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NEW PROJECTION ALGORITHM FOR VARIATIONAL INEQUALITY PROBLEM OVER FIXED POINT SET OF A QUASI-NONEXPANSIVE MAPPING IN HILBERT SPACE

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Abstract. Hybrid steepest descent methods are used to solve a variational inequality problem (VIP) over the fixed point set of (quasi) nonexpansive mapping in the literature. In these results, the mapping involved in VIP is required to be Lipschitz continuous and strongly monotone. In this paper, we propose a new projection algorithm for solving a pseudomonotone VIP over the fixed point set of a quasi-nonexpansive mapping in Hilbert spaces. Compared the previous methods, the mapping is not assumed to be Lipschitz continuous, strongly monotone or uniformly continuous in our algorithm. We prove the weak convergence and estimate the convergence rate of the proposed algorithm. Some numerical examples are given to illustrate the effectiveness of our algorithm and compare the computed results with other related algorithms.
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1. Introduction

Let C is a nonempty closed convex subset in a real Hilbert space H and $A : H \to H$ is a single-valued mapping. Consider the following variational inequality problem (VIP): find a point $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0 \ \ \forall x \in C, \tag{1.1}$$

Denote the solution set of the problem (1.1) by $VI(C, A)$.

Variational inequalities play an important role in many fields, such as mathematical programming, transportation research, game theory, machine learning [1, 11, 23]. Thanks to its wide applications, variational inequality theory has become an important area of investigation in the past several decades, and has been extensively studied for both finite and infinite dimensional spaces; see, e.g., [13, 16, 3], and the references therein.

One of the most interesting and important problems in the variational inequality theory is to develop efficient iterative algorithms for approximating the solutions of the problem (1.1). The simplest iterative algorithm for solving the problem (1.1) is the following projection method:

$$x_{k+1} = P_C(x_k - \lambda A x_k),\ k \geq 1. \tag{1.2}$$

If A is L-Lipschitz continuous and strongly monotone, then the sequence $\{x_k\}$ generated by (1.3) converges to the solution of $VI(C, A)$ when $\lambda \in (0, \frac{2\alpha}{L^2})$.

To weak the strong monotonicity of A in (1.2), Korpelevich [17] and Antipin [4] proposed the following extragradient method in the finite dimensional Euclidean space:

$$\begin{cases} x_1 \in C, \\ y_k = P_C(x_k - \lambda_k A x_k), \\ x_{k+1} = P_C(x_k - \lambda_k A y_k),\ k \geq 1, \end{cases} \tag{1.3}$$

where A is monotone and L-Lipschitz continuous and $\{\lambda_k\} \subset (0, 1/L)$. The sequence $\{x_k\}$ generated by (1.3) converges to the solution of $VI(C, A)$ if $VI(C, A)$ is nonempty. In recent years, the extragradient method and its modifications have been extended to infinite dimensional spaces; see, e.g., [9, 8, 10, 12] and the references therein.

When the Lipschtiz constant L of A can not be computed accurately, the algorithm (1.3) including its modifications where the step-sizes are determined by L are hard to use. To overcome the shortcoming, Iusem [15] proposed an iterative algorithm in the finite dimensional Euclidean space for solving the problem (1.1) as follows:

Algorithm 1.1

Initialization: Given $l \in (0, 1)$, $\mu \in (0, 1)$, $\gamma > 0$. Take an arbitrary initial point $x_1 \in C$.

Iterative Steps: Given the current iterate x_k, calculate x_{k+1} as follows:

Step 1. Compute

$$y_k = P_C(x_k - \gamma_n A x_k)$$

where $\gamma_k = \gamma l^{j_k}$ with j_k is the smallest non-negative integer j satisfying

$$\gamma l^j \|Ax_k - Ay_l\| \leq \mu \|x_k - y_k\|.$$

Step 2. Compute

$$x_{k+1} = P_C(x_k - \lambda_k A y_k),$$

where $\lambda_k = \frac{\langle Ay_k, x_k - y_k \rangle}{\|Ay_k\|^2}$.

In Algorithm 1.1, although the operator A is still required to be Lipschitz continuous, the Lipschitz constant of A need not to be known that is such that Algorithm 1.1 can be used conveniently. Few years later, this method was improved by Solodov and Svaiter [24].

Very recently, some new iterative algorithms for solving the problem (1.1) with non-Lipschitz continuous operator A were proposed in Hilbert space. Anh and Vinh [2] constructed a self-adaptive inertial gradient projection algorithm for the the monotone or strongly pseudomonotone problem (1.1). Vuong and Shehu [27] constructed a Halpern algorithm to solve the pseudomonotone problem (1.1). Thong et al. [25] proposed a viscosity algorithm to solve the pseudomonotone problem (1.1). In [27, 25], the operator A is required to uniformly continuous and sequentially weakly continuous on bounded subsets of C. Replacing the sequentially weakly continuity of A on bounded subsets of C with a weaker condition, Reich et al. [22] introduced the following algorithm to solve the pseudomotone problem (1.1):

