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EXISTENCE OF SOLUTIONS TO A FOURTH ORDER $\phi-$LAPLACIAN BVP ON THE HALF-LINE USING NONORDERED UPPER AND LOWER SOLUTIONS

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Abstract. In this paper, we consider the following fourth-order boundary value problem on the half line of Sturm-Liouville type associated with a $\phi-$Laplacian operator

$$\begin{cases} (\phi(u'''))'(t) = g(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0, +\infty), \\ u(0) = A, u'(0) = B, \quad u''(0) + au'''(0) = C, \quad u'''(+\infty) = D. \end{cases}$$

The existence of solutions is obtained by using the Schauder fixed point theorem together with nonordered lower and upper solution method.

Key Words and Phrases: Boundary value problem, differential equations, one-sided Nagumo condition, lower and upper solutions, a priori estimates.

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1. Introduction

The present paper is concerned with the existence of solutions to the following fourth order $\phi-$Laplacian equation:

$$(\phi(u'''))'(t) = g(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0, +\infty), \tag{1.1}$$

subject to the boundary conditions

$$u(0) = A, u'(0) = B, \quad u''(0) + au'''(0) = C, \quad u'''(+\infty) = D, \tag{1.2}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism map such that $\phi(0) = 0$. The nonlinearity $g : [0, +\infty) \times \mathbb{R}^4 \to \mathbb{R}$ is an L^1-Carathéodory function, A, B, C, D are real numbers and the coefficient a is negative. We note that $u'''(+\infty) := \lim_{t\to+\infty} u'''(t)$.

In the last decades, $\phi-$Laplacian equations, which is the natural generalization to problems associated to the $p-$Laplacian operator $\phi_p(t) = |t|^{p-2}t\,(p > 1)$, have been extensively studied by different authors with many different methods. The homeomorphisms ϕ is in particular motivated by the one dimensional version of mean curvature problems (or capillary surfaces) for which $\phi(v) = \dfrac{v}{\sqrt{1+v^2}}$. One of the first researcher who has studied this type of problems is D. O'Regan (see [13]).

Boundary value problems for ordinary differential equations (bvp for short) appear in several applications. They are used to describe a large number of physical problems ranging from physical sciences to engineering. Fourth order differential equations arise, e.g., in some problems of deformation of structures, elasticity, fluid dynamics, electromagnetism, quantum mechanics,... etc. This explains the great interest for this type of problems which can be set either on bounded or unbounded domains. For further reading, we refer, .e.g, to [1]-[9], [11], [12], [14]-[16] and the references therein.

The main idea of this work is to reduce the boundary value problem (1.1)-(1.2) to a fixed point problem in a suitable function space. Then, the existence of fixed point is obtained by the Schauder fixed point theorem combined with the method of nonordered lower and upper solutions. Also, an unilateral Nagumo-type condition will allow us to get a priori estimates on the solutions.

Since we consider a $\phi-$Laplacian boundary value problem on an unbounded domain, this work improves some existing results in the literature such as [4].

The paper has four sections. Section 2 is devoted to the definition of the appropriate function space, the weighted norms, and the unilateral Nagumo conditions. In Section 3, we prove the main result. Finally, in Section 4, we provide an example showing the applicability of the existence theorem.

2. Definitions and preliminary results

Consider the set

$$X = \left\{x \in C^3[0, +\infty) : \lim_{t\to+\infty} x'''(t) \text{ exists in } \mathbb{R}\right\}$$

and the norme $\|x\|_X := \max\{\|x\|_0, \|x'\|_1, \|x''\|_2, \|x'''\|_3\}$, where

$$\|x^{(i)}\|_i = \sup_{0\le t<+\infty} \left|\frac{x^{(i)}(t)}{1+t^{3-i}}\right|,\ i = 0, 1, 2, 3.$$

The following lemma can be easily proved using standard arguments.

Lemma 2.1 *Let $u \in X$, then*

$$\lim_{t\to+\infty} \frac{u''(t)}{1+t} = \lim_{t\to+\infty} u'''(t),\ \lim_{t\to+\infty} \frac{u'(t)}{1+t^2} = \frac{1}{2} \lim_{t\to+\infty} u'''(t)$$

and

$$\lim_{t\to+\infty} \frac{u(t)}{1+t^3} = \frac{1}{6} \lim_{t\to+\infty} u'''(t).$$

Using this lemma, we can deduce that $(X, \|.\|_X)$ is a Banach space (see [10]).

Definition 2.2 A function $g : [0, +\infty) \times \mathbb{R}^4 \to \mathbb{R}$ is called an L^1-Carathéodory function if it satisfies

(i) for each $(x, y, z, w) \in \mathbb{R}^4$, $t \mapsto g(t, x, y, z, w)$ is measurable on $[0, +\infty)$;

(ii) for almost every $t \in [0, +\infty)$, $(x, y, z, w) \mapsto g(t, x, y, z, w)$ is continuous in $\mathbb{R}^4$;

(iii) for each $\rho > 0$, there exists a positive function $\varphi_\rho \in L^1[0, +\infty)$ such that for all $(x(t), y(t), z(t), w(t)) \in \mathbb{R}^4$ with

$$\max\left\{\sup_{0\le t<+\infty}\left|\frac{x(t)}{1+t^3}\right|, \sup_{0\le t<+\infty}\left|\frac{y(t)}{1+t^2}\right|, \sup_{0\le t<+\infty}\left|\frac{z(t)}{1+t}\right|, \sup_{0\le t<+\infty}\left|\frac{w(t)}{2}\right|\right\} < \rho,$$

then

$$|g(t, x(t), y(t), z(t), w(t))| \le \varphi_\rho(t), \quad \text{a.e. } t \in [0, +\infty).$$

The next lemma will be useful for the fixed point formulation of the problem.

Lemma 2.3 *Let $\eta \in L^1[0, +\infty)$. Then, the linear boundary value problem*

$$(\phi(u'''))'(t) + \eta(t) = 0, \quad t \in [0, +\infty), \tag{2.1}$$

with boundary conditions (1.2) *has a unique solution in X. Moreover, this solution can be expressed as*

$$\begin{aligned} u(t) =& A + Bt + \frac{C - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty}\eta(s)ds\right)}{2}t^2 \\ &+ \int_0^t\left(\frac{t^2+s^2}{2} - ts\right)\phi^{-1}\left(\phi(D) + \int_s^{+\infty}\eta(\tau)d\tau\right)ds. \end{aligned} \tag{2.2}$$

Now, we define the upper and lower solutions of the problem (1.1), (1.2).

