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FIXED POINTS OF α-NONEXPANSIVE AND α-ASIMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, inspired by the concept of α-admissibility developed in metric fixed point theory, we introduce the concepts of α-nonexpansive and α-asymptotically nonexpansive mappings in normed spaces and present some fixed point theorems for such mappings. The approach here carries forward and improves the results obtained for monotone nonexpansive and monotone asymptotically nonexpansive mappings.

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1. Introduction

The concept of nonexpansive mapping is an important one for metric fixed point theory. The study of fixed point theory for such mappings is not as simple as the study of contraction or contraction type mappings. It is commonly known that a self mapping T on a metric space (X, d) is called a Lipschitz mapping if there exists $\lambda \geq 0$ such that

$$d(Tx, Ty) \leq \lambda d(x, y)$$

holds for all $x, y \in X$. If we denote the smallest of the λ numbers that satisfy this inequality with L, L is called the Lipschitz constant of T. If $L < 1$, T is called contraction mapping, and if $L = 1$, T is called nonexpansive mapping. Additionally, if

$$d(Tx, Ty) < d(x, y)$$

holds for all $x, y \in X$ with $x \neq y$, then T is called contractive mapping.

According to the Banach fixed point theorem, if X is complete and T is a contraction mapping, then T has a unique fixed point, and every Picard iteration sequence (the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for $x_0 \in X$) converges to this fixed point. If T is merely a contractive mapping, completeness of X is not sufficient to guarantee a fixed point. However, if X is compact, then every contractive mapping has a unique fixed point, and the Picard iteration sequence still converges to it.

In contrast, even the compactness of the space does not ensure the existence of fixed points for nonexpansive mappings, as illustrated by the simple example $T : \{0,1\} \to \{0,1\}$, defined by $T(0) = 1$, $T(1) = 0$. Clearly, additional conditions are needed. For instance, the Schauder fixed point theorem ensures that every nonexpansive mapping on a nonempty, compact, and convex subset of a normed space has at least one fixed point. However, in this case, the fixed point may not be unique, and the Picard iteration sequence may not converge.

Previous studies have explored how the compactness condition on X can be weakened while still guaranteeing the existence of fixed points for nonexpansive mappings. In 1965, Browder [8], Göhde [9], and Kirk [11] independently proved that every nonexpansive self-mapping on a nonempty, closed, bounded, and convex subset of a Hilbert space (or even a uniformly convex Banach space) has at least one fixed point.

While research on the existence of fixed points for nonexpansive mappings continues, attention has also turned to understanding the behavior of iteration sequences that converge to these fixed points, the conditions under which convergence occurs, and how the rates of convergence compare.

In addition to the Picard iteration sequence mentioned above, Mann iteration sequence, Ishikawa iteration sequence and Krasnoselskii iteration sequence, which we will consider in this study, are some of them (see [1, 7] for more information). Let C be a nonempty convex subset of a normed space and let $T : C \to C$ be a map. Then, the iterative sequence $\{x_n\}$ defined by

$$x_{n+1} = \lambda x_n + (1-\lambda)Tx_n$$

for $\lambda \in (0,1)$ and $x_0 \in X$, is the Krasnoselskii iteration sequence.

On the other hand, the fixed point theorems obtained for Lipschitz mappings have been given a different perspective by imposing an order relation on the space. For this purpose, Ran and Reurings [13] presented a theorem that guarantees the existence of a fixed point if X is a partially ordered complete metric space and the contraction inequality holds for elements comparable in order, and this theorem was extended to other contraction type inequalities (see [3, 4, 6, 10, 12]). This idea was taken into account by Bachar and Khamsi [5] for nonexpansive mappings and by Alfuraidan and Khamsi [2] for asymptotically nonexpansive mappings and some fixed point theorems for monotone (asymptotically) nonexpansive mappings in Banach spaces was obtained.

