}), \tag{4.5}$$

$$\theta(\chi(f_i \circ \chi(x)), t) \geq \theta(\chi(x), \omega_i t), \tag{4.6}$$

$$0 < \frac{|\sum_{i \notin I} \alpha_i|^{\beta+1}}{|\alpha_i| \omega_i |M|^\beta} < \frac{1}{2^\beta}, \tag{4.7}$$

and

$$K_i(y) = \alpha_i y. \tag{4.8}$$

If $F : X \to Y$ satisfies

$$\mu \left(\sum_{i=1}^{m} K_i(F(f_i(\overline{x}))) - H(\overline{x}), t \right) \geq \theta(\overline{x}, t), \quad \overline{x} \in X^n, t > 0, \tag{4.9}$$

then there exists $G : X \to Y$, *a unique solution of* (1.1) *fulfilling:*

$$\mu\left(G(x) - F(x), t\right) \geq \theta\left(\chi(x), \frac{(1-2^{\beta}q)|M|^{\beta}t}{2^{2\beta+1}}\right), \quad x \in X, \ t > 0, \tag{4.10}$$

with $\alpha_0\omega_0 := \min_{i\notin I}|\alpha_i|\omega_i$ *and* $q := \frac{|\sum_{i\notin I}\alpha_i|^{\beta+1}}{\alpha_0\omega_0|M|^{\beta}}$.

Proof. Let $x \in X$. Denoting $\chi(x)$ by $\overline{x} = (x_1, \dots, x_n)$ in (4.9) and using (4.3), we have

$$\mu\Big(M\big(F(x) - \frac{-1}{M}\Big(\sum_{i\notin I} K_i(F(f_i \circ \chi(x)) - H(\chi(x)))\Big), t\Big) \geq \theta(\chi(x), t),$$

for all $x \in X$ and $t > 0$. Then

$$\mu\left(F(x) - \frac{-1}{M}\left(\sum_{i\notin I} K_i(F(f_i \circ \chi(x)) - H(\chi(x))\right), \frac{t}{|M|^{\beta}}\right) \geq \theta(\chi(x), t).$$

Equivalently,

$$\mu\left(F(x) - \frac{-1}{M}\left(\sum_{i\notin I} K_i(F(f_i \circ \chi(x)) - H(\chi(x))\right), t\right) \geq \theta(\chi(x), |M|^{\beta}t). \tag{4.11}$$

Define

$$\mu_0(g, t) := \frac{t}{t + \varrho(g)}, \quad g \in Y^X, \ t > 0,$$

with

$$\varrho(g) := \inf\{a > 0, \ \mu(g(x), as) \geq \theta(\chi(x), |M|^{\beta}s), \quad x \in X, \ s > 0\}.$$

Then, ϱ is a modular on Y^X. Therefore, (Y^X, μ_0, τ_m) is a FM-space. Let us show that (Y^X, μ_0, τ_m) is complete. Notice first that by (4.2), for arbitrary $x \in X$ and $\lambda \in]0, 1[$, there exists $t_0 > 0$ so that

$$\theta(\chi(x), |M|^{\beta}t_0) > 1 - \lambda.$$

Let $(g_n)_n$ be a Cauchy sequence in Y^X, $\delta > 0$ and $\epsilon > 0$ be given, such that $\epsilon t_0 < \delta(1-\epsilon)$. Then there exists $n_0 \in \mathbb{N}$ satisfying

$$\rho(g_n - g_m) < \frac{\epsilon t_0}{1-\epsilon}, \quad m, n \geq n_0.$$

Therefore

$$\begin{aligned} \mu(g_n(x) - g_m(x), \delta) &\geq \mu\left(g_n(x) - g_m(x), \frac{\epsilon t_0}{1-\epsilon}\right) \\ &\geq \theta\left(\chi(x), t_0|M|^{\beta}\right) \\ &> 1 - \lambda, \quad x \in X, \ n, m \geq n_0. \end{aligned}$$

Then, $(g_n(x))_n$ is a Cauchy sequence in $(Y, \mu, \star_m)$. Since the space is complete, $(g_n(x))_n$ is convergent in $(Y, \mu, \star_m)$ for every $x \in X$. Define $g : X \to Y$ by

$$g(x) := \lim_{n\to+\infty} g_n(x), \qquad x \in X.$$

Since μ is lower semicontinuous, then

$$\begin{aligned}\mu\left(g_n(x)-g(x),\frac{\epsilon t}{1-\epsilon}\right) &\geq \lim_{m\to+\infty}\sup_{k\geq m}\mu\left(g_n(x)-g_k(x),\frac{\epsilon t}{1-\epsilon}\right)\\ &\geq \theta(\chi(x),|M|^{\beta}t),\quad x\in X,\ t>0.\end{aligned}$$