Definition 2.4 Given $a < 0$ and $A, B, C, D \in \mathbb{R}$, a function $\alpha \in X$ with $\phi(\alpha''') \in C^1[0, +\infty)$ is said to be a lower solution of problem (1.1), (1.2) if

$$(\phi(\alpha'''))'(t) \ge g(t, \overline{\alpha}(t), \alpha'(t), \alpha''(t), \alpha'''(t)), \quad t \in [0, +\infty),$$

and

$$\alpha'(0) \le B, \quad \alpha''(0) + a\ \alpha'''(0) \le C, \quad \alpha'''(+\infty) < D, \tag{2.3}$$

where $\overline{\alpha}(t) := \alpha(t) - \alpha(0) + A$.

A function β is an upper solution if it satisfies the reversed inequalities with $\overline{\beta}(t) := \beta(t) - \beta(0) + A$, where $\phi(\beta''') \in C^1([0, +\infty)$.

In order to have an *a priori* estimation for the third derivative u''', we need a growth condition on the nonlinear function.

Let $\gamma_i, \Gamma_i \in C[0, +\infty), \gamma_i(t) \le \Gamma_i(t),\ i = 0, 1, 2$ with $\lim_{t\to+\infty}\frac{\gamma_2(t)}{1+t}$ and $\lim_{t\to+\infty}\frac{\Gamma_2(t)}{1+t}$ exist in $\mathbb{R}$. Define the set

$$E = \left\{(t, x_0, x_1, x_2, x_3) \in [0, +\infty) \times \mathbb{R}^4 : \gamma_i(t) \le x_i \le \Gamma_i(t), i = 0, 1, 2\right\}.$$

Definition 2.5 We say that an L^1-Carathéodory function, $g : E \to \mathbb{R}$ satisfy the one-sided Nagumo-type growth condition in E if it satisfies either

$$g(t, x, y, z, w) \le \psi(t)h(|w|), \quad \forall (t, x, y, z, w) \in E, \tag{2.4}$$

or

$$g(t,x,y,z,w) \geq -\psi(t)h(|w|), \quad \forall (t,x,y,z,w) \in E, \tag{2.5}$$

for some positive continuous functions ψ, h and some $\nu > 1$ such that

$$\begin{gathered} \sup_{0 \leq t < +\infty} \psi(t)(1+t)^{\nu} < +\infty, \int^{+\infty} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))} ds = +\infty, \\ \int^{+\infty} \frac{\phi^{-1}(-s)}{h(|\phi^{-1}(-s)|)} ds = -\infty. \end{gathered} \tag{2.6}$$

Remark 2.6 We want to point out that the condition

$$\int_0^{+\infty} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))} ds = +\infty$$

does not imply that

$$\int_0^{+\infty} \frac{\phi^{-1}(-s)}{h(|\phi^{-1}(-s)|)} ds = -\infty.$$

In fact, if we consider the functions ϕ and h as

$$\phi(x) = \begin{cases} x^3, \; x \geq 0 \\ x^{1/7}, \; x \leq 0 \end{cases} \qquad \text{and } h(x) = x^2 + 1.$$

We have

$$\int_0^{+\infty} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))} ds = \int_0^{+\infty} \frac{s^{1/3}}{s^{2/3}+1} = +\infty,$$

while

$$\int_0^{+\infty} \frac{\phi^{-1}(-s)}{h(|\phi^{-1}(-s)|)} ds = \int_0^{+\infty} -\frac{s^7}{s^{14}+1} ds \cong -0.23.$$

Next lemma provides an *a priori* bound estimation for u'''.

Lemma 2.7 *Let $g : [0,+\infty) \times \mathbb{R}^4 \to \mathbb{R}$ be an* L^1-Carathéodory *function satisfying* (2.4) *and* (2.6), *or* (2.5) *and* (2.6) *in E. Then, for every $\rho > 0$, there exists $N > 0$ such that every solution noted u of* (1.1), (1.2) *such that*

$$\gamma_0(t) \leq u(t) \leq \Gamma_0(t), \gamma_1(t) \leq u'(t) \leq \Gamma_1(t), \gamma_2(t) \leq u''(t) \leq \Gamma_2(t), \tag{2.7}$$

for $t \in [0,+\infty)$, satisfies $\|u'''\|_3 < N$. Note that N do not depend on the solution u.

Proof. Let u be a solution of (1.1), (1.2) such that (2.7) holds. Let $\rho > 0$ be such that

$$\rho > \max\Big\{ \Big|\frac{C-\Gamma_2(0)}{a}\Big|, \Big|\frac{C-\gamma_2(0)}{a}\Big|, |D| \Big\}. \tag{2.8}$$

By this inequality we cannot have $|u'''(t)| > \rho$ for all $t \in [0,+\infty)$, because

$$|u'''(0)| = \Big|\frac{C-u''(0)}{a}\Big| \leq \max\Big(\Big|\frac{C-\Gamma_2(0)}{a}\Big|, \Big|\frac{C-\gamma_2(0)}{a}\Big| \Big) < \rho \tag{2.9}$$

and $|u'''(+\infty)| = |D| < \rho$.

If $|u'''(t)| \leq \rho$ for all $t \in [0,+\infty)$, to complete the proof it is sufficient to take $N > \rho/2$ since

$$\|u'''\|_3 = \sup_{0 \leq t < +\infty} |\frac{u'''(t)}{2}| \leq \frac{\rho}{2} < N.$$

If there exists $t \in (0, +\infty)$ such that $|u'''(t)| > \rho$, then by (2.6) we can consider $N > \rho$ such that

$$\int_{\phi(\rho)}^{\phi(N)} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))} ds > M\Big(M_1 + \sup_{0 \le t < +\infty} \frac{|\Gamma_2(t)|}{1+t} \frac{\nu}{\nu - 1}\Big)$$

and

$$\int_{\phi(-N)}^{\phi(-\rho)} \frac{\phi^{-1}(s)}{h(|\phi^{-1}(s)|)} ds < M\Big(-M_1 + \inf_{0 \le t < +\infty} \frac{-|\gamma_2(t)|}{1+t} \frac{\nu}{\nu - 1}\Big)$$

with

$$M := \sup_{0 \le t < +\infty} \psi(t)(1+t)^{\nu}$$

and

$$M_1 := \sup_{0 \le t < +\infty} \frac{\Gamma_2(t)}{(1+t)^{\nu}} - \inf_{0 \le t < +\infty} \frac{\gamma_2(t)}{(1+t)^{\nu}}.$$

In this proof we assume that the function g satisfy the growth condition (2.4). (If g satisfy the growth condition (2.5) then the result follow in the same way).