The condition on the aforementioned inequalities can also be weakened by using an α function to cover it instead of the order relation given on the space. For this purpose, Samet et al. [14] introduced the notion of α-admissible mapping and guaranteed the existence of a fixed point if the contraction inequality is satisfied by points satisfying the property $\alpha(x,y) \geq 1$. If X is endowed with an order relation $\preceq$ and the function $\alpha : X \times X \to \mathbb{R}$ is defined in the form

$$\alpha(x,y) = \begin{cases} 1 & , \quad x \preceq y \\ 0 & , \quad x \npreceq y \end{cases}, \tag{1.1}$$

the condition that points satisfying the property $\alpha(x,y) \geq 1$ satisfy the contraction inequality coincides with the condition that points of the form $x \preceq y$ satisfy the same inequality. Thus, in this special case of α function, the idea of Ran and Reurings is obtained.

In this paper, the method of α-admissibility is applied to both nonexpansive and asymptotically nonexpansive mappings. Based on this approach, we introduce the concepts of α-nonexpansive and α-asymptotically nonexpansive mappings, and establish several new fixed point theorems for such mappings in Banach spaces.

2. Preliminaries

Let (X,d) be a metric space, $T : X \to X$ be a mapping and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be α-admissible if

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1$$

for every $x, y \in X$. (see [14] for more information).

Now let us introduce the following definition, which forms the main idea of this study.

Definition 2.1 Let (X,d) be a metric space, $T : X \to X$ be a mapping and $\alpha : X \times X \to \mathbb{R}$ be a function. Then, T is called α-Lipschitzian mapping if it is α-admissible and there exists $K \geq 0$ such that

$$d(Tx,Ty) \leq K d(x,y)$$

for every $x, y \in X$ with $\alpha(x,y) \geq 1$. Especially, if $K < 1$ then T is said to be α-contaction and if $K \leq 1$ then T is said to be α-nonexpansive mapping. Also T is said to be α-asymptotically nonexpansive mapping if T is α-admissible and there exists a sequence $\{K_n\}$ of real numbers such that $\lim K_n = 1$ and

$$d(T^n x, T^n y) \leq K_n d(x,y)$$

for every $x, y \in X$ with $\alpha(x,y) \geq 1$.

If X is equipped with a partial order relation, say $\preceq$, then taking into account the function specified in (1.1), the concept of α-Lipschitzian (resp. α-nonexpansive, α-asymptotically nonexpansive) mapping becomes monotone Lipschitzian (resp. monotone nonexpansive, monotone asymptotically nonexpansive) mapping.

Let $(X, \|\cdot\|)$ be a normed space and $\alpha : X \times X \to \mathbb{R}$ be a function. Throughout this paper we will consider the following subsets of X :

$$\begin{aligned} U(\alpha,a) &= \{x \in X : \alpha(a,x) \geq 1\}, \\ L(\alpha,a) &= \{x \in X : \alpha(x,a) \geq 1\} \end{aligned}$$

for $a \in X$ and we will call $U(\alpha,a)$ and $L(\alpha,a)$ upper and lower level sets of a, respectively. It is clear that for $a, b \in X$

$$b \in U(\alpha,a) \Leftrightarrow a \in L(\alpha,b).$$

Definition 2.2 Let X be a nonempty set and $\alpha : X \times X \to \mathbb{R}$ be a function. We will say that α has nested property with respect to upper (resp. lower) level sets if for every $a, b \in X$ such that $b \in U(\alpha, a)$ implies $U(\alpha, b) \subseteq U(\alpha, a)$ (resp. $b \in L(\alpha, a)$ implies $L(\alpha, b) \subseteq L(\alpha, a)$). Also we will say that α is reflexive if for every $a \in X$ such that $\alpha(a, a) \geq 1$.

Remark 2.3 Let α be a reflexive function on a nonempty set X. Then α has the nested property with respect to upper level sets if and only if it has the nested property with respect to lower level sets.

Definition 2.4 Let X be a nonempty set and $\alpha : X \times X \to \mathbb{R}$ be a function. We will say that α has equivalent property if for every $a, b \in X$ such that $b \in U(\alpha, a)$ and $a \in U(\alpha, b)$ implies $a = b$.

3. α-Nonexpansive Mappings

Before giving our main theorem, let us express and prove some lemmas needed in the theorem.