Algorithm 1.2

Initialization: Given $\mu > 0, l \in (0,1), \lambda \in \left(0, \frac{1}{\mu}\right)$, let $x_1 \in C$ be arbitrary.
Iterative Steps: Given the current iterate x_k, calculate x_{k+1} as follows:
Step 1. Compute

$$z_k = P_C(x_k - \lambda A x_k)$$

and $r_\lambda(x_k) := x_k - z_k$. If $r_\lambda(x_k) = 0$, then stop and x_k is a solution of $VI(C, A)$. Otherwise,
Step 2. Compute

$$y_k = x_k - \tau_k r_\lambda(x_k),$$

where $\tau_k = l^{j_k}$ and j_k is the smallest non-negative integer j satisfying

$$\langle Ax_k - A(x_k - l^j r_\lambda(x_k)), r_\lambda(x_k)\rangle \leq \frac{\mu}{2}\|r_\lambda(x_k)\|^2.$$

Step 3. Compute $x_{n+1} = P_{C_k}(x_k)$,
where $\quad C_k := \{x \in C : h_k(x) \leq 0\}$
and $\quad h_k(x) = \langle Ay_k, x - x_k\rangle + \frac{\tau_k}{2\lambda}\|r_\lambda(x_k)\|^2.$
Set $k = k + 1$ and go to **Step 1**.

On the other hand, an interesting problem to solve a VIP on the fixed point set of a nonlinear mapping. In [28, 30], the hybrid steepest descent methods were introduced for finding a point in $VI(Fix(T), A)$, where T is a nonexpansive mapping on H. The central results on its convergence are as follows.

Theorem 1.1 [28, 30] *Let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Suppose that a mapping $A : H \to H$ is κ-Lipschitzian and η-strongly monotone over $T(H)$. Then, by using any sequence $\{\lambda_k\} \subset [0, \infty)$ satisfying*

$$\lim_{k\to\infty} \lambda_k = 0, \ \sum_{k=1}^{\infty} \lambda_k = \infty, \ \sum_{k=1}^{\infty} |\lambda_{k+1} - \lambda_k| < \infty.$$

Then sequence $\{x_k\}$ generated, with arbitrary $x_0 \in H$, by

$$x_{k+1} = Tx_k - \lambda_{k+1} A(Tx_k)$$

converges strongly to the unique point in $VI(Fix(T), A)$.

In 2004, Yamada and Ogura [29] improved Theorem 1 by replacing the nonexpansive mapping with quasi-nonexpansive mapping and approximated the solution of the convex optimization problem over the fixed point set of wide range of subgradient projection operators in real Hilbert space.

In this paper, we continue to study the VIP over the fixed point set of a quasi-nonexpansive mapping. We introduce a new projection algorithm to find a solution of the considered problem and prove the weak convergence of the proposed algorithm. The contributions of this paper are as follows:

1. Compared with [29], the operator A is pseudomonotone and non-Lipschtiz continuous, and hence the condition on A is relaxed.
2. the operator A is not required to be uniformly continuous which is necessary in [25, 22, 27].
3. the Armijo-type line search used in [25, 22, 27] is omitted.
4. the convergence rate of the proposed algorithm is shown.

The paper is organized as follows. We first recall some basic definitions and results in section 2. The convergence and convergence rate of the proposed algorithm are proved and analyzed in section 3 and 4, respectively. In section 5, we present some numerical experiments which demonstrate the proposed algorithm performances as well as provide a preliminary computational overview by comparing it with some related algorithms. Finally, we give a conclusion in section 6.

2. Preliminaries and notation

Let C be a nonempty closed convex subset of a real Hilbert space H. We use $x_k \rightharpoonup x$ to denote that the sequence $\{x_k\}$ converges weakly to x as $k \to \infty$ and $x_k \to x$ to denote that the sequence $\{x_k\}$ converges weakly to x as $k \to \infty$, respectively.

Definition 2.1 Let $A : C \to H$ be an operator. Then

1. A is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0,\ \forall x, y \in C;$$

2. A is called pseudomonotone if

$$\langle Ax, y - x\rangle \geq 0 \Longrightarrow \langle Ay, y - x\rangle \geq 0,\ \forall x, y \in C.$$

3. A is called sequentially weakly continuous if $x_k \rightharpoonup x \Longrightarrow Ax_k \rightharpoonup Ax$ for all $\{x_k\} \subset C$ and $x \in H$.

Let $T : C \to H$ be a mapping and $Fix(T)$ denote the set of fixed points of T. Let I be the identity mapping on H.

Definition 2.2 Let $T : C \to H$ be a mapping. Then

1. T is called nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|,\ \forall x, y \in C;$$

2. T is called quasi-nonexpansive if

$$\|Tx - y\| \leq \|x - y\|,\ \forall x \in C,\ \forall y \in Fix(T);$$

3. $I - T$ is said be to demiclosed at 0 if $x_k \rightharpoonup x$ with $\{x_k\} \subset C$ and $x \in H$ and $(I - T)x_k \to 0$ implies that $x = Tx$.

It is known that $Fix(T)$ is closed and convex if $T : C \to H$ is a quasi-nonexpansive mapping and $Fix(T) \neq \emptyset$; see [5].

For any $x \in H$, there exists a unique element $z \in C$, denoted by $P_C(x)$, such that $\|z-x\| = \inf_{y \in C} \|y-x\|$. The mapping $P_C : H \to C$ is called a metric projection from H onto C. It is known that P_C is nonexpansive. The following lemma characterizes the part properties of P_C.