By (2.8), suppose that there exist positive numbers t_* and t_+ such that $u'''(t_*) = \rho$, $u'''(t) > \rho$ for all $t \in (t_*, t_+]$. Then,

$$\begin{aligned}
\int_{\phi(u'''(t_*))}^{\phi(u'''(t_+))} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))} ds &= \int_{t_*}^{t_+} \frac{u'''(s)}{h(u'''(s))} (\phi(u'''))'(s) ds \\
&= \int_{t_*}^{t_+} \frac{g(s, u(s), u'(s), u''(s), u'''(s))}{h(u'''(s))} u'''(s) ds \\
&\le \int_{t_*}^{t_+} \psi(s)\ u'''(s)\ ds \le M \int_{t_*}^{t_+} \frac{u'''(s)}{(1+s)^{\nu}}\ ds \\
&= M \int_{t_*}^{t_+} \Big(\frac{u''(s)}{(1+s)^{\nu}}\Big)' + \frac{\nu u''(s)}{(1+s)^{1+\nu}} ds \\
&= M\Big(\frac{u''(t_+)}{(1+t_+)^{\nu}} - \frac{u''(t_*)}{(1+t_*)^{\nu}} + \int_{t_*}^{t_+} \frac{\nu u''(s)}{(1+s)^{1+\nu}} ds\Big) \\
&\le M\Big(M_1 + \sup_{0 \le t < +\infty} \frac{|\Gamma_2(t)|}{1+t} \int_0^{+\infty} \frac{\nu}{(1+s)^{\nu}} ds\Big) \\
&< \int_{\phi(\rho)}^{\phi(N)} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))} ds.
\end{aligned}$$

Since $u'''(t_+) < N$ and as t_*, t_+ are arbitrary in $(0, +\infty)$, we have $u'''(t) < N$ for all $t \in [0, +\infty)$. By (2.8), suppose that there exist t_* and $t_- \in (0, +\infty)$ such that

$u'''(t_*) = -\rho$, $u'''(t) < -\rho$ for all $t \in [t_-, t_*)$. Then,

$$\begin{aligned}
\int_{\phi(u'''(t_-))}^{\phi(u'''(t_*))} \frac{\phi^{-1}(s)}{h(|\phi^{-1}(s)|)} ds &= \int_{t_-}^{t_*} \frac{u'''(s)}{h(|u'''(s)|)} (\phi(u'''))'(s) ds \\
&= \int_{t_-}^{t_*} \frac{g(s, u(s), u'(s), u''(s), u'''(s))}{h(|u'''(s)|)} u''(s) ds \\
&\geq \int_{t_-}^{t_*} \psi(s) u'''(s) ds \geq M \int_{t_-}^{t_*} \frac{u'''(s)}{(1+s)^\nu} ds \\
&= M \int_{t_-}^{t_*} \Big(\frac{u''(s)}{(1+s)^\nu}\Big)' + \frac{\nu u''(s)}{(1+s)^{1+\nu}} ds \\
&= M \Big(\frac{u''(t_*)}{(1+t_*)^\nu} - \frac{u''(t_-)}{(1+t_-)^\nu} + \int_{t_-}^{t_*} \frac{\nu u''(s)}{(1+s)^{1+\nu}} ds \Big) \\
&\geq M \Big(-M_1 + \inf_{0 \leq t < +\infty} \frac{-|\gamma_2(t)|}{1+t} \int_0^{+\infty} \frac{\nu}{(1+s)^\nu} ds \Big) \\
&> \int_{\phi(-N)}^{\phi(-\rho)} \frac{\phi^{-1}(s)}{h(|\phi^{-1}(s)|)} ds.
\end{aligned}$$

Since $u'''(t_-) > -N$ and as t_*, t_- are arbitrary in $(0, +\infty)$, we have $u'''(t) > -N$ for all $t \in [0, +\infty)$. Therefore $\|u'''\|_3 < N/2 < N$ for all $t \in [0, +\infty)$.

The next lemma concerns the fixed point operator.

Lemma 2.8 [1] *A set $U \subset X$ is relatively compact if the following three conditions hold:*

(1) *all functions from U are uniformly bounded;*
(2) *all functions from U are equicontinuous on any compact interval of $[0, +\infty)$;*
(3) *all functions from U are equiconvergent at infinity, that is, for given $\epsilon > 0$, there exists $t_\epsilon > 0$ such that*

$$\Big|\frac{x^{(i)}(t)}{1+t^{3-i}} - \lim_{t \to +\infty} \frac{x^{(i)}(t)}{1+t^{3-i}}\Big| < \epsilon, \ \textit{for all } t > t_\epsilon, x \in U \textit{ and } i = 0, 1, 2, 3.$$

We end this section with the Schauder Fixed Point Theorem.

Theorem 2.9 [17] *Let E be a Banach space, and $U \subset E$ a nonempty, bounded, closed, and convex subset of E. Let $T : U \to E$ be a completely continuous operator with $T(U) \subset U$. Then T has a fixed point.*

3. Main result

We are now in position to prove the existence of at least one solution for the problem (1.1), (1.2).

Theorem 3.1 *Let $g : [0, +\infty) \times \mathbb{R}^4 \to \mathbb{R}$ be an* L^1-Carathéodory *function, and α, β lower and upper solutions of* (1.1), (1.2)*, respectively, such that*

$$\alpha''(t) \leq \beta''(t), \quad \forall t \in [0, +\infty). \tag{3.1}$$

Assume that g satisfies the one-sided Nagumo condition (2.4)*, or* (2.5)*, in the set*

$$E_* = \Big\{(t, x, y, z, w) \in [0, +\infty) \times \mathbb{R}^4 : \overline{\alpha}(t) \le x \le \overline{\beta}(t),\ \alpha'(t) \le y \le \beta'(t),$$
$$\alpha''(t) \le z \le \beta''(t)\Big\},$$

and

$$g(t, \overline{\alpha}(t), \alpha'(t), z, w) \ge g(t, x, y, z, w) \ge g(t, \overline{\beta}(t), \beta'(t), z, w), \tag{3.2}$$

for (t, z, w) fixed and $\overline{\alpha}(t) \le x \le \overline{\beta}(t),\ \alpha'(t) \le y \le \beta'(t)$.
Then, problem (1.1)*,* (1.2) *has at least a solution $u \in X$ and there exists $R > 0$ such that*

$$\overline{\alpha}(t) \le u(t) \le \overline{\beta}(t), \alpha'(t) \le u'(t) \le \beta'(t),$$
$$\alpha''(t) \le u''(t) \le \beta''(t), |u'''(t)| < R, \quad \forall t \in [0, +\infty).$$