Lemma 3.1 *Let $(X, \|\cdot\|)$ be a Banach space, τ be a Hausdorff topology on X, and let $\alpha : X \times X \to \mathbb{R}$ be a reflexive function having the nested and equivalent properties, such that its upper and lower level sets are convex and τ-closed. Let $C \subseteq X$ be a convex and bounded subset not reduced to a one point, and let $T : C \to C$ be an α-admissible mapping. Then, for the Krasnoselskii iteration sequence $\{x_n\}$ starting at x_0 with $\alpha(x_0, Tx_0) \geq 1$, we have*

$$U(\alpha, Tx_{n+1}) \subseteq U(\alpha, Tx_n) \subseteq U(\alpha, x_{n+1}) \subseteq U(\alpha, x_n), \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Moreover, if $\{x_n\}$ has two subsequences that τ-converge to z and w, respectively, then $z = w$.

Proof. First, recall that since α is reflexive, we have $\alpha(x, x) \geq 1$ for all $x \in X$, which implies $x \in U(\alpha, x)$ and $x \in L(\alpha, x)$. Moreover, by definition, for any $x, y \in C$ with $\alpha(x, y) \geq 1$, we have $y \in U(\alpha, x)$ and $x \in L(\alpha, y)$. Since the upper and lower level sets are convex, we deduce that

$$\lambda x + (1 - \lambda) y \in U(\alpha, x) \cap L(\alpha, y), \quad \forall \lambda \in [0, 1]. \tag{3.2}$$

We now show by induction that $\alpha(x_n, Tx_n) \geq 1$ for all $n \in \mathbb{N}$.
Base case: By the hypothesis, we have $\alpha(x_0, Tx_0) \geq 1$.
Inductive step: Assume that $\alpha(x_m, Tx_m) \geq 1$ for some $m \geq 0$. Then, by (3.2), we have

$$x_{m+1} = \lambda x_m + (1 - \lambda) Tx_m \in U(\alpha, x_m) \cap L(\alpha, Tx_m).$$

This means $x_{m+1} \in U(\alpha, x_m)$, and $Tx_m \in U(\alpha, x_{m+1})$ (since $x_{m+1} \in L(\alpha, Tx_m)$).

Then, using the nestedness of α, we obtain

$$U(\alpha, Tx_m) \subseteq U(\alpha, x_{m+1}) \subseteq U(\alpha, x_m).$$

Moreover, since T is α-admissible and $\alpha(x_m, Tx_m) \geq 1$, we have $\alpha(Tx_m, Tx_{m+1}) \geq 1$, implying $Tx_{m+1} \in U(\alpha, Tx_m)$. But since $U(\alpha, Tx_m) \subseteq U(\alpha, x_{m+1})$, it follows that:

$$Tx_{m+1} \in U(\alpha, x_{m+1}) \Rightarrow \alpha(x_{m+1}, Tx_{m+1}) \geq 1.$$

Thus, by induction, we have $\alpha(x_n, Tx_n) \geq 1$ for all $n \in \mathbb{N}$. Applying (3.2) again, and using the same reasoning, we get the nested inclusions:

$$U(\alpha, Tx_{n+1}) \subseteq U(\alpha, Tx_n) \subseteq U(\alpha, x_{n+1}) \subseteq U(\alpha, x_n), \quad \forall n.$$

Now suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ that τ-converges to $z \in X$. Since the sets $U(\alpha, x_n)$ form a decreasing sequence of closed sets, and each neighborhood $N_z \in \tau$ contains infinitely many x_n, we conclude that

$$z \in \bigcap_n U(\alpha, x_n),$$

i.e., $z \in U(\alpha, x_n)$ for all n, which implies $\alpha(x_n, z) \geq 1$, or equivalently, $x_n \in L(\alpha, z)$ for all n.

Let now w be the τ-limit of another subsequence $\{x_{m_k}\}$ of $\{x_n\}$. Since $x_{m_k} \in L(\alpha, z)$ and $L(\alpha, z)$ is τ-closed, we get $w \in L(\alpha, z)$, hence $\alpha(w, z) \geq 1$.

Repeating the same argument with the roles of z and w reversed, we obtain $\alpha(z, w) \geq 1$. Then, by the equivalence property of α, we conclude $z = w$. $\square$

Remark 3.2 Under the assumptions of Lemma 3.1, if we assume $\alpha(Tx_0, x_0) \geq 1$, then we will have

$$U(\alpha, x_n) \subseteq U(\alpha, x_{n+1}) \subseteq U(\alpha, Tx_n) \subseteq U(\alpha, Tx_{n+1})$$

for any $n \in \mathbb{N}$.

