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MULTIPLICITY OF POSITIVE SOLUTIONS FOR CONVOLUTION EQUATIONS WITH NONLOCAL BOUNDARY CONDITION

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Abstract. This paper is devoted to studying convolution equations with nonlocal boundary condition. By means of the theory of fixed point index, some existence, multiplicity, and nonexistence results for positive solutions are derived in terms of different values of parameter λ.
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1. Introduction

In this paper, we study the following convolution equations with nonlocal boundary condition

$$\begin{cases} (a * u'')(t) + \lambda f(t, u(t)) = 0, \quad t \in (0,1), \\ u'(0) = 0, \quad u(1) = \varphi(u), \end{cases} \tag{1.1$_\lambda$}$$

where $\lambda > 0$ is a parameter, $\varphi(u) = \int_0^1 u(s)d\alpha(s)$ is a Stieltjes integral with the function α which is of bounded variation and monotone increasing on $[0,1]$, and $f : [0,1]\times[0,+\infty) \to (0,+\infty)$ is continuous. There are two different nonlocal elements in problem $(1.1)_\lambda$: one is the convolution $a * u''$ (see [10]), which appears in the equation itself; the other is Stieltjes integral $\varphi(u)$, which occurs in the boundary condition.

In recent years, many papers have been devoted to the study of nonlocal differential equations boundary value problems, see, such as, [6]-[11], [14] and [17]. Convolution equations have recently gained much attention, on account of their applications in wave propagation dynamics, image and signal processing, geophysics, univariate splines over uniform knots and B-splines, potential theory, evolution of populations and mathematical analysis (see [1], [3]-[5], [7], [13]).

We recall that both the elliptic equation and the fractional derivative can be realized in the form of a particular convolution. If u'' in the equation of $(1.1)_\lambda$ is replaced by Δu and $a = \delta_0$ the Dirac delta with mass concentrated at zero, we have the typical elliptic boundary value problem (see [2], [15], [16], [18]):

$$\Delta u(x) + \lambda f(x, u(x)) = 0, \quad x \in (0,1),$$

which is related to the equilibrium equations of continuum mechanics-bars, beams, strings. Besides, if $a(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}$, $1 < \alpha \leq 2$, we have the Caputo fractional order time-derivative and the nonlinear fractional evolution equation:

$$D_t^\alpha u(t) + \lambda f(t, u(t)) = 0, \quad t \in (0,1).$$

To the best of our knowledge, there are few papers dealing with the combination of convolution equations and nonlocal boundary conditions. In [11], the authors studied the existence of at least a positive solution and monotonicity of solutions of problem $(1.1)_\lambda$, without considering the influence of the choice of λ on the solution. In [8], the author considered nonlocal differential equations with convolution coefficients of the form

$$-M((a * u^q)(1))u''(t) = \lambda f(t, u(t)), \quad t \in (0,1), \tag{1.2}$$

where $\lambda > 0$ and $q \geq 1$ are parameters and both $M : [0,+\infty) \to \mathbb{R}$ and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous functions. Considering the influence of the choice of λ on the solution, the author demonstrated that, subject to given boundary data, problem (1.2) will not admit a positive solution when λ is sufficiently large. Greatly inspired by above works, in this paper, we study the positive solutions of problem $(1.1)_\lambda$. We demonstrate the existence, multiplicity, and nonexistence results for positive solutions of problem $(1.1)_\lambda$ for different values of λ. Furthermore, it is worth mentioning that the rang of solutions in this paper is different from that in [11].

The paper is organized as follows. In Section 2, we state some notations as well as recall Lemmas which will be used later, and we give some properties of the associated Green function. Section 3 is concerned with the existence, multiplicity and nonexistence of positive solutions for $(1.1)_\lambda$. An example is also given to illustrate the main results in Section 4.