Proof. Integrating (3.1) and using (2.3), we have $\alpha'(t) \le \beta'(t)$ and $\overline{\alpha}(t) \le \overline{\beta}(t)$, for $t \in [0, +\infty)$. Therefore, we can consider the i-modified and perturbed equation

$$\begin{aligned}(\phi(u'''))'(t) &= g(t, \delta_0(t, u), \delta_1(t, u'), \delta_2(t, u''), \delta_{3i}(t, u'''(t))) \\ &\quad + \frac{1}{1+t^2}\frac{u''(t) - \delta_2(t, u'')}{1 + |u''(t) - \delta_2(t, u'')|}, \quad t \in [0, +\infty),\end{aligned} \tag{3.3}$$

where the functions $\delta_j, \delta_{3i} : [0, +\infty) \times \mathbb{R} \to \mathbb{R}, j = 0, 1, 2,\ i = 1, 2$ are given by

$$\delta_0(t, x) = \begin{cases} \overline{\beta}(t), & x > \overline{\beta}(t), \\ x, & \overline{\alpha}(t) \le x \le \overline{\beta}(t), \\ \overline{\alpha}(t), & x < \overline{\alpha}(t), \end{cases}$$

$$\delta_j(t, y_j) = \begin{cases} \beta^{(j)}(t), & y_j > \beta^{(j)}(t), \\ y_j, & \alpha^{(j)}(t) \le y_j \le \beta^{(j)}(t), \\ \alpha^{(j)}(t), & y_j < \alpha^{(j)}(t), \end{cases} \qquad j = 1, 2,$$

$$\delta_{31}(t, w) = \begin{cases} N, & w > N, \\ w, & -N \le w \le N, \\ -N, & w < -N, \end{cases}$$

where $N > \max\{\sup_{0 \le t < +\infty} |\alpha'''(t)|, \sup_{0 \le t < +\infty} |\beta'''(t)|\}$, and

$$\delta_{32}(t, w) = w.$$

For a better understanding, we divide the proof into several steps.

Step 1: Every solution of (3.3), (1.2) satisfies $\alpha''(t) \le u''(t) \le \beta''(t)$ for all $t \in [0, +\infty)$. Let u_i be a solution of the i-modified problem (3.3), (1.2) for $i = 1, 2$ and suppose, by contradiction, that there exists $t \in (0, +\infty)$ such that $\alpha''(t) > u_i''(t)$. Therefore,

$$\inf_{0 \le t < +\infty} (u_i''(t) - \alpha''(t)) < 0.$$

By (2.3) this infimum cannot be attained at $+\infty$. In fact,

$$\inf_{0\le t<+\infty}(u_i''(t)-\alpha''(t)) := u_i''(+\infty)-\alpha''(+\infty)<0$$

and

$$u_i'''(+\infty)-\alpha'''(+\infty)\le 0.$$

We reach the contradiction

$$0\ge u_i'''(+\infty)-\alpha'''(+\infty)>D-D=0.$$

If

$$\inf_{0\le t<+\infty}(u_i''(t)-\alpha''(t)) := u_i''(0^+)-\alpha''(0^+)<0,$$

then again a contradiction is reached

$$\begin{aligned} 0\le u_i'''(0^+)-\alpha'''(0^+) &\le \frac{C-u_i''(0)}{a}+\frac{\alpha''(0)-C}{a}\\ &=-\frac{1}{a}(u_i''(0)-\alpha''(0))<0. \end{aligned}$$

If there is $t_*\in(0,+\infty)$ then, define

$$\min_{0\le t<+\infty}(u_i''(t)-\alpha''(t)) := u_i''(t_*)-\alpha''(t_*)<0,$$

Then there exists $t_*<\bar{t}$, such that

$$u_i''(t)-\alpha''(t)<0,\ u_i'''(t)-\alpha'''(t)\ge 0,\ \text{for all}\, t\in(t_*,\bar{t}).$$

Therefore by (3.2) and Definition 2.4, we get the contradiction for $i=1,2$

$$\begin{aligned} (\phi(u_i'''))'(t)-(\phi(\alpha'''))'(t) &= g(t,\delta_0(t,u_i(t)),\delta_1(t,u_i'(t)),\delta_2(t,u_i''(t)),\delta_{3i}(t,u_i'''(t))\\ &\quad+\frac{1}{1+t^2}\frac{u_i''(t)-\delta_2(t,u_i''(t))}{1+|u_i''(t)-\delta_2(t,u_i''(t))|}-(\phi(\alpha'''))'(t)\\ &= g(t,\delta_0(t,u_i(t)),\delta_1(t,u_i'(t)),\alpha''(t),\alpha'''(t))\\ &\quad+\frac{1}{1+t^2}\frac{u_i''(t)-\alpha''(t)}{1+|u_i''(t)-\alpha''(t)|}-(\phi(\alpha'''))'(t)\\ &\le \frac{1}{1+t^2}\frac{u_i''(t)-\alpha''(t)}{1+|u_i''(t)-\alpha''(t)|}<0,\ a.e.,\ t\in(t_*,\bar{t}) \end{aligned}$$

Hence the function $\phi(u_i'''(t))-\phi(\alpha'''(t))$ is decreasing for all $t\in(t_*,\ \bar{t})$. If $t\in(t_*,\ \bar{t})$, then,

$$0=\phi(u_i'''(t_*))-\phi(\alpha'''(t_*))>\phi(u_i'''(t))-\phi(\alpha'''(t))$$

and $u_i'''(t)-\alpha'''(t)<0$. Therefore $u_i''(t)-\alpha''(t)$ is decreasing in $(t_*,\ \bar{t})$, which is a contradiction.