Lemma 3.3 *Under the assumptions of Lemma 3.1, for any $i, n \in \mathbb{N}$, we have*

$$(1+n(1-\lambda)) \|Tx_i - x_i\| \leq \|Tx_{n+i} - x_i\| + \lambda^{-n}(\|Tx_i - x_i\| - \|Tx_{n+i} - x_{n+i}\|). \tag{3.3}$$

As a consequence, it follows that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Proof. We begin by recalling that the iteration sequence $\{x_n\}$ is defined by

$$x_{n+1} = \lambda x_n + (1-\lambda)Tx_n.$$

Let us examine the difference between successive iterates:

$$\|x_{n+2} - x_{n+1}\| = \|\lambda(x_{n+1} - x_n) + (1-\lambda)(Tx_{n+1} - Tx_n)\|.$$

Using the triangle inequality, we get

$$\|x_{n+2} - x_{n+1}\| \leq \lambda\|x_{n+1} - x_n\| + (1-\lambda)\|Tx_{n+1} - Tx_n\|.$$

Since T is α-nonexpansive and $\alpha(x_n, Tx_n) \geq 1$, from Lemma 3 we know that

$$\|Tx_{n+1} - Tx_n\| \leq \|x_{n+1} - x_n\|.$$

Hence,

$$\|x_{n+2} - x_{n+1}\| \leq \lambda\|x_{n+1} - x_n\| + (1-\lambda)\|x_{n+1} - x_n\| = \|x_{n+1} - x_n\|.$$

This shows that the sequence $\{\|x_{n+1} - x_n\|\}$ is non-increasing.

Now, we can write the inequality (3.3) as follows:

$$\|Tx_{n+i} - x_i\| \geq \lambda^{-n}(\|Tx_{n+i} - x_{n+i}\| - \|Tx_i - x_i\|) + (1+n(1-\lambda)) \|Tx_i - x_i\|. \tag{3.4}$$

To show accuracy the inequality (3.4), we will use the mathematical induction on the index n. First note that, since

$$\|Tx_i - x_i\| \geq \|Tx_i - x_i\| - \|Tx_i - x_i\| + \|Tx_i - x_i\|,$$

then (3.4) is true for $n = 0$ and for all $i \in \mathbb{N}$. Now, assume (3.4) is true for some $n \in \mathbb{N}$ and for all $i \in \mathbb{N}$, and prove it is true for $n+1$. In this case by taking $i+1$ instead of i, we have

$$\begin{aligned} \|Tx_{n+i+1} - x_{i+1}\| &\geq \lambda^{-n}(\|Tx_{n+i+1} - x_{n+i+1}\| - \|Tx_{i+1} - x_{i+1}\|) \\ &\quad +(1+n(1-\lambda))\|Tx_{i+1} - x_{i+1}\| \end{aligned} \tag{3.5}$$

for some $n \in \mathbb{N}$. On the other hand, since T is α-nonexpansive, we have

$$\begin{aligned} \|Tx_{n+i+1} - x_{i+1}\| &= \|Tx_{n+i+1} - (\lambda x_i + (1-\lambda)Tx_i)\| \\ &= \|(1-\lambda+\lambda)Tx_{n+i+1} - (\lambda x_i + (1-\lambda)Tx_i)\| \\ &= \|(1-\lambda)Tx_{n+i+1} + \lambda Tx_{n+i+1} - \lambda x_i - (1-\lambda)Tx_i\| \\ &= \|(1-\lambda)(Tx_{n+i+1} - Tx_i) + \lambda(Tx_{n+i+1} - x_i)\| \\ &\leq (1-\lambda)\|Tx_{n+i+1} - Tx_i\| + \lambda\|Tx_{n+i+1} - x_i\| \\ &\leq (1-\lambda)\left\{ \begin{array}{l} \|Tx_{n+i+1} - Tx_{n+i}\| + \|Tx_{n+i} - Tx_{n+i-1}\| \\ + \cdots + \|Tx_{i+1} - Tx_i\| \end{array} \right\} \\ &\quad +\lambda\|Tx_{n+i+1} - x_i\| \\ &\leq \lambda\|Tx_{n+i+1} - x_i\| + (1-\lambda)\sum\nolimits_{k=0}^{n}\|Tx_{i+k+1} - Tx_{i+k}\| \\ &\leq \lambda\|Tx_{n+i+1} - x_i\| + (1-\lambda)\sum\nolimits_{k=0}^{n}\|x_{i+k+1} - x_{i+k}\|. \end{aligned} \tag{3.6}$$