Then, $\mu_0(g_n-g,t)>1-\epsilon$, for all $n\geq n_0$. Thus $(g_n)_n$ is a convergent sequence in (Y^X,μ_0,τ_m). Therefore, (Y^X,μ_0,τ_m) is complete. Now, we show that μ_0 is β-homogeneous. Let $g\in Y^X$, $t>0$ and $\lambda\in\mathbb{K}\setminus\{0\}$. Since μ is β-homogeneous, we get

$$\mu_0(\lambda g,t)=\frac{t}{t+\mid\lambda\mid^{\beta}\varrho(g)}=\mu_0\left(g,\frac{t}{\mid\lambda\mid^{\beta}}\right).$$

Then, μ_0 is β-homogeneous. Also, we get $\lim_{t\to+\infty}\mu_0(g,t)=1$, for every $g\in Y^X$. Let $\xi\in Y^X$, define $J\xi: X\mapsto Y$ by

$$J\xi(x):=\frac{-1}{M}\left(\sum_{i\notin I}K_i(\xi(f_i\circ\chi(x)))-H(\chi(x))\right),\quad \xi\in Y^X,\ x\in X. \tag{4.12}$$

Let $g,h\in Y^X$ and a is a positive constant so that $\rho(g-h)<a$. Then

$$\mu(g(x)-h(x),t)\geq\theta\left(\chi(x),\frac{|M|^{\beta}t}{a}\right),\quad x\in X,\ t>0.$$

For every $x\in X$ and $t>0$,

$$\begin{aligned}\mu(Jg(x)-Jh(x),t) &\geq \min_{i\notin I}\mu\left(g(f_i\circ\chi(x))-h(f_i\circ\chi(x)),\frac{t|M|^{\beta}|\alpha_i|}{|\sum_{j\notin I}\alpha_j|^{\beta+1}}\right)\\ &\geq \min_{i\notin I}\theta\left(\chi(f_i\circ\chi(x)),\frac{t|M|^{\beta}|\alpha_i||M|^{\beta}}{a|\sum_{j\notin I}\alpha_j|^{\beta+1}}\right)\\ &\geq \theta\left(\chi(x),\frac{t|M|^{\beta}\alpha_0\omega_0|M|^{\beta}}{a|\sum_{j\notin I}\alpha_j|^{\beta+1}}\right).\end{aligned}$$

Therefore

$$\mu_0\left(Jg-Jh,\frac{t|\sum_{j\notin I}\alpha_j|^{\beta+1}}{\alpha_0\omega_0|M|^{\beta}}\right)\geq\mu_0(g-h,t),\quad g,h\in Y_f^X,\ t>0.$$

That means that J is a μ_0-inner contraction with $q=\frac{|\sum_{j\notin I}\alpha_j|^{\beta+1}}{\alpha_0\omega_0|M|^{\beta}}$ and $\alpha_0\omega_0=\min_{i\notin I}|\alpha_i|\omega_i$. Therefore, by Theorem 3.2, there exists a unique mapping $G\in Y^X$ so that $JG=G$, $\lim_{n\to+\infty}\mu_0(T^nF-G,t)=1$ and for every $t>0$,

$$\mu_0(F-G,t)\geq\mu_0\left(F-JF,\frac{(1-2^{\beta}q)t}{2^{2\beta+1}}\right). \tag{4.13}$$

Since $\lim_{n\to+\infty}\mu_0(J^nF-G,t)=1$, it follows μ–$\lim_{n\to+\infty}(J^nF)(x)=G(x)$, for every $x\in X$. Now, we prove that G satisfies equation (1.1). First, observe that

$$\begin{aligned}\sum_{i=1}^{m}K_i\left(G(f_i(\overline{x}))\right)&=\sum_{i=1}^{m}K_i\left(\lim_{k\to+\infty}(J^kF)(f_i(\overline{x}))\right)\\&=\lim_{k\to+\infty}\left(\sum_{i=1}^{m}K_i\left((J^kF)(f_i(\overline{x}))\right)\right).\end{aligned}$$