So $u_i''(t)\ge\alpha''(t),\forall t\in[0,+\infty)$. Analogously it can be shown that $u_i''(t)\le\beta''(t),\forall t\in[0,+\infty)$ for $i=1,2$. Hence, $\alpha''(t)\le u_i''(t)\le\beta''(t),\forall t\in[0,+\infty),\ i=1,2.$

As $\alpha'(0) \leq B \leq \beta'(0)$ and $u_i'(0) = B$, integrating on $[0,t]$ for $i = 1,2$, we get

$$\alpha'(t) - \alpha'(0) = \int_0^t \alpha''(s)ds \leq \int_0^t u_i''(s)ds = u_i'(t) - B \leq \int_0^t \beta''(s)ds = \beta'(t) - \beta'(0),$$
$$\alpha'(t) \leq \alpha'(t) - \alpha'(0) + B \leq u_i'(t) \leq \beta'(t) - \beta'(0) + B \leq \beta'(t),$$
$$\alpha(t) - \alpha(0) = \int_0^t \alpha'(s)ds \leq \int_0^t u_i'(s)ds = u_i(t) - A \leq \int_0^t \beta'(s)ds = \beta(t) - \beta(0),$$
$$\overline{\alpha}(t) \leq u_i(t) \leq \overline{\beta}(t),\ i = 1,2.$$

Step 2: If u is a solution of the 2-modified problem (3.3), (1.2), then u is a solution for the initial problem. By Lemma 2.7, there exists $R_1 > 0$ not depending on u such that

$$\|u'''\|_3 < R_1.$$

Now, we need to consider $N = N_1$, where

$$N_1 > \max\{2R_1, \sup_{0\leq t<+\infty} |\alpha'''(t)|, \sup_{0\leq t<+\infty} |\beta'''(t)|\}.$$

If the 1-modified problem (3.3), (1.2) has a solution u, then u is solution of problem (1.1), (1.2), where

$$\|u'''\|_3 < R_1 < \frac{N_1}{2} < N_1.$$

Step 3: The 1-modified problem (3.3), (1.2) has at least one solution. Let us define the operator $T : X \to X$ by

$$\begin{aligned} Tu(t) = A + Bt + &\frac{C - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty} G(u(s))ds\right)}{2}t^2 \\ &+ \int_0^t \left(\frac{t^2+s^2}{2} - ts\right)\phi^{-1}\left(\phi(D) + \int_s^{+\infty} G(u(\tau))d\tau\right)ds, \end{aligned}$$

with

$$\begin{aligned} -G(u(\tau)) := &\, g(\tau, \delta_0(\tau,u), \delta_1(\tau,u'), \delta_2(\tau,u''), \delta_{31}(\tau,u''')) \\ &+ \frac{1}{1+\tau^2}\frac{u''(\tau) - \delta_2(\tau,u'')}{1 + |u''(\tau) - \delta_2(\tau,u'')|}. \end{aligned}$$

Using Lemma 2.3, the fixed points of T are solutions of 1-modified (3.3), (1.2). So it is sufficient to prove that T has a fixed point in X. For this purpose, it is enough prove that the operator T satisfies the conditions of the Schauder Fixed Point Theorem 2.9. The proof is split into three steps.

(i) $T : X \to X$ is well defined. Let $u \in X$. As g is an L^1-Carathéodory function, we take

$$\rho > \max\{N_1, \|\overline{\alpha}\|_0, \|\overline{\beta}\|_0, \|\alpha'\|_1, \|\beta'\|_1, \|\alpha''\|_2, \|\beta''\|_2\},$$

we obtain by Definition 2.2

$$
\begin{aligned}
\int_0^{+\infty} |G(u(s))| ds &\leq \int_0^{+\infty} \varphi_\rho(s) + \frac{1}{1+s^2} \frac{|u''(s) - \delta_2(s,u'')|}{1+|u''(s) - \delta_2(s,u'')|} ds \\
&\leq \int_0^{+\infty} \varphi_\rho(s) + \frac{1}{1+s^2} ds = M_\rho < +\infty.
\end{aligned} \tag{3.4}
$$

Hence, G is also an L^1-Carathéodory function. We have,

$$
\begin{aligned}
\lim_{t\to+\infty} (Tu)'''(t) &= \lim_{t\to+\infty} \phi^{-1}\left(\phi(D) + \int_t^{+\infty} G(u(\tau))d\tau\right) \\
&= D.
\end{aligned}
$$

Therefore, $Tu \in X$.

(ii) T is continuous. Let $(u_n) \subset X$, such that $u_n \to u$ in X. There exists $r > 0$ such that

$$\|u_n\|_X < r, \ \forall n \in \mathbb{N}.$$

We have to prove that

$$\|Tu_n - Tu\|_X \overset{n\to+\infty}{\longrightarrow} 0.$$

To this end, we can see that

$$\|(Tu_n)''' - (Tu)'''\|_3 \overset{n\to+\infty}{\longrightarrow} 0,$$

$$\|(Tu_n)'' - (Tu)''\|_2 \overset{n\to+\infty}{\longrightarrow} 0,$$

$$\|(Tu_n)' - (Tu)'\|_1 \overset{n\to+\infty}{\longrightarrow} 0$$

and

$$\|Tu_n - Tu\|_0 \overset{n\to+\infty}{\longrightarrow} 0.$$

We have,

$$
\begin{aligned}
&\sup_{0\leq t<+\infty} |\phi((Tu_n)''')(t) - \phi((Tu)''')(t)| \\
&= \sup_{0\leq t<+\infty} \Big| \int_t^{+\infty} G(u_n(\mu))d\mu - \int_t^{+\infty} G(u(\mu))d\mu \Big| \\
&\leq \int_0^{+\infty} \Big| G(u_n(\mu)) - G(u(\mu)) \Big| d\mu \leq 2M_\rho < +\infty.
\end{aligned}
$$

From Lebesgue Dominated Convergence Theorem, G is an L^1-Carathéodory function. Hence,

$$\int_0^{+\infty} \Big| G(u_n(\mu)) - G(u(\mu)) \Big| d\mu \to 0,$$

as $n \to +\infty$. Since ϕ homeomorphism, we obtain that

$$\|(Tu_n)''' - (Tu)'''\|_3 \to 0,$$

as $n \to +\infty$.