Lemma 2.1 [14] *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $x \in H$, the following hold:*

(i) $z = P_C(x)$ *if and only if* $\langle x - z, z - y \rangle \geq 0,\ \forall y \in C$;
(ii) $\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_C(x)\|^2$, $\forall y \in C$.

Lemma 2.2 ([21]) *Let $\{a_k\}$ and $\{b_k\}$ be two sequences of nonnegative real numbers satisfying*

$$a_{k+1} \leq a_k + b_k,\ \forall k \geq 1,$$

where $\sum b_k < \infty$. Then $\lim_{k\to\infty} a_k$ exists.

Lemma 2.3 ([6]) *Let D be a nonempty set of H and $\{x_k\}$ be a sequence in H such that the following two conditions hold:*

(i) *for all $x \in D$, $\lim_{k\to\infty} \|x_k - x\|$ exists;*
(ii) *every sequential weak cluster point of $\{x_k\}$ is in D.*

Then the sequence $\{x_k\}$ converges weakly to a point in D.

3. Main results

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $A : C \to H$ be a nonlinear mapping and $T : C \to H$ be a quasi-nonexpansive mapping. In this section, we consider the following problem: find a point $\bar{x} \in Fix(T)$ such that

$$\langle A\bar{x}, y - \bar{x} \rangle \geq 0,\ \forall y \in Fix(T). \tag{Γ}$$

Denote the set of solutions of the above problem by Ω.

We present the following algorithm to solve the problem (Γ).

Algorithm 3.1 Initialization Choose the initial point $x_1 \in C$ and the sequences $\{\beta_k\} \subset [0, \beta']$ with $\beta' < 1$, $\{\gamma_k\} \subset (0, 1)$ and $\{\alpha_k\} \subset (0, +\infty)$ satisfying

$$\sum_{k=1}^{\infty} \alpha_k \gamma_k = \infty \text{ and } \sum_{k=1}^{\infty} \alpha_k^2 < \infty. \tag{3.1}$$

Set $k = 1$.

Step 1: For the current itrtate x_k, compute

$$z_k = \beta_k x_k + (1 - \beta_k) T x_k$$

and construct the subset

$$C_k = \{z \in C : \|z_k - z\| \leq \|x_k - z\|\}.$$

Step 2: Set

$$\eta_k = \max\{1, \|Ax_k\|\}$$

and compute

$$\begin{cases} y_k = P_{C-x_k}\left(-\dfrac{\alpha_k}{\eta_k}Ax_k\right), \\ x_{k+1} = P_{C_k}(x_k + \gamma_k y_k), \end{cases}$$

Step 3: If $y_k = 0$ and $x_k = z_k$, then the algorithm stops and $x_k \in \Omega$; otherwise, set $k = k+1$ and go to **Step 1**.

Remark 3.1 It is easy to see that $x_k \in C$ for each $k \in \mathbb{N}$.

The following remark shows that the stopping criterion of Algorithm 3.1 can well work.

Remark 3.2 Assume that $y_k = 0$ and $z_k = x_k$ for some $k \in \mathbb{N}$. For any $y \in Fix(T)$, since $y - x_k \in C - x_k$, by Lemma 2.1 (i) we have

$$-\frac{\alpha_k}{\eta_k}\langle Ax_k, x_k - y\rangle \geq 0$$

and hence $\langle Ax_k, y - x_k\rangle \geq 0$. On the other hand, from $z_k = x_k$ it follows that $x_k = Tx_k$, i.e., $x_k \in Fix(T)$. This with the arbitrariness of $y \in Fix(T)$ implies that $x_k \in \Omega$.

For showing the convergence of Algorithm 3.1, we assume that the stopping criterion of Algorithm 3.1 does not hold for all $k \in \mathbb{N}$. Also assume that the following conditions hold:

(A1) $\Omega \neq \emptyset$.

(A2) $I - T$ is demiclosed at 0.

(A3) A is bounded and sequentially weakly continuous on bounded subset of C.

(A4) A is pseudomonotone on C.

Lemma 3.1 *The sequence $\{x_k\}$ is well defined.*

Proof. Since C_k closed convex, we only need to show that C_k is nonempty for each $k \in \mathbb{N}$. In fact, for any $y \in Fix(T)$, since T is quasi-nonexpansive, we have

$$\|z_k - y\| \leq \beta_k\|x_k - y\| + (1-\beta_k)\|Tx_k - y\| \leq \|x_k - y\|,$$

which implies that $y \in C_k$. It follows that

$$Fix(T) \subset C_k \tag{3.2}$$

and hence C_k is nonempty for each $k \in \mathbb{N}$. Therefore, the sequence $\{x_k\}$ is well defined. This completes the proof. □