2. Preliminaries and lemmas

Here we present some basic knowledge and definitions which will be used in the sequel. As we all know, we denote the finite convolution by $*$, namely,

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds, \quad t \geq 0,$$

where $f, g : [0,+\infty) \to \mathbb{R}$. Suppose $\beta > 0$, and g_β is the standard kernel, namely,

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0.$$

Especially, $g_1(t) = 1$, $g_2(t) = t$. We say that $a \in L^1_{Loc}(\mathbb{R}_+)$ is of type $\mathcal{PC}_1$ (see [11]), if the following condition is satisfied

($\mathcal{PC}_1$): There exists a nonnegative kernel $c \in AC(\mathbb{R}_+)$ such that $c * a = g_1$ on $(0, +\infty)$.

We will use the following assumption:

(H_1) $a \in AC(\mathbb{R}_+)$ is of type $\mathcal{PC}_1$ and non-increasing.

For $f(t,x) : [0,1] \times [0,+\infty) \to (0,+\infty)$ a continuous function and given numbers $0 \le a < b \le 1$ and $0 \le c < d < +\infty$, we will denote, respectively, the numbers

$$f^m_{[a,b]\times[c,d]} = \min_{(t,y)\in[a,b]\times[c,d]} f(t,y), \quad f^M_{[a,b]\times[c,d]} = \max_{(t,y)\in[a,b]\times[c,d]} f(t,y).$$

Remark 2.1 If a is of type $\mathcal{PC}_1$, there exists a nonnegative kernel $c \in AC(\mathbb{R}_+)$ such that $c * a = g_1$. Let $b = g_1 * c$. Then $b \in AC^1(\mathbb{R}_+)$ is nonnegative and monotone increasing. Besides, it follows from $(b * a)(t) = (g_1 * c * a)(t) = (g_1 * g_1)(t) = g_2(t)$ that $b * a = g_2$.

Lemma 2.2 [11] *Assume that* (H_1) *holds. Then a function* $u \in C^2([0,1])$ *is a solution of* $(1.1)_\lambda$ *if and only if it satisfies the integral equation*

$$u(t) = \varphi(u) + \lambda \int_0^1 G(t,s) f(s,u(s)) ds, \quad t \in [0,1], \tag{2.1}$$

where

$$G(t,s) = \begin{cases} b(1-s) - b(t-s), & 0 \le s \le t \le 1, \\ b(1-s), & 0 \le t \le s \le 1, \end{cases}$$

and b *is given in Remark 2.1.*

Lemma 2.3 *Suppose that* (H_1) *holds. Then the function* $G(t,s)$ *emerged in Lemma* 2.2 *satisfies the following properties:*

(1) $G : [0,1] \times [0,1] \to [0,+\infty)$ *is continuous and* $G(1,s) = G(t,1) = 0$*;*

(2) $\mathcal{G}(s) := \max_{t\in[0,1]} G(t,s) = b(1-s),\ s \in [0,1]$*;*

(3) $G_t(t,s)$ *is bounded and* $G(t,s)$ *is non-increasing with respect to* t*.*

Proof. By direct computations we obtain (1) and (2). Since $b(t) = (g_1 * c)(t) = \int_0^t c(s) ds$, we have $b'(t) = c(t)$. Thus,

$$G_t(t,s) = \begin{cases} -\, b_t(t-s) = -c(t-s), & 0 \le s \le t \le 1. \\ 0, & 0 \le t \le s \le 1. \end{cases}$$

By using the properties of c, $G_t(t,s)$ is bounded, and $G(t,s)$ is non-increasing with respect to t. The proof of (3) is finished.

Set $E = C([0,1])$. Obviously, E is a Banach space with the usual supremum norm $\|\cdot\|$. We define the operator $F_\lambda : E \to E$ as

$$(F_\lambda u)(t) = \varphi(u) + \lambda \int_0^1 G(t,s) f(s,u(s)) ds, \quad t \in [0,1].$$

Let

$$K = \{u \in E : u \geqslant 0,\ \varphi(u) \geqslant C_0 \|u\|\}, \quad B(0,\rho) = \{u \in K : \|u\| < \rho\},$$

where

$$C_0 = \min\left\{\varphi(\mathbf{1}), \inf_{s\in(0,1)} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)d\alpha(t)\right\}.$$

Obviously, the problem of finding a solution to $(1.1)_\lambda$ is equivalent to the problem of finding a fixed point to the operator equation $F_\lambda u = u$. It is easy to prove that K is a cone, and $B(0,\rho)$ is a open set.