Next, we demonstrate that for every $k\in\mathbb{N}$,

$$\mu\left(\sum_{p=1}^{m}K_p\circ(J^kF)\circ f_p(\overline{x})-H(\overline{x}),t\right)\geq\theta(\overline{x},\frac{t}{q^k}),\quad \overline{x}\in X^n,\ t>0. \tag{4.14}$$

The case $k=0$ is just (4.9). Now, we fix $k\in\mathbb{N}$ and suppose that (4.14) holds. Then for every $\overline{x}\in X^n$,

$$\begin{aligned}\sum_{p=1}^{m}&K_p\circ(J^{k+1}F)\circ f_p(\overline{x})=\sum_{p=1}^{m}K_p\circ(J(J^kF))\circ f_p(\overline{x})\\&=\sum_{p=1}^{m}K_P\left[\frac{-1}{M}\left(\sum_{i\notin I}K_i\left((J^kF)(f_i(\chi(f_p(\overline{x}))))-H(\chi(f_p(\overline{x})))\right)\right)\right]\\&=\frac{-1}{M}\sum_{i\notin I}K_i\left(\sum_{p=1}^{m}K_p\left((J^kF)(f_p(\overline{f_i\circ\chi(\overline{x})})\right)\right)+\frac{1}{M}\sum_{p=1}^{m}K_p\left(H(\chi(f_p(\overline{x})))\right)\\&=\frac{-1}{M}\sum_{i\notin I}K_i\left(\sum_{p=1}^{m}K_p\left((J^kF)(f_p(\overline{f_i\circ\chi(\overline{x})})\right)\right)\\&+\frac{1}{M}\sum_{p=1}^{m}K_p\left(H(f_p\circ\chi(x_1),\ldots,f_p\circ\chi(x_n))\right)\\&=\frac{-1}{M}\sum_{i\notin I}K_i\left(\sum_{p=1}^{m}K_p\left((J^kF)(f_p(\overline{f_i\circ\chi(\overline{x})})))\right)\right)\\&+\frac{1}{M}\left[\sum_{i\notin I}K_i\left(K(\overline{f_i\circ\chi(\overline{x})})\right)+\sum_{i\in I}K_i\left(H(\overline{x})\right)\right]\\&=\frac{-1}{M}\sum_{i\notin I}K_i\left(\sum_{p=1}^{m}K_p\left((J^kF)(f_p(\overline{f_i\circ\chi(\overline{x})})))\right)-H(\overline{f_i\circ\chi(\overline{x})})\right)+H(\overline{x}),\end{aligned}$$

with for every $i\notin I$, $\overline{f_i\circ\chi(\overline{x})}:=(f_i\circ\chi(x_1),\ldots,f_i\circ\chi(x_n)))$. Hence,

$$\mu\left(\sum_{p=1}^{m}K_p\circ(J^{k+1}f)\circ f_p(\overline{x})-H(\overline{x}),t\right)$$

$$
\begin{aligned}
&= \mu\left(\frac{-1}{M}\sum_{i\notin I} K_i \circ \left[\sum_{p=1}^{m} K_p((J^kF)(f_p(\overline{f_i\circ\chi(\overline{x})})) - H(\overline{f_i\circ\chi(\overline{x})})\right], t\right)\\
&\geq \min_{i\notin I}\mu\left(\sum_{i\notin I}\alpha_i\left[\sum_{p=1}^{m} K_p((J^kF)(f_p(\overline{f_i\circ\chi(\overline{x})})) - H(\overline{f_i\circ\chi(\overline{x})})\right], t|M|^{\beta} f_i\right)\\
&\geq \min_{i\notin I}\theta\left(\overline{\varphi_i\circ\chi(\overline{x})}, \frac{t|M|^{\beta}|\alpha_i|}{q^k}\right)\\
&\geq \min_{i\notin I}\theta\left(\overline{x}, \frac{t|M|^{\beta}|\alpha_i|\omega_i}{q^k}\right)\\
&\geq \theta\left(\overline{x}, \frac{t|M|^{\beta}\alpha_0\omega_0}{q^k}\right) = \theta\left(\overline{x}, \frac{t}{q^{k+1}}\right).
\end{aligned}
$$

Then (4.14) holds for every $k \in \mathbb{N}$. Letting $k \to +\infty$ in (4.14), we get:

$$\lim_{k\to+\infty}\sum_{p=1}^{m} K_p \circ (J^kF)\circ f_p(\overline{x}) = H(\overline{x}), \quad \overline{x}\in X^n.$$