$$
\begin{aligned}
\sup_{0\le t<+\infty}\Big|\frac{(Tu_n)''(t)}{1+t}-\frac{(Tu)''(t)}{1+t}\Big| &= \sup_{0\le t<+\infty}\Bigg|-a\frac{\phi^{-1}\left(\phi(D)+\int_0^{+\infty}G(u_n(s))ds\right)}{1+t}\\
&\quad+\frac{\int_0^t\phi^{-1}\left(\phi(D)+\int_s^{+\infty}G(u_n(\tau))d\tau\right)ds}{1+t}\\
&\quad+a\frac{\phi^{-1}\left(\phi(D)+\int_0^{+\infty}G(u(s))ds\right)}{1+t}\\
&\quad-\frac{\int_0^t\phi^{-1}\left(\phi(D)+\int_s^{+\infty}G(u(\tau))d\tau\right)ds}{1+t}\Bigg|\\
&\le \sup_{0\le t<+\infty}\frac{|a|}{1+t}\Bigg|\phi^{-1}\left(\phi(D)+\int_0^{+\infty}G(u(s))ds\right)\\
&\quad-\phi^{-1}\left(\phi(D)+\int_0^{+\infty}G(u_n(s))ds\right)\Bigg|\\
&\quad+\sup_{0\le t<+\infty}\Bigg[\frac{1}{1+t}\\
&\quad\int_0^t\Bigg|\phi^{-1}\left(\phi(D)+\int_s^{+\infty}G(u_n(\tau))d\tau\right)\\
&\quad-\phi^{-1}\left(\phi(D)+\int_s^{+\infty}G(u(\tau))d\tau\right)\Bigg|ds\Bigg]\\
&\le \sup_{0\le t<+\infty}\frac{2|a|}{1+t}\|(Tu_n)'''-(Tu)'''\|_3\\
&\quad+\sup_{0\le t<+\infty}\frac{2}{1+t}\int_0^t\|(Tu_n)'''-(Tu)'''\|_3ds\\
&\le 2|a|\|(Tu_n)'''-(Tu)'''\|_3\\
&\quad+2\|(Tu_n)'''-(Tu)'''\|_3\to 0,
\end{aligned}
$$

as $n\to+\infty$.

$$
\begin{aligned}
\sup_{0\le t<+\infty}\Big|\frac{(Tu_n)'(t)}{1+t^2}-\frac{(Tu)'(t)}{1+t^2}\Big| &= \sup_{0\le t<+\infty}\Bigg|-at\frac{\phi^{-1}\left(\phi(D)+\int_0^{+\infty}G(u_n(s))ds\right)}{1+t^2}\\
&+\frac{\int_0^t(t-s)\phi^{-1}\left(\phi(D)+\int_s^{+\infty}G(u_n(\tau))d\tau\right)ds}{1+t^2}\\
&+at\frac{\phi^{-1}\left(\phi(D)+\int_0^{+\infty}G(u(s))ds\right)}{1+t^2}
\end{aligned}
$$

$$-\frac{\int_0^t (t-s)\phi^{-1}\left(\phi(D)+\int_s^{+\infty} G(u(\tau))d\tau\right)ds}{1+t^2}\Bigg|$$

$$\leq \sup_{0\leq t<+\infty} \frac{|a|t}{1+t^2}\left|\phi^{-1}\left(\phi(D)+\int_0^{+\infty} G(u(s))ds\right)\right.$$

$$\left.-\phi^{-1}\left(\phi(D)+\int_0^{+\infty} G(u_n(s))ds\right)\right|$$

$$+\sup_{0\leq t<+\infty} \frac{1}{1+t^2}\int_0^t (t-s)\Bigg|$$

$$\phi^{-1}\left(\phi(D)+\int_s^{+\infty} G(u_n(\tau))d\tau\right)-\phi^{-1}\left(\phi(D)+\int_s^{+\infty} G(u(\tau))d\tau\right)\Bigg|ds$$

$$\leq \sup_{0\leq t<+\infty} \frac{2|a|t}{1+t^2}\|(Tu_n)'''-(Tu)'''\|_3$$

$$+\sup_{0\leq t<+\infty} \frac{2}{1+t^2}\int_0^t (t-s)\|(Tu_n)'''-(Tu)'''\|_3 ds$$

$$\leq 2|a|\|(Tu_n)'''-(Tu)'''\|_3+\sup_{0\leq t<+\infty} \frac{2}{1+t^2}\int_0^t t\|(Tu_n)'''-(Tu)'''\|_3 ds$$

$$\leq 2|a|\|(Tu_n)'''-(Tu)'''\|_3+2\|(Tu_n)'''-(Tu)'''\|_3\to 0,$$

as $n\to+\infty$.

By the same arguments, we can prove that

$$\sup_{0\leq t<+\infty}\left|\frac{(Tu_n)(t)}{1+t^3}-\frac{(Tu)(t)}{1+t^3}\right|\to 0$$

as $n\to+\infty$.

(iii) T is completely continuous. We apply Lemma 2.8, where M_ρ is defined in (3.4).

In what follows, we consider the following constant

$$L_\rho=\max\left(\phi^{-1}(|\phi(D)|+M_\rho),\left|\phi^{-1}(-|\phi(D)|-M_\rho)\right|\right)$$

Let $U\subset X$ be any bounded subset. Then there is $r>0$ such that $\|u\|_X<r$ for all $u\in U$. For each $u\in U$, one has

$$\begin{aligned}\|Tu\|_0&=\sup_{0\leq t<+\infty}\frac{|Tu(t)|}{1+t^3}\\&\leq |A|+|B|+\frac{|C|}{2}+\frac{|a|L_\rho}{2}+\sup_{0\leq t<+\infty} L_\rho\int_0^t\frac{\frac{1}{2}(t-s)^2}{1+t^3}ds\\&\leq |A|+|B|+\frac{|C|}{2}+\frac{|a|L_\rho}{2}+\frac{L_\rho}{6}<+\infty,\end{aligned}$$

$\|(Tu)\|_0<+\infty$.

Also,

$$\begin{aligned}\|(Tu)'\|_1 &= \sup_{0\leq t<+\infty}\frac{|(Tu)'(t)|}{1+t^2}\leq |B|+|C|+|a|L_\rho\\ &\quad+\sup_{0\leq t<+\infty}\int_0^t\frac{t-s}{1+t^2}\Big|\phi^{-1}\left(\phi(D)+\int_s^{+\infty}G(u(\mu))d\mu\right)\Big|ds\\ &\leq |B|+|C|+|a|L_\rho+\sup_{0\leq t<+\infty}L_\rho\int_0^t\frac{t-s}{1+t^2}ds\\ &\leq |B|+|C|+|a|L_\rho+\frac{L_\rho}{2}<+\infty,\end{aligned}$$

$\|(Tu)'\|_1<+\infty$.