Now from (3.6), we have

$$\lambda\|Tx_{n+i+1} - x_i\| \geq \|Tx_{n+i+1} - x_{i+1}\| - (1-\lambda)\sum\nolimits_{k=0}^{n}\|x_{i+k+1} - x_{i+k}\|,$$

and taking into account (3.5) we get

$$\begin{aligned} \lambda\|Tx_{n+i+1} - x_i\| &\geq \lambda^{-n}(\|Tx_{n+i+1} - x_{n+i+1}\| - \|Tx_{i+1} - x_{i+1}\|) \\ &\quad +(1+n(1-\lambda))\|Tx_{i+1} - x_{i+1}\| \\ &\quad -(1-\lambda)\sum\nolimits_{k=0}^{n}\|x_{i+k+1} - x_{i+k}\|. \end{aligned}$$

Dividing both sides of this inequality by λ we get

$$\begin{aligned} \|Tx_{n+i+1} - x_i\| &\geq \lambda^{-(n+1)}(\|Tx_{n+i+1} - x_{n+i+1}\| - \|Tx_{i+1} - x_{i+1}\|) \\ &\quad +(1+n(1-\lambda))\lambda^{-1}\|Tx_{i+1} - x_{i+1}\| \\ &\quad -(1-\lambda)\lambda^{-1}\sum\nolimits_{k=0}^{n}\|x_{i+k+1} - x_{i+k}\|. \end{aligned}$$

Finally, since the inequality $1+n(1-\lambda) \leq \lambda^{-n}$ is true and the sequence $\{\|Tx_n - x_n\|\}$ is nonincreasing we have

$$\begin{aligned}
\|Tx_{n+i+1} - x_i\| &\geq \lambda^{-(n+1)}(\|Tx_{n+i+1} - x_{n+i+1}\| - \|Tx_{i+1} - x_{i+1}\|) \\
&\quad +(1+n(1-\lambda))\lambda^{-1}\|Tx_{i+1} - x_{i+1}\| \\
&\quad -(1-\lambda)\lambda^{-1}\sum_{k=0}^{n}\|x_{i+k+1} - x_{i+k}\| \\
&\geq \lambda^{-(n+1)}(\|Tx_{n+i+1} - x_{n+i+1}\| - \|Tx_{i+1} - x_{i+1}\|) \\
&\quad +(1+n(1-\lambda))\lambda^{-1}\|Tx_{i+1} - x_{i+1}\| \\
&\quad -(1-\lambda)^2\lambda^{-1}(n+1)\|Tx_i - x_i\| \\
&= \lambda^{-(n+1)}\{\|Tx_{n+i+1} - x_{n+i+1}\| - \|Tx_{i+1} - x_{i+1}\| \\
&\quad -\|Tx_i - x_i\| + \|Tx_i - x_i\|\} \\
&\quad +(1+n(1-\lambda))\lambda^{-1}\|Tx_{i+1} - x_{i+1}\| \\
&\quad -(1-\lambda)^2\lambda^{-1}(n+1)\|Tx_i - x_i\| \\
&= \lambda^{-(n+1)}\{\|Tx_{n+i+1} - x_{n+i+1}\| - \|Tx_i - x_i\|\} \\
&\quad +[(1+n(1-\lambda))\lambda^{-1} - \lambda^{-(n+1)}]\|Tx_{i+1} - x_{i+1}\| \\
&\quad +[\lambda^{-(n+1)} - (1-\lambda)^2\lambda^{-1}(n+1)]\|Tx_i - x_i\| \\
&\geq \lambda^{-(n+1)}\{\|Tx_{n+i+1} - x_{n+i+1}\| - \|Tx_i - x_i\|\} \\
&\quad +[(1+n(1-\lambda))\lambda^{-1} - \lambda^{-(n+1)}]\|Tx_i - x_i\| \\
&\quad +[\lambda^{-(n+1)} - (1-\lambda)^2\lambda^{-1}(n+1)]\|Tx_i - x_i\| \\
&= \lambda^{-(n+1)}\{\|Tx_{n+i+1} - x_{n+i+1}\| - \|Tx_i - x_i\|\} \\
&\quad +(1+(n+1)(1-\lambda))\|Tx_i - x_i\|.
\end{aligned}$$