Lemma 3.2 *The limit of $\{\|x_k - z\|\}$ exists for any $z \in \Omega$.*

Proof. By the definition of y_k we have

$$\begin{aligned} y_k &= P_{C-x_k}\left(-\frac{\alpha_k}{\eta_k}Ax_k\right) \\ &= \text{argmin}_{v\in C-x_k}\frac{1}{2}\left\|v-(-\frac{\alpha_k}{\eta_k}Ax_k)\right\|^2 \\ &= \text{argmin}_{v\in C-x_k}\left\{\frac{\alpha_k}{\eta_k}\langle Ax_k, v\rangle + \frac{1}{2}\left(\|v\|^2+\frac{\alpha_k^2}{\eta_k^2}\|Ax_k\|^2\right)\right\} \\ &= \text{argmin}_{v\in C-x_k}\left\{\frac{\alpha_k}{\eta_k}\langle Ax_k, v\rangle + \frac{\|v\|^2}{2}\right\},\ \forall k\in\mathbb{N}. \end{aligned} \tag{3.3}$$

Note that $0 \in C - x_k$ for each $k \in \mathbb{N}$. So by (3.3) we get

$$\frac{\alpha_k}{\eta_k}\langle Ax_k, y_k\rangle + \frac{\|y_k\|^2}{2} \le 0,\ \forall k\in\mathbb{N}. \tag{3.4}$$

From (3.4) it follows that

$$\begin{aligned} \|y_k\|^2 &\le -\frac{2\alpha_k}{\eta_k}\langle Ax_k, y_k\rangle \le \frac{2\alpha_k}{\eta_k}\|Ax_k\|\|y_k\| \\ &\le 2\alpha_k\|y_k\|, \end{aligned}$$

which implies that

$$\|y_k\| \le 2\alpha_k,\ \forall k\in\mathbb{N}. \tag{3.5}$$

Let $u_k = x_k + \gamma_k y_k$ for each $k \in \mathbb{N}$. By (3.5) it is easy to obtain that

$$\|u_k - x_k\| = \gamma_k\|y_k\| \le 2\alpha_k,\ \forall k\in\mathbb{N}. \tag{3.6}$$

For any $y \in Fix(T)$, since $y - x_k \in C - x_k$, by Lemma 2.1(i) we have

$$\left\langle -\frac{\alpha_k}{\eta_k}Ax_k - y_k, y - x_k - y_k\right\rangle \le 0$$

and hence

$$\langle y_k, x_k - y\rangle \le -\|y_k\|^2 + \frac{\alpha_k}{\eta_k}\langle Ax_k, y-x_k\rangle - \frac{\alpha_k}{\eta_k}\langle Ax_k, y_k\rangle,\ \forall k\in\mathbb{N}. \tag{3.7}$$

By (3.2), (3.5)-(3.7) we get

$$\begin{aligned} \|x_{k+1}-y\|^2 &= \|P_{C_k}u_k - P_{C_k}y\|^2 \le \|u_k - y\|^2 \\ &= \|u_k-x_k\|^2 + \|x_k-y\|^2 + 2\langle u_k-x_k, x_k-y\rangle \\ &= \|u_k-x_k\|^2 + \|x_k-y\|^2 + 2\gamma_k\langle y_k, x_k-y\rangle \\ &\le 4\alpha_k^2 + \|x_k-y\|^2 + 2\gamma_k\left[\frac{\alpha_k}{\eta_k}\langle Ax_k, y-x_k\rangle - \frac{\alpha_k}{\eta_k}\langle Ax_k, y_k\rangle\right] \\ &\le 4\alpha_k^2 + \|x_k-y\|^2 + \frac{2\gamma_k\alpha_k}{\eta_k}\langle Ax_k, y-x_k\rangle + \frac{2\gamma_k\alpha_k}{\eta_k}\|Ax_k\|\|y_k\| \\ &\le 4\alpha_k^2 + \|x_k-y\|^2 + \frac{2\gamma_k\alpha_k}{\eta_k}\langle Ax_k, y-x_k\rangle + 4\gamma_k\alpha_k^2 \\ &\le \|x_k-y\|^2 + \frac{2\gamma_k\alpha_k}{\eta_k}\langle Ax_k, y-x_k\rangle + 8\alpha_k^2, \forall k\in\mathbb{N}. \end{aligned} \tag{3.8}$$

For $z \in \Omega$, by (A4) we have $\langle Ax_k, z - x_k\rangle \le 0$. So replacing y with z in (3.8), we have

$$\|x_{k+1} - z\|^2 \le \|x_k - z\|^2 + 8\alpha_k^2, \forall k \in \mathbb{N}. \tag{3.9}$$

Applying Lemma 2.2 and (3.1) to (3.9), we obtain that the limit of $\{\|x_k - z\|\}$ exists. This completes the proof. □

Lemma 3.3 *It holds that* $\lim_{k\to\infty} \|z_k - x_k\| = 0$.