Consequently,

$$\sum_{p=1}^{m} K_p \circ G \circ f_p(\overline{x}) = H(\overline{x}), \quad \overline{x} := (x_1, \dots, x_n) \in X^n.$$

Now, we are in a position to prove that (4.10) holds true. By (4.11) we get

$$\mu(F(x) - JF(x), t) \geq \theta(\chi(x), |M|^{\beta}t), \quad x \in X, \quad t > 0.$$

Then $\rho(F - JF) \leq 1$. Therefore, by using (4.13), we obtain

$$\rho(G - F) \leq \frac{2^{2\beta+1}}{1 - 2^{\beta}q}\rho(F - TF).$$

This means that $\rho(G - F) \leq \frac{2^{2\beta+1}}{1-2^{\beta}q}$, whereby we get (4.10). □

Using Theorem 4.1, we get the stability for numerous equations. One of them is Cauchy inhomogeneous equation

$$F(x_1 + x_2) - F(x_1) - F(x_2) = H(x_1, x_2). \tag{4.15}$$

Corollary 4.1. *Let $(X, \|\|)$ be a normed space, $L, L^{'} \in \mathbb{R}_+$, $p \in]-\infty, 0[$. Assume that $H : X^2 \to Y$ is symmetric and biadditive, $F : X \to Y$ satisfies:*

$$\mu\left(F(x_1 + x_2) - F(x_1) - F(x_2) - H(x_1, x_2), t\right) \geq \frac{t}{t + L\|x_1\|^p + L^{'}\|x_2\|^p},$$

for every $x_1, x_2 \in X$ and every $t > 0$. Then, there exists a unique solution $G : X \to Y$ to (4.15) so that:

$$\mu(G(x) - F(x), t) \geq \frac{(1 - 2^p)t}{(1 - 2^p)t + 2^{p+1}(L + L^{'})\|x\|^p}, \quad x \in X, \ t > 0. \tag{4.16}$$

Example 4.1. Let $L, L^{'} \in \mathbb{R}_+$ and $p \in \mathbb{R}_-^\star$. Assume that $X = Y = \mathbb{R}$ and $(\mathbb{R}, \mu, \tau_m)$ is the FM-space with

$$\mu(x,t) := \frac{t}{t+|x|}, \quad x \in \mathbb{R}, t > 0.$$

Suppose that $H : \mathbb{R}^2 \to \mathbb{R}$ define by $H(x,y) = xy$ for every $x, y \in \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ satisfy

$$\frac{1}{t + |F(x_1+x_2) - F(x_1) - F(x_2) - x_1x_2|} \geq \frac{1}{t + L|x_1|^p + L'|x_2|^p}$$

for every $x_1, x_2 \in \mathbb{R}$ and every $t > 0$. Then, using Corollary 4.1, there is a unique function $G : \mathbb{R} \to \mathbb{R}$ such that

$$G(x_1 + x_2) - G(x_1) - G(x_2) = x_1x_2$$

and

$$\frac{1}{t - |G(x) - F(x)|} \geq \frac{1 - 2^p}{(1-2^p)t + 2^{p+1}(L + L^{'})|x|^p}, \quad x \in \mathbb{R}.$$

Now, using Theorem 4.1, we show the stability of Jordan-von Neumman inhomogeneous equation:

$$F(x_1 + x_2) + F(x_1 - x_2) - 2F(x_1) - 2F(x_2) = H(x_1, x_2). \tag{4.17}$$

Set:

$$HF(x_1, x_2) := F(x_1 + x_2) + F(x_1 - x_2) - 2F(x_1) - 2F(x_2) - H(x_1, x_2).$$

Corollary 4.2. *Let $\| \ \|$ is a norm on X, $L, L^{'} \in \mathbb{R}_+$, $p, \omega_2 \in \mathbb{R}$ such that $p < -1$ and $\omega_2 > 2$. Assume also that H is a biadditive given mapping and $F : X \to Y$ satisfy:*

$$\mu\left(HF(x_1, x_2), t\right) \geq \frac{t}{t + L\|x_1\|^p + L'\|x_2\|^p},$$

for every $x_1, x_2 \in X$ and every $t > 0$. Then, there exists a unique solution $G : X \to Y$ of (4.17) *such that:*

$$\mu(G(x) - F(x), t) \geq \frac{t}{t + K\|x\|^p}, \quad x \in X, \ t > 0, \tag{4.18}$$

with $K := \frac{(L+L^{'})}{2^{\beta-2}(\omega_0 - 2)}$ and $\omega_0 := \min(2^{-p}, \omega_2)$.