This shows that (3.4) is true for $(n+1)$. Now assume $\lim_{n\to\infty}\|x_n - Tx_n\| = R$. Then by taking limit as $i \to \infty$ in (3.4) we have

$$\delta(C) \geq (1+n(1-\lambda))R$$

for all $n \in \mathbb{N}$, where

$$\delta(C) = \sup\{\|x-y\| : x, y \in C\}.$$

Hence, we get

$$R \leq \frac{\delta(C)}{(1+n(1-\lambda))}$$

for all $n \in \mathbb{N}$. This shows that $R = 0$, that is, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. □

Definition 3.4 [5] Let $(X, \|\cdot\|)$ be a normed space, τ be a Hausdorff topology on X. Then X is said to have τ-Opial condition if for any sequence $\{x_n\}$ in X which τ-converges to x, we have

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - z\|$$

for any $z \in X$ with $z \neq x$.

Now we are ready to state the main result for α-nonexpansive mapping.

Theorem 3.5 *Let $(X, \|\cdot\|)$ be a Banach space, τ be a Hausdorff topology on X such that X satisfies the τ-Opial condition. Let $\alpha : X \times X \to \mathbb{R}$ be a reflexive function which has nested and equivalent properties such that upper and lower level sets are convex and τ-closed. Let C be a nonempty, convex, bounded and τ-compact subset of X and $T : C \to C$ be an α-nonexpansive mapping. If there exists $x_0 \in C$ such that $Tx_0 \in U(\alpha, x_0) \cup L(\alpha, x_0)$, then T has a fixed point.*

Proof. Let $x_0 \in C$ such that $Tx_0 \in U(\alpha, x_0) \cup L(\alpha, x_0)$. Without loss of generality, assume $Tx_0 \in U(\alpha, x_0)$. Consider the Krasnoselskii iteration sequence $\{x_n\}$ which starts at x_0 in C defined by $x_{n+1} = \lambda x_n + (1 - \lambda)Tx_n$ for $n \in \mathbb{N}$. Since C is τ-compact, then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which τ-converges to some point $z \in C$. Therefore, Lemma 3.1 implies $\{x_n\}$ is also τ-converges to z and $z \in U(\alpha, x_n)$ for any $n \in \mathbb{N}$. Consider the type function

$$r(x) = \limsup_{n\to\infty} \|x_n - x\|$$

for $x \in C$. Then, Lemma 3.3 implies

$$r(x) = \limsup_{n\to\infty} \|Tx_n - x\|$$

for any $x \in C$. Since T is α-nonexpansive, we have

$$\begin{aligned} r(Tz) &= \limsup_{n\to\infty} \|x_n - Tz\| \\ &= \limsup_{n\to\infty} \|Tx_n - Tz\| \\ &\leq \limsup_{n\to\infty} \|x_n - z\| \\ &= r(z). \end{aligned}$$

Finally, since X satisfies the τ-Opial condition, then we have $Tz = z$. □

In Theorem 3.5, by taking the α function as in (1.1), we can obtain the following corollary, which is actually the main theorem of [5]. Recall that an order interval in a partially ordered space $(X, \preceq)$ is any of the subsets

$$\begin{aligned} &[a, b] = \{x \in X : a \preceq x \preceq b\} \\ &[a, \to) = \{x \in X : a \preceq x\} \\ &(\leftarrow, b] = \{x \in X : x \preceq b\} \end{aligned}$$

for any $a, b \in X$.

Corollary 3.6 *Let $(X, \|\cdot\|)$ be a Banach space, τ be a Hausdorff topology on X such that X satisfies the τ-Opial condition. Let $\preceq$ be a partial order on X such that order intervals are convex and τ -closed. Let C be a nonempty, convex, bounded and τ-compact subset of X and $T : C \to C$ be a monotone nonexpansive mapping. If there exists $x_0 \in C$ such that x_0 and Tx_0 are comparable. Then T has a fixed point.*

4. α-Asymptotically Nonexpansive Mappings

In this section we will present a fixed theorem for α-asymptotically nonexpansive mapping. For this we need the following technical lemma.