From now on in this paper, we always assume that $C_0 > 0$ and $0 < \varphi(\mathbf{1}) < 1$.

Lemma 2.4 [11] *Assume that* (H_1) *holds. Then any solution* $u \in K$ *of* $(1.1)_\lambda$ *is non-increasing.*

Lemma 2.5 *Assume that* (H_1) *holds. Then* $F_\lambda : K \to K$ *is a completely continuous operator.*

Proof. For $u \in K$, by the continuity of $G(t,s)$, we know $F_\lambda u \in E$. It follows from Lemma 2.3 that

$$(F_\lambda u)(t) \geq C_0\|u\| + \lambda \int_0^1 G(t,s)f(s,u(s))ds \geq 0, \quad t \in [0,1].$$

Applying φ on both sides of the expression of F_λ, we have

$$\begin{aligned}
\varphi(F_\lambda u) &= \varphi(u)\varphi(\mathbf{1}) + \lambda \int_0^1 \int_0^1 G(t,s)f(s,u(s))dsd\alpha(t) \\
&\geq \varphi(u)C_0 + \lambda \inf_{s\in(0,1)} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)d\alpha(t) \int_0^1 \mathcal{G}(s)f(s,u(s))ds \\
&\geq C_0\left[\varphi(u) + \lambda \int_0^1 \mathcal{G}(s)f(s,u(s))ds\right] \\
&\geq C_0\|F_\lambda u\|.
\end{aligned}$$

Therefore, $F_\lambda(K) \subset K$.

Next, we show that F_λ is compact. Let D be a bounded set in K, and then there exists $0 < Q < +\infty$ such that $\|u\| \leqslant Q$ for each $u \in D$. Then

$$\begin{aligned}
||F_\lambda u|| &\leq \varphi(\mathbf{1})\|u\| + \max_{t\in[0,1]} \lambda \int_0^1 G(t,s)f(s,u(s))ds \\
&\leq \varphi(\mathbf{1})Q + \lambda f^M_{[0,1]\times[0,Q]} \int_0^1 \mathcal{G}(s)ds < +\infty.
\end{aligned}$$

Consequently, $F_\lambda(D)$ is uniformly bounded.

Since $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $s \in [0,1]$ and $t_1,\ t_2 \in [0,1]$, when $|t_2 - t_1| < \delta$, we have

$$|G(t_2,s) - G(t_1,s)| < \frac{\varepsilon}{\lambda f^M_{[0,1]\times[0,Q_1]}}.$$

Then, for any $u \in D$, when $|t_2 - t_1| < \delta$ and $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}|(F_\lambda u)(t_2) - (F_\lambda u)(t_1)| &\le \lambda \int_0^1 |G(t_2,s) - G(t_1,s)| f(s,u(s))ds \\ &\le \lambda f^M_{[0,1]\times[0,Q_1]} \int_0^1 |G(t_2,s) - G(t_1,s)|ds < \varepsilon.\end{aligned}$$

This implies that $F_\lambda(D)$ is equicontinuous. Hence, by the Arzela-Ascoli theorem, we know that F_λ is compact.

Finally, we show that F_λ is continuous. Assume $u_n, u \in K$ and $\|u_n - u\| \to 0$ $(n \to \infty)$. Then

$$\begin{aligned}|(F_\lambda u_n)(t) - (F_\lambda u)(t)| &\le |\varphi(u_n) - \varphi(u)| + \lambda \int_0^1 G(t,s)|f(s,u_n(s)) - f(s,u(s))|ds \\ &\le \varphi(\mathbf{1})\|u_n - u\| + \lambda \int_0^1 \mathcal{G}(s)|f(s,u_n(s)) - f(s,u(s))|ds \\ &\to 0, \quad n \to \infty.\end{aligned}$$

This implies that F_λ is continuous. Therefore, $F_\lambda : K \to K$ is completely continuous. This completes the proof.