Proof. From Lemma 3.2 it follows that $\{x_k\}$ is bounded. Fix $y \in \Omega$. Let $M > 0$ such that $\sup_{k\in\mathbb{N}} \|x_k - y\| < M$. By Lemma 2.1 (ii) and (3.6) we have

$$\begin{aligned}\|x_{k+1} - y\|^2 &= \|P_{C_k} u_k - y\|^2 \\ &\le \|u_k - y\|^2 - \|x_{k+1} - u_k\|^2 \\ &= \|u_k - x_k\|^2 + \|x_k - y\|^2 + 2\langle u_k - x_k, x_k - y\rangle - \|x_{k+1} - u_k\|^2 \\ &\le 4\alpha_k^2 + \|x_k - y\|^2 + 4M\alpha_k - \|x_{k+1} - u_k\|^2\end{aligned}$$

and hence

$$\|x_{k+1} - u_k\|^2 \le 4\alpha_k^2 + 4M\alpha_k + \|x_k - y\|^2 - \|x_{k+1} - y\|^2,\ \forall k \in \mathbb{N}. \tag{3.10}$$

Note that the limit of $\{\|x_k - y\|^2\}$ exists by Lemma 3.2. Letting $k \to \infty$ in (3.10) , by (3.1) we get

$$\lim_{k\to\infty} \|x_{k+1} - u_k\| = 0. \tag{3.11}$$

Furthermore, from (3.6) and (3.11) it follows that

$$\|x_{k+1} - x_k\| \le \|x_{k+1} - u_k\| + \|u_k - x_k\| \le \|x_{k+1} - u_k\| + 2\alpha_k \to 0,\ \text{as } k \to \infty. \tag{3.12}$$

Since $x_{k+1} \in C_k$, by (3.12) we have

$$\|z_k - x_{k+1}\| \le \|x_k - x_{k+1}\| \to 0,\ \text{as } k \to \infty. \tag{3.13}$$

Combining (3.12) and (3.13) we obtain

$$\|z_k - x_k\| \le \|z_k - x_{k+1}\| + \|x_k - x_{k+1}\| \to 0,\ \text{as } k \to \infty. \tag{3.14}$$

This completes the proof. □

Theorem 3.1 *The sequence* $\{x_k\}$ *converges weakly to a point* $\bar{x}$ *in* Ω.

Proof. For any $y \in Fix(T)$, by (3.8) we have

$$\frac{2\alpha_k\gamma_k}{\eta_k}\langle Ax_k, x_k - y\rangle \le \|x_k - y\|^2 - \|x_{k+1} - y\|^2 + 8\alpha_k^2,\ \forall k \in \mathbb{N}. \tag{3.15}$$

Adding with k from 1 to l in (3.15), we get

$$\begin{aligned}2\sum_{k=1}^{l} \frac{\alpha_k\gamma_k}{\eta_k}\langle Ax_k, x_k - y\rangle &\le \|x_1 - y\|^2 - \|x_{l+1} - y\|^2 + 8\sum_{k=1}^{l}\alpha_k^2 \\ &\le \|x_1 - y\|^2 + 8\sum_{k=1}^{l}\alpha_k^2.\end{aligned} \tag{3.16}$$

Letting $l \to \infty$ in (3.16), by (3.1) we obtain

$$2\sum_{k=1}^{\infty} \frac{\alpha_k \gamma_k}{\eta_k} \langle Ax_k, x_k - y\rangle < \infty. \tag{3.17}$$

Since $\{x_k\}$ is bounded, from (A3) it holds that $\{Ax_k\}$ is bounded, which leads to $\{\eta_k\}$ is also bounded. So there exists $M' > 0$ such that $\sup_{k\geq 1} \eta_k < M'$. It follows that

$$\sum_{k=1}^{\infty} \frac{\alpha_k \gamma_k}{\eta_k} \geq \sum_{k=1}^{\infty} \frac{\alpha_k \gamma_k}{M'},$$

which together with (3.1) implies that

$$\sum_{k=1}^{\infty} \frac{\alpha_k \gamma_k}{\eta_k} = \infty.$$

This together with (3.17) leads to

$$\liminf_{k\to\infty} \langle Ax_k, x_k - y\rangle \leq 0.$$

Hence there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that

$$\lim_{j\to\infty} \langle Ax_{k_j}, x_{k_j} - y\rangle \leq 0. \tag{3.18}$$

Since $\{x_k\}$ is bounded, there exists a subsequence of $\{x_k\}$ converging weakly to a point $\bar{x}$. Without loss of generality, we may assume that $x_{k_j} \rightharpoonup \bar{x}$ with $j \to \infty$. Since A is weakly sequence continuous, by (3.18) we get

$$\langle A\bar{x}, \bar{x} - y\rangle \leq 0. \tag{3.19}$$

We prove that $\bar{x} \in Fix(T)$. By the definition of z_k and (3.14) we have

$$\|x_{k_j} - Tx_{k_j}\| = \frac{1}{1-\beta_{k_j}} \|x_{k_j} - z_{k_j}\| \leq \frac{1}{1-\beta'} \|x_{k_j} - z_{k_j}\| \to 0, \text{ as } j \to \infty. \tag{3.20}$$

From (3.20) and (A2) we get $\bar{x} \in Fix(T)$, which together with the arbitrariness of $y \in Fix(T)$ and (3.19) implies that $\bar{x} \in \Omega$.