Also,

$$\begin{aligned}\|(Tu)''\|_2 &= \sup_{0\leq t<+\infty}\frac{|(Tu)''(t)|}{1+t}\leq |C|+|a|L_\rho\\ &\quad+\sup_{0\leq t<+\infty}\int_0^t\frac{1}{1+t}\Big|\phi^{-1}\left(\phi(D)+\int_s^{+\infty}G(u(\mu))d\mu\right)\Big|ds\\ &\leq |C|+|a|L_\rho+\sup_{0\leq t<+\infty}L_\rho\int_0^t\frac{1}{1+t}ds\\ &\leq |C|+|a|L_\rho+L_\rho<+\infty,\end{aligned}$$

$\|(Tu)''\|_2<+\infty$. Also,

$$\|(Tu)'''\|_3=\frac{1}{2}\sup_{0\leq t<+\infty}\Big|\phi^{-1}\left(\phi(D)+\int_t^{+\infty}G(u(\mu))d\mu\right)\Big|\leq L_\rho<+\infty,$$

$\|(Tu)'''\|_3<+\infty$. And, therefore,

$$\begin{aligned}\|Tu\|_X &\leq |A|+|B|+|C|+|a|L_\rho+L_\rho\\ &\leq |A|+|B|+|C|+(|a|+1)L_\rho<+\infty.\end{aligned}$$

That is, TU is uniformly bounded.

To prove that TU is equicontinuous, let $L > 0$ and $t_1, t_2 \in [0, L]$ with $t_1 < t_2$, we have

$$\begin{aligned}
\left|\frac{Tu(t_2)}{1+t_2^3} - \frac{Tu(t_1)}{1+t_1^3}\right| &= \Bigg|\frac{2A + 2Bt_2 + Ct_2^2 - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty} G(u(s))ds\right) t_2^2}{2(1+t_2^3)} \\
&\quad - \frac{2A + 2Bt_1 + Ct_1^2 - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty} G(u(s))ds\right) t_1^2}{2(1+t_1^3)} \\
&\quad + \int_0^{t_2} \frac{\frac{t_2^2+s^2}{2} - t_2 s}{1+t_2^3}\phi^{-1}\left(\phi(D) + \int_s^{+\infty} G(u(\tau))d\tau\right) ds \\
&\quad - \int_0^{t_1} \frac{\frac{t_1^2+s^2}{2} - t_1 s}{1+t_1^3}\phi^{-1}\left(\phi(D) + \int_s^{+\infty} G(u(\tau))d\tau\right) ds\Bigg| \\
&\le \Bigg|\frac{2A + 2Bt_2 + Ct_2^2 - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty} G(u(s))ds\right) t_2^2}{2(1+t_2^3)} \\
&\quad - \frac{2A + 2Bt_1 + Ct_1^2 - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty} G(u(s))ds\right) t_1^2}{2(1+t_1^3)}\Bigg| \\
&\quad + L_\rho \int_0^{t_1} \left|\frac{\frac{t_2^2+s^2}{2} - t_2 s}{1+t_2^3} - \frac{\frac{t_1^2+s^2}{2} - t_1 s}{1+t_1^3}\right| ds \\
&\quad + L_\rho \int_{t_1}^{t_2} \left|\frac{\frac{t_2^2+s^2}{2} - t_2 s}{1+t_2^3}\right| ds \to 0,
\end{aligned}$$

as $t_1 \to t_2$.

$$\begin{aligned}
\left|\frac{(Tu)'(t_2)}{1+t_2^2} - \frac{(Tu)'(t_1)}{1+t_1^2}\right| &= \Bigg|\frac{B + Ct_2 - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty} G(u(s))ds\right) t_2}{1+t_2^2} \\
&\quad - \frac{B + Ct_1 - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty} G(u(s))ds\right) t_1}{1+t_1^2} \\
&\quad + \int_0^{t_2} \frac{t_2 - s}{1+t_2^2}\phi^{-1}\left(\phi(D) + \int_s^{+\infty} G(u(\tau))d\tau\right) ds \\
&\quad - \int_0^{t_1} \frac{t_1 - s}{1+t_1^2}\phi^{-1}\left(\phi(D) + \int_s^{+\infty} G(u(\tau))d\tau\right) ds\Bigg| \\
&\le \Bigg|\frac{B + Ct_2 - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty} G(u(s))ds\right) t_2}{1+t_2^2} \\
&\quad - \frac{B + Ct_1 - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty} G(u(s))ds\right) t_1}{1+t_1^2}\Bigg|
\end{aligned}$$

$$+L_\rho \int_0^{t_1} \left|\frac{t_2 - s}{1+t_2^2} - \frac{t_1 - s}{1+t_1^2}\right| ds + L_\rho \int_{t_1}^{t_2} \left|\frac{t_2 - s}{1+t_2^2}\right| ds \to 0,$$

as $t_1 \to t_2$.

$$\begin{aligned}
\left|\frac{(Tu)''(t_2)}{1+t_2} - \frac{(Tu)''(t_1)}{1+t_1}\right| &= \left|\frac{C - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty} G(u(s))ds\right)}{1+t_2}\right. \\
&\quad - \frac{C - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty} G(u(s))ds\right)}{1+t_1} \\
&\quad + \int_0^{t_2} \frac{1}{1+t_2}\phi^{-1}\left(\phi(D) + \int_s^{+\infty} G(u(\tau))d\tau\right) ds \\
&\quad \left. - \int_0^{t_1} \frac{1}{1+t_1}\phi^{-1}\left(\phi(D) + \int_s^{+\infty} G(u(\tau))d\tau\right) ds\right| \\
&\leq \left|\frac{C - a\phi^{-1}\left(\phi(D + \int_0^{+\infty} G(u(s))ds\right)}{1+t_2}\right. \\
&\quad \left. - \frac{C - a\phi^{-1}\left(\phi(D) + \int_0^{+\infty} G(u(s))ds\right)}{1+t_1}\right| \\
&\quad + L_\rho \int_0^{t_1} \left|\frac{1}{1+t_2} - \frac{1}{1+t_1}\right| ds + L_\rho \int_{t_1}^{t_2} \left|\frac{1}{1+t_2}\right| ds \to 0,
\end{aligned}$$

as $t_1 \to t_2$.