Lemma 2.6 [11], [12] *Let U be a bounded open set, and with K a cone in a real Banach space X, suppose both that $U_K = U \cap K \supseteq \{0\}$ and that $\overline{U_K} \neq K$. Assume that $T : \overline{U_K} \to K$ is completely continuous such that $x \neq Tx$, for any $x \in \partial U_K$. Then the fixed point index $i_K(T, U_K)$ has the following properties.*
(1) If there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for each $x \in \partial U_K$ and $\lambda > 0$, then $i_K(T, U_K) = 0$.
(2) If $\mu x \neq Tx$ for $x \in \partial U_K$ and $\mu \ge 1$, then $i_K(T, U_K) = 1$.
(3) If $i_K(T, U_K) \neq 0$, then T has a fixed point in U_K.
(4) Let U_K^1 be a open set in X with $U_K^1 \subseteq U_K$. If $i_K(T, U_K) = 0$ and $i_K(T, U_K^1) = 1$, then T has a fixed point in $U_K \setminus \overline{U_K^1}$. The same result holds if $i_K(T, U_K) = 1$ and $i_K(T, U_K^1) = 0$.

3. Main results

Theorem 3.1 *Suppose that (H_1) holds; in addition, assume that*
(H_2)

$$\liminf_{x\to+\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x} = +\infty.$$

Then there exist $0 < \lambda_* \le \lambda^*$ such that
(i) $(1.1)_\lambda$ has at least two positive solutions in K if $0 < \lambda < \lambda_*$;
(ii) $(1.1)_\lambda$ has no positive solutions in K if $\lambda > \lambda^*$;
(iii) $(1.1)_\lambda$ has at least one positive solution in K if $\lambda = \lambda_*$ and $\lambda = \lambda^*$.

Proof. By $0 < \varphi(\mathbf{1}) < 1$, there exists $\lambda_0 > 0$ sufficiently small such that

$$\varphi(\mathbf{1}) + \lambda_0 f^M_{[0,1]\times[0,1]} \int_0^1 \mathcal{G}(s)ds < 1. \tag{3.1}$$

Fix $\lambda \in (0, \lambda_0]$. By (H_2), there exists $R_\lambda > 1$ such that

$$f(t,x) > N_0 x, \quad x \geq C_0 R_\lambda,\ 0 \leq t \leq 1,$$

where

$$N_0 > \left(C_0 \lambda \max_{t\in[0,1]} \int_0^1 G(t,s)ds \right)^{-1}.$$

We next prove that a fixed point of F_λ is on $\partial B(0,1)$ or in $B(0,1)$. We may suppose that F_λ has no fixed points on $\partial B(0,1)$ (otherwise, the proof is finished). We will utilize Lemma 2.6 to prove that F_λ has a fixed point in $B(0,1)$. Assume by contradiction that there exist $u \in \partial B(0,1)$ and $\mu \geqslant 1$ such that $\mu u = F_\lambda u$. For $\lambda \in (0, \lambda_0]$, we conclude

$$1 = \|u\| = \frac{1}{\mu}\|F_\lambda u\| \leq \|F_\lambda u\| \leq \varphi(\mathbf{1}) + \lambda_0 f^M_{[0,1]\times[0,1]} \int_0^1 \mathcal{G}(s)ds,$$

which contradicts (3.1). Consequently, we have

$$i_k(F_\lambda, B(0,1)) = 1.$$

It follows from Lemma 2.6 that F_λ has a fixed point in $B(0,1)$. Consequently, F_λ has a fixed point on $\partial B(0,1)$ or in $B(0,1)$, which is a solution of $(1.1)_\lambda$. Now we prove $u \in K$ is a positive solution. It follows from Lemma 2.4 that u is non-increasing. Thus, $u(t) \geq u(1) = \varphi(u) \geq C_0\|u\|,\ t \in [0,1]$. If $\|u\| = 0$, then $u(t) \equiv 0$ for $t \in [0,1]$, and

$$0 = (a * u'')(t) = -\lambda f(t, u(t)) < 0,$$

a contradiction. Therefore, $\|u\| > 0$ and $u(t) \geq C_0\|u\| > 0,\ t \in [0,1]$.