Now, we will give some comments about the condition (4.1).

Remark 4.1. 1) The constant functions from X^n into Y satisfy (4.1).

2) If $H_1, H_2 : X^n \to Y$ satisfy (4.1), then $\alpha_1 H_1 + \alpha_2 H_2$ also satisfy (4.1), for every fixed scalars α_1, α_2.

3) If we take the situation in Corollary 4.1, then the condition (4.1) has the form

$$H(x_1 + x_2, x_1 + x_2) - H(x_1, x_1) - H(x_2, x_2) = H(2x_1, 2x_2) - 2H(x_1, x_2). \tag{4.19}$$

Let $\rho : X \to Y$. Then, $H : X^2 \to Y$ defined by

$$H(x_1, x_2) = \rho(x_1 + x_2) - \rho(x_1) - \rho(x_2) \tag{4.20}$$

is a solution to (4.19). Note that if H is symmetric and biadditive (i.e $H(x_1, x_2) = H(x_2, x_1)$ and $H(x_1, x_2 + x_3) = H(x_1, x_2) + H(x_1, x_2)$), then (4.20) holds with $\rho(x) = \frac{1}{2}H(x, x)$ for $x \in X$. Therefore (4.19) is satisfied for every symmetric and biadditive $H : X^2 \to Y$.

4) Consider the situation in Corollary 4.2, then the condition (4.1) has the form

$$\begin{aligned} H(x_1 + x_2, x_1 + x_2) + H(x_1 - x_2, x_1 - x_2) - 2H(x_1, x_1) - 2H(x_2, x_2) \\ = H(2x_1, 2x_2) + H(0, 0) - 4H(x_1, x_2). \end{aligned} \tag{4.21}$$

Note that if H is biadditive, therefore we get (4.21) holds true.

4.1. **Open Problem.** Determine whether Theorem 4.1 can be extended to settings where condition (4.1) (respectively, inequality (4.7)) is replaced by a weaker assumption. More precisely, is it possible to establish an analogous stability result under less restrictive conditions?

5. Further Discussions

The study of Ulam stability in fuzzy modular spaces is not only a theoretical pursuit but also has potential for practical use. Fuzzy modular structures naturally appear in situations where uncertainty, vagueness, or incomplete information cannot be ignored. In such cases, stability results play an important role, since they ensure that approximate solutions of functional equations remain close to exact ones, even when the available data is imprecise or affected by noise.

More specifically, these results may be useful in information theory and decision-making models, where fuzzy data must be handled carefully to ensure reliability and consistency. In approximation theory, stability provides a safeguard that approximate functions or operators preserve essential properties under small perturbations, which is particularly valuable in numerical analysis with uncertain data. Likewise, in control theory and dynamical systems, Ulam stability helps guarantee that small changes in parameters or initial conditions do not lead to significant deviations, supporting the development of robust and reliable models.

From a broader mathematical perspective, Ulam stability in fuzzy modular spaces can be regarded as a general framework that extends earlier results from normed, modular, and probabilistic metric spaces. This opens several directions for further work, such as exploring stability in Banach algebras, operator equations, or stochastic models involving fuzzy components.

5.1. **Open problem.** A natural question for future research is whether the methods used in this paper can be adapted to study Ulam-type stability of functional equations in probabilistic fuzzy modular spaces, where fuzziness and randomness are combined. Addressing this problem would expand the reach of stability theory to models that incorporate both uncertainty and stochastic effects.

6. Conclusion

In this paper, we examined the Ulam stability of a general linear functional equation within the framework of fuzzy modular spaces, using an analogue of the Banach contraction principle. Our results contribute to the understanding of how the Ulam stability concepts can be extended to fuzzy modular frameworks, highlighting the flexibility of these spaces in handling functional equations under uncertainty.

For future research, it would be interesting to study the Ulam stability of similar equations in broader contexts that generalize fuzzy modular spaces. Examples include intuitionistic fuzzy modular spaces, probabilistic modular spaces, and others. Such investigations could deepen our insight into Ulam stability behavior in more complex fuzzy environments and expand potential applications in areas such as fuzzy analysis and information science.

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