Finally, we prove that $\{x_k\}$ converges weakly to $\bar{x}$. In fact, by the argument above, we have show that every sequential weak cluster point of $\{x_k\}$ is in Ω. Hence by Lemma 3.2 and Lemma 2.3 with $D = \Omega$ we obtain that $\{x_k\}$ converges weakly to $\bar{x}$. This completes the proof. □

Remark 3.3 As the application of Algorithm 3.1, we may solve an optimization problem with constraint. Let $f : C \to \mathbb{R}$ be a differentiable function and $g : C \to \mathbb{R}$ be a convex function. Let $A = \nabla f$. It is known that $A : C \to H$ is monotone and hence pseudomontone. Finding the solution x^* of the optimization problem $\min_{x\in C} f(x)$ is equivalent to find $x^* \in VI(C, \nabla f)$. Define a mapping $S_g : C \to H$ by

$$S_g x = \begin{cases} x - \dfrac{g(x)}{\|u\|^2} u, & \text{if } g(x) > 0, \\ x, & \text{otherwise,} \end{cases}$$

where u is any vector of $\partial g(x)$. Let $T = 2S_g - I$. It follows that T is a quasi-nonexpansive mapping and $Fix(S_g) = \{x \in C : g(x) \leq 0\}$; see [29]. Especially when H is a finite dimension space, A and T satisfy the conditions (A2)-(A4). In this case, we may use Algorithm 3.1 to solve the following optimization problem with constraint:

$$\begin{aligned} &\min_{x\in C} f(x) \\ &\text{s.t. } g(x) \leq 0. \end{aligned}$$

If $T = I$ in Algorithm 3.1, we have $z_k = x_k$, $C_k = C$ and $x_k + \gamma_k y_k \in C$ for each $k \in \mathbb{N}$. Therefore, x_{k+1} in Algorithm 3.1 is $x_k + \gamma_k y_k$. In this situation, we obtain a new iterative algorithm for solving the problem (1.1). More precisely, we have the following result.

Theorem 3.2 *Assume that $VI(C, A) \neq \emptyset$ and the conditions* (A3) *and* (A4) *hold. Let the sequences $\{\gamma_k\} \subset (0, 1)$ and $\{\alpha_k\} \subset (0, +\infty)$ satisfy*

$$\sum_{k=1}^{\infty} \alpha_k \gamma_k = \infty \text{ and } \sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

Let $\{x_k\}$ be the sequence generated by the manner: $x_1 \in C$ and

$$\begin{cases} y_k = P_{C-x_k}\left(-\dfrac{\alpha_k}{\eta_k} A x_k\right), \\ x_{k+1} = x_k + \gamma_k y_k, \end{cases} \tag{3.21}$$

where $\eta_k = \max\{1, \|Ax_k\|\}$. The sequence $\{x_k\}$ generated by (3.21) *converges weakly to a point $\bar{x} \in VI(C, A)$.*

Remark 3.4 In Theorem 3.1, since the mapping A is not required to be Lipschitz continuous and strongly monotone, our result improves the result of Theorem 1 and the result in [29] by relaxing the condition on the mapping. In the algorithm (3.21), we do not use the line search rule and our algorithm is simpler than Algorithm 1.1 and 1.2. Moreover, in our results the mapping A does not need to be uniformly continuous that improves the corresponding ones in [22] and [25].

4. Convergence rate in the nonasymptotic sense

In this section, we focus on estimating the convergence rate of Algorithm 3.1 in the nonasymptotic sense. This seems to be first result for the algorithms of solving the pseudomonotone variational inequality problems and fixed point problems.

Let $M' > 0$ be defined as in the proof of Theorem 3.1.

Theorem 4.1 (convergence rate) *Let $\{x_k\}$ be the sequence generated by Algorithm* 3.1. *Assume the conditions* (A1)-(A4) *and* (3.2) *hold. Then, for any integer $l \in \mathbb{N}$, we have*

$$\min_{1\leq k\leq l} b_k \leq \frac{1}{\Psi_l}(\|x_1 - y\|^2 + \Lambda),$$

where $b_k = \max\left\{\frac{1}{M'}\langle Ax_k, x_k - y\rangle, \langle Ax_k, x_k - y\rangle\right\}$ *and*

$$\Lambda = \sum_{k=1}^{\infty} 8\alpha_k^2,\ \ \Psi_l = \sum_{k=1}^{l} 2\gamma_k\alpha_k.$$

Proof. We estimate $\langle Ax_k, x_k - y\rangle \geq 0$ by the following cases:

1. if $\langle Ax_k, x_k - y\rangle \geq 0$, then

$$\frac{1}{\eta_k}\langle Ax_k, x_k - y\rangle \geq \frac{1}{M'}\langle Ax_k, x_k - y\rangle;$$

2. if $\langle Ax_k, x_k - y\rangle < 0$, then

$$\frac{1}{\eta_k}\langle Ax_k, x_k - y\rangle > \langle Ax_k, x_k - y\rangle.$$

So we alway have

$$2\alpha_k\gamma_k \max\left\{\frac{1}{M'}\langle Ax_k, x_k - y\rangle, \langle Ax_k, x_k - y\rangle\right\} \leq \frac{2\alpha_k\gamma_k}{\eta_k}\langle Ax_k, x_k - y\rangle,\ \ \forall k \in \mathbb{N}. \tag{4.1}$$

From (3.15) and (4.1) it follows that

$$\begin{aligned} &2\alpha_k\gamma_k \max\left\{\frac{1}{M'}\langle Ax_k, x_k - y\rangle, \langle Ax_k, x_k - y\rangle\right\} \\ &\leq \|x_k - y\|^2 - \|x_{k+1} - y\|^2 + 8\alpha_k^2,\ \ \forall k \in \mathbb{N}. \end{aligned} \tag{4.2}$$