Lemma 4.1 [2] *Let C be a nonempty closed convex subset of uniformly convex Banach space $(X, \|\cdot\|)$. Let $\tau : C \to [0, \infty)$ be a type function. Then τ has a unique minimum point $z \in C$ such that*

$$\tau(z) = \inf\{\tau(x) : x \in C\} = \tau_0.$$

Moreover, if $\{z_n\}$ is a minimizing sequence in C, the $\{z_n\}$ converges strongly to z.

Theorem 4.2 *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and $\alpha : X \times X \to \mathbb{R}$ be a function having nested property such that upper and lower level sets are closed and convex. Let C be a nonempty closed, convex and bounded subset of X and $T : C \to C$ be a continuous α-asymptotically nonexpansive mapping. If there exists $x_0 \in C$ such that $Tx_0 \in U(\alpha, x_0) \cup L(\alpha, x_0)$, then T has a fixed point.*

Proof. Let $x_0 \in C$ be such that $Tx_0 \in U(\alpha, x_0) \cup L(\alpha, x_0)$. Without loss of generality, assume $Tx_0 \in U(\alpha, x_0)$. Since T is α-admissible, we have $T^{n+1}x_0 \in U(\alpha, T^n x_0)$ for every $n \in \mathbb{N}$. Since α has nested property we have $U(\alpha, T^{n+1}x_0) \subseteq U(\alpha, T^n x_0)$ and they are also closed, convex and X is reflexive, we conclude that

$$C_\alpha = \bigcap_{n \geq 0} U(\alpha, T^n x_0) \cap C = \bigcap_{n \geq 0} \{x \in C : \alpha(T^n x_0, x) \geq 1\} \neq \emptyset.$$

Let $x \in C_\alpha$, then $x \in U(\alpha, T^n x_0)$ for all $n \geq 0$ and so $\alpha(T^n x_0, x) \geq 1$. Since T is α-admissible we have $\alpha(T^{n+1}x_0, Tx) \geq 1$, that is, we have $Tx \in U(\alpha, T^{n+1}x_0) \subseteq U(\alpha, T^n x_0)$ for all $n \geq 0$. This shows that $Tx \in C_\alpha$ and so $T(C_\alpha) \subseteq C_\alpha$. Now consider the type function $\tau : C_\alpha \to [0, \infty)$ generated by $\{T^n x_0\}$, that is,

$$\tau(x) = \limsup_{n\to\infty} \|T^n x_0 - x\|.$$

Considering Lemma 4.1 we have a unique $z \in C_\alpha$ such that

$$\tau(z) = \inf\{\tau(x) : x \in C_\alpha\} = \tau_0.$$

Since $z \in C_\alpha$, we have $T^p z \in C_\alpha$ for every $p \in \mathbb{N}$. Let $p, n \in \mathbb{N}$ with $n > p$. Then, since $z \in C_\alpha$, we have $z \in U(\alpha, T^{n-p}x_0)$ and so $\alpha(T^{n-p}x_0, z) \geq 1$. Since T is α-asymptotically nonexpansive we have

$$\|T^n x_0 - T^p z\| = \left\|T^p T^{n-p} x_0 - T^p z\right\| \leq K_p \left\|T^{n-p} x_0 - z\right\|, \tag{4.1}$$

where $\{K_p\}$ is given in Definition 2.1 such that $\lim_{p\to\infty} K_p = 1$. Therefore using (4.1) we have

$$\begin{aligned}
\tau_0 &\leq \tau(T^p z) \\
&= \limsup_{n\to\infty} \|T^n x_0 - T^p z\| \\
&\leq K_p \limsup_{n\to\infty} \left\|T^{n-p} x_0 - z\right\| \\
&= K_p \limsup_{n\to\infty} \|T^n x_0 - z\| \\
&= K_p \tau(z) = K_p \tau_0.
\end{aligned} \tag{4.2}$$

Hence from (4.2) we have

$$\lim_{p\to\infty} \tau(T^p z) = \tau_0.$$

This shows that $\{T^p z\}$ is a minimizing sequence of τ. Using Lemma 4.1, we obtain that $\{T^p z\}$ converges to z. Since T is continuous we have $Tz = z$. □

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