Using L^1-Carathéodory function G and the fact that ϕ is a homeomorphism,

$$\begin{aligned}
&\left|\phi((Tu)'')(t_2) - \phi((Tu)'')(t_1)\right| \\
&= \left|\phi(D) + \int_{t_2}^{+\infty} G(u(\tau))d\tau - \left(\phi(D) + \int_{t_1}^{+\infty} G(u(\tau))d\tau\right)\right| \\
&= \left|\int_{t_2}^{+\infty} G(u(\tau))d\tau - \int_{t_1}^{t_2} G(u(\tau))d\tau - \int_{t_2}^{+\infty} G(u(\tau))d\tau\right| \\
&= \left|\int_{t_1}^{t_2} G(u(\tau))d\tau\right| \to 0,
\end{aligned}$$

Moreover, $TU \subset X$ is equiconvergent at infinity. To see this, we use that the function G is L^1-Carathéodory and ϕ^{-1} is continuous. From Lemma 2.1, for all $u \in U$, $\lim_{t\to+\infty}(Tu)'''(t) = D$. Then, $\lim_{t\to+\infty} \frac{(Tu)''(t)}{1+t} = D$, $\lim_{t\to+\infty} \frac{(Tu)'(t)}{1+t^2} = \frac{D}{2}$ and $\lim_{t\to+\infty} \frac{(Tu)(t)}{1+t^3} = \frac{D}{6}$

$$|(Tu)'''(t) - D| = \left|\phi^{-1}\left(\phi(D) + \int_t^{+\infty} G(u)(\mu)d\mu\right) - D\right| \to 0,$$

as $t \to +\infty$. Regarding to the second derivatives, we have

$$\begin{aligned}\lim_{t\to+\infty}\left|\frac{(Tu)''(t)}{1+t}-D\right| &= \lim_{t\to+\infty}\Big|\frac{1}{1+t}\Big(C-a\phi^{-1}\Big(\phi(D)+\int_0^{+\infty}G(u(s))ds\Big)\\
&\quad+\int_0^t\phi^{-1}\Big(\phi(D)+\int_s^{+\infty}G(u(\tau))d\tau\Big)ds\Big)-D\Big|\\
&\le \lim_{t\to+\infty}\Big|\frac{1}{1+t}\Big(C-a\phi^{-1}\Big(\phi(D)+\int_0^{+\infty}G(u(s))ds\Big)\Big)\Big|\\
&\quad+\lim_{t\to+\infty}\Big|\frac{1}{1+t}\int_0^t\phi^{-1}\Big(\phi(D)+\int_s^{+\infty}G(u(\tau))d\tau\Big)ds-D\Big|\\
&= \lim_{t\to+\infty}\Big|\frac{1}{1+t}\Big(C-a\phi^{-1}\Big(\phi(D)+\int_0^{+\infty}G(u(s))ds\Big)\Big)\Big|\\
&\quad+\lim_{t\to+\infty}\Big|\phi^{-1}\Big(\phi(D)+\int_t^{+\infty}G(u(\tau))d\tau\Big)-D\Big|=0.\end{aligned}$$

Since,

$$\begin{aligned}&\lim_{t\to+\infty}\frac{1}{1+t}\int_0^t\phi^{-1}\Big(\phi(D)+\int_s^{+\infty}G(u(\tau))d\tau\Big)ds\\
&=\lim_{t\to+\infty}\phi^{-1}\Big(\phi(D)+\int_t^{+\infty}G(u(\tau))d\tau\Big)=D.\end{aligned}$$

By the same technique, one can show that

$$\Big|\frac{(Tu)'(t)}{1+t^2}-\frac{D}{2}\Big|\to 0,\quad \Big|\frac{(Tu)(t)}{1+t^3}-\frac{D}{6}\Big|\to 0,$$

as $t \to +\infty$. By Lemma 2.8, we conclude that the set TU is relatively compact.

Consider the constant

$$R_2=\max\{\rho, |A|+|B|+|C|+(|a|+1)L_\rho\}$$

and let $\overline{B}\subset X$ be the closed bounded ball of radius $r\ge R_2$. Then, $T\overline{B}\subset\overline{B}$. Since T is completely continuous, by Theorem 2.9, T has at least one fixed point $u\in X$ such that

$$\begin{aligned}\overline{\alpha}(t)\le u(t)\le\overline{\beta}(t),&\qquad \alpha'(t)\le u'(t)\le\beta'(t),\\
\alpha''(t)\le u''(t)\le\beta''(t),&\qquad -2R_1<u'''(t)<2R_1,\end{aligned}$$

for all $t\in[0,+\infty)$.

4. Example

Consider the fourth-order differential equation

$$(1+t^3)(2+t^3)^4((u''')^3)'(t)=|u''(t)-12t-4|(-u'(t))+(-u(t)+3)|u'''(t)-12|,\quad t>0, \tag{4.1}$$

with the boundary conditions

$$u(0)=3,\ u'(0)=0,\quad u''(0)+au'''(0)=1,\quad u'''(+\infty)=D, \tag{4.2}$$

where $-\frac{1}{4} \leq a < 0$ and $0 < D < 12$.

Notice that the above problem is a particular case of (1.1), (1.2) with $A = 3$, $B = 0$, $C = 1$ and $D = 5$,

$$g(t, x, y, z, w) = \frac{\left|z - 12t - 4\right|(-y) + (-x + 3)\left|w - 12\right|}{(1 + t^3)(2 + t^3)^4}. \tag{4.3}$$

g is an L^1-Carathéodory function. Moreover, the functions $\alpha(t) \equiv 1$ and $\beta(t) = 2t^3 + 2t^2 + 2$ are, respectively, nonordered lower and upper solutions for (4.1), (4.2)), with $\overline{\alpha}(t) = 3$ and $\overline{\beta}(t) = 2t^3 + 2t^2 + 3$. The nonlinearity g satisfies the one-sided Nagumo-type growth condition (2.4) with

$$\psi(t) = \frac{1}{1 + t^3}, 1 < \nu < 3, h(|w|) = 1,$$

over the set

$$E_0 = \Big\{(t, x, y, z, w) \in [0, +\infty) \times \mathbb{R}^4 : 3 \leq x \leq 2t^3 + 2t^2 + 3,\ 0 \leq y \leq 6t^2 + 4t,$$
$$0 \leq z \leq 12t + 4\Big\}.$$

The assumptions of Theorem 3.1 are fulfilled. Therefore, there is at least a one solution u of (4.1), (4.2), and there is $R > 0$, such that

$$3 \leq u(t) \leq 2t^3 + 2t^2 + 3, \quad 0 \leq u'(t) \leq 6t^2 + 4t,$$
$$0 \leq u''(t) \leq 12t + 4, \quad \|u'''\|_3 \leq R, \quad \forall t \in [0, +\infty).$$

Note that g does not satisfy the usual bilateral Nagumo condition.

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