Adding k from 1 to l in (4.2) we get

$$\begin{aligned} 2\min_{1\leq k\leq l} b_k \sum_{k=1}^{l}\alpha_k\gamma_k &\leq 2\sum_{k=1}^{l}\alpha_k\gamma_k b_k \\ &\leq \|x_1 - y\|^2 - \|x_{l+1} - y\|^2 + 8\sum_{k=1}^{l}\alpha_k^2 \\ &\leq \|x_1 - y\|^2 + \Lambda \end{aligned}$$

and hence

$$\min_{1\leq k\leq l} b_k \leq \frac{1}{2\sum_{k=1}^{l}\alpha_k\gamma_k}(\|x_1 - y\|^2 + \Lambda),\ \ \forall l \in \mathbb{N}.$$

The desired result is obtained. This completes the proof. □

Corollary 4.1 *Let* $\{x_k\}$ *be the sequence generated by Algorithm* 3.1. *Assume the conditions* (A1)-(A4) *hold. In the ergodic sense, Algorithm* 3.1 *has the* $O\left(\frac{1}{k^{1-\alpha}}\right)$ *convergence rate if* $\alpha_k = \frac{1}{k^\alpha}$ *with* $\frac{1}{2} < \alpha \leq 1$ *and* $\gamma_k = \gamma \in (0, 1)$ *for each* $k \in \mathbb{N}$.

Proof. For each $k \in \mathbb{N}$, we have

$$\frac{2\gamma}{1-\alpha}[(k+1)^{1-\alpha} - k^{1-\alpha}] \leq \frac{2\gamma_k}{k^\alpha} = 2\gamma_k\alpha_k,$$

which leads to

$$\frac{2\gamma}{1-\alpha}[(l+1)^{1-\alpha}-1] \leq \sum_{k=1}^{l} 2\gamma_k\alpha_k, \ \forall l \in \mathbb{N}.$$

Hence

$$\Psi_l \geq \frac{2\gamma}{1-\alpha}[(l+1)^{1-\alpha}-1] \geq \frac{2\gamma}{1-\alpha}(l^{1-\alpha}-1),$$

which implies that Algorithm 3.1 has the $O\left(\frac{1}{k^{1-\alpha}}\right)$ convergence rate. This completes the proof. $\square$

5. Numerical experiment

In this section, we present some numerical examples to illustrate the convergence of our algorithms and compare the computed results with other related algorithms. All the codes were written by Matlab 2016b and all the numerical experiments were conducted on a PC Intel(R) Core (TM) i5-4260U CPU, 2.00 GHz, Ram 4.00 GB.

We first use the following fractional programming problem to illustrate Algorithm 3.1. The problem is a modification of the one considered in [7].

Example 5.1 Consider the constrained quadratic fractional programming problem with the feasible set $C = \{x \in \mathbb{R}^5 : 1 \leq x_i \leq 3, i = 1, 2, \cdots, 5\}$:

$$\begin{aligned} &\min_{x\in C} \ f(x) = \frac{x^T Q x + a^T x + a_0}{b^T x + b_0}, \\ &\text{s.t. } g(x) \leq 0, \end{aligned} \tag{5.1}$$

where

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 & 2 \\ -1 & 6 & -1 & 3 & 0 \\ 2 & -1 & 3 & 0 & 1 \\ 0 & 3 & 0 & 5 & 0 \\ 2 & 0 & 1 & 0 & 4 \end{pmatrix}, a = \begin{pmatrix} 2 \\ 2 \\ -1 \\ -2 \\ 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, a_0 = -2, b_0 = 20$$

and

$$g(x) = 2\|x\|^2 - 2x^T u - 1$$

with $u = (0.2, 0.2, 0.2, 0.2, 0.2)^T$.

By a simple calculation, we have

$$\nabla f(x) = \frac{(b^T x + b_0)(2Qx + a) - b(x^T Q x + a^T x + a_0)}{(b^T x + b_0)^2}.$$

Let $Ax = \nabla f(x)$ for all $x \in \mathbb{R}^5$. Then A is Lipschitz continuous and pseudomonotone on C. Define a mapping $S_g : C \to \mathbb{R}^5$ by

$$S_g x = \begin{cases} x - \dfrac{g(x)\nabla g(x)}{\|\nabla g(x)\|^2}, & \text{if } g(x) > 0, \\ x, & \text{otherwise.} \end{cases}$$

It is known that S_g is a quasi-firmly nonexpansive mapping. Let $T = 2S_g - I$. Then T is a quasi-nonexpansive mapping with $Fix(T) = \{x \in \mathbb{R}^5 : g(x) \leq 0\}$. The exact solution of problem (5.1) is $x^* = (1, \cdots, 1)^T$.

We use Algorithm 3.1 to approximate x^* by setting $\beta_k = \frac{1}{2}$, $\gamma_k = \frac{1}{2} - \frac{(-1)^k}{5}$, and $\alpha_k = \frac{1}{k}$ for each $k \in \mathbb{N}$. We use the initial point x_1 which is randomly taken from $l \times rand(5,1)$ in Matlab. We terminate the iterations if $\|x_k - x^*\| \leq 10^{-4}$ or if the number of iterations ≥ 200. The computed results are presented in Figure 1.

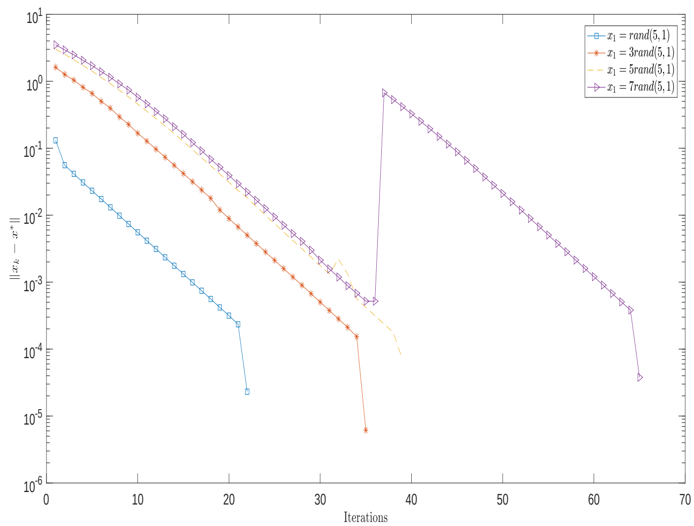

FIGURE 1. Computed results for Example 5.1 with different initial point x_1

Next we use the following examples to compare our algorithm (3.21) (also called Algorithm 3.2) with Algorithm 1.1, Algorithm 1.2. We take $\gamma_k = \frac{1}{2} + \frac{(-1)^k}{5}$ and $\alpha_k = \frac{1}{k^{0.6}}$ for our algorithm (3.21), $l = 0.5$, $\mu = 0.5$ and $\gamma = 2$ for Algorithm 1.1, $\lambda = 1.8$, $\mu = 0.5$ and $l = 0.5$ for Algorithm 1.2.

Example 5.2 This example was presented by [19] (also considered in [20]). The feasible set is $C = \mathbb{R}^m$ (for some positive even integer m) and $A = (a_{ij})_{1 \leq i,j \leq m}$ is the $m \times m$ square matrix the terms of which are given by

$$a_{ij} = \begin{cases} -1, & \text{if } j = m+1-i > i, \\ 1, & \text{if } j = m+1-i < i, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that the zero vector $x^* = (0, \cdots, 0)^T$ is the solution of the corresponding variational inequality problem. The initial point for all algorithms is $x_1 = (1, \cdots, 1)^T$.

We terminate the iterations if $\|x_k - x^*\| \leq 10^{-4}$ or if the number of iterations ≥ 1000. The results are presented in Table 1 where Sec denotes the costed CPU time (s) and Iter denotes the numbers of iterations.

TABLE 1. Numerical results for Example 5.2

Algorithms	m = 10		m = 50		m = 100	
	Sec.	Iter.	Sec.	Iter.	Sec.	Iter.
Algorithm 1.1	1.3056	1000	1.9487	1000	5.2199	1000
Algorithm 1.2	0.0266	65	0.0535	70	0.0743	72
Algorithm 3.2	0.0113	16	0.0111	16	0.0308	17

Example 5.3 Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $Ax = Mx$ with $M = NN^T + S + D$, where N is an $m \times m$ matrix, S is an $m \times m$ skew-symmetric matrix, D is an $m \times m$ diagonal matrix, whose diagonal entries are positive (so M is positive define), and the feasible set

$$C = \{x \in \mathbb{R}^m : -5 \leq x_i \leq 5, i = 1, \cdots, m\}.$$

It is clear that A is monotone and Lipschitz continuous with a Lipschitz constant $L = \|M\|$. The solution of the corresponding variational inequality problem is $x^* = (0, \cdots, 0)^T$.

In this example, all the entries of N, S and D are generated randomly in the interval (-2,2) and those of D are in the interval (0,1). We take the initial point $x_1 = (1, \cdots, 1)^T$ and use the stopping rule $\|x_k - x^*\| \leq 10^{-4}$ or the number of iteration ≥ 1000 for all the algorithms. The numerical results are presented in Table 2 where Sec denotes the costed CPU time (s) and Iter denotes the numbers of iterations.

TABLE 2. Numerical results for Example 5.3

Algorithms	m = 10		m = 30		m = 50	
	Sec.	Iter.	Sec.	Iter.	Sec.	Iter.
Algorithm 1.1	142.1506	210	337.9765	312	581.2995	533
Algorithm 1.2	44.7937	156	186.5969	459	465.9734	1000
Algorithm 3.2	24.2290	150	68.0919	373	96.3342	444

From the computed results of Example 5.2 and 5.3, Algorithm 3.2, our algorithm use the less CPU times and numbers of iterations and hence the performance is better than that of Algorithm 1.1 and 1.2.

6. Conclusion

In this paper, we have proposed new projection algorithm for solving a VIP over fixed point set of a quasi-nonexpansive mapping in real Hilbert spaces. Compared with some related results, the conditions imposed on the mapping in our algorithm is simpler. Since the line search rule is not used, the structure of our algorithm is simpler than the several existing algorithms. We proved the weak convergence and estimate the convergence rate in the nonasymptotic sense of the proposed algorithm. Some numerical experiments are given to illustrate the effectiveness of our new algorithm.

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