Next, we are going to prove that another fixed point of F_λ is on $\partial B(0, R_\lambda)$ or in $B(0,R_\lambda)\backslash\overline{B(0,1)}$. We suppose that $F_\lambda x \neq x$ for $x \in \partial B(0,R_\lambda)$ (otherwise, the proof is finished). We will show that F_λ has a fixed point in $B(0,R_\lambda)\backslash\overline{B(0,1)}$. Suppose there exist $u \in \partial B(0,R_\lambda)$ and $\eta_2 > 0$ such that $u = F_\lambda u + \eta_2\mathbf{1}$. It follows that $\varphi(u) \geqslant C_0\|u\| = C_0 R_\lambda > 0$. Then

$$u(t) = (F_\lambda u)(t) + \eta_2 > (F_\lambda u)(t) \geq \varphi(u) \geq C_0\|u\| = C_0 R_\lambda, \quad t \in [0,1],$$

and

$$\begin{aligned} R_\lambda &= \|u\| \\ &= \|F_\lambda u + \eta_2 \mathbf{1}\| \\ &> \lambda \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,u(s))ds \\ &\geq \lambda \max_{t\in[0,1]} \int_0^1 G(t,s) N_0 u(s) ds \\ &\geq \lambda N_0 \max_{t\in[0,1)} \int_0^1 G(t,s) C_0 R_\lambda ds \\ &> R_\lambda, \end{aligned}$$

which is a contradiction. Hence, we deduce that

$$i_K(F_\lambda, B(0, R_\lambda)) = 0.$$

By Lemma 2.6, there exists a fixed point of F_λ on $B(0, R_\lambda)\backslash\overline{B(0,1)}$. In conclusion, for $\lambda \in (0, \lambda_0]$, one fixed point of F_λ is on $\partial B(0,1)$ or in $B(0,1)$, and another fixed point of F_λ is on $\partial B(0, R_\lambda)$ or in $B(0, R_\lambda)\backslash\overline{B(0,1)}$. Hence, $(1.1)_\lambda$ has at least two positive solutions in K for $\lambda \in (0, \lambda_0]$.

Denote

$$S = \{\lambda' > 0 : (1.1)_\lambda \text{ has at least two positive solutions for } \lambda \in (0, \lambda']\}, \tag{3.2}$$

$$S_1 = \{\lambda > 0 : (1.1)_\lambda \text{ has at least one positive solution}\}. \tag{3.3}$$

We note that $S \neq \varnothing$ since $\lambda_0 \in S$. It follows that $S_1 \neq \varnothing$. Define $\lambda_* = \sup S$, $\lambda^* = \sup S_1$. It is obvious that $\lambda^* \geq \lambda_*$.

Now we claim that $\lambda^* < +\infty$. Indeed, if $\lambda^* = +\infty$, there exists $\{(\lambda_n, u_n)\}$ such that $\{\lambda_n\}$ is an increasing positive sequence, $\lambda_n \to \infty$ as $n \to \infty$, and

$$u_n(t) = \varphi(u_n) + \lambda_n \int_0^1 G(t,s) f(s, u_n(s)) ds, \quad t \in [0,1].$$

Define

$$\alpha = \inf\{\|u_n\|\}, \quad \beta = \sup\{\|u_n\|\}.$$

We show below that $\alpha > 0$ and $\beta < +\infty$.

(i) If $\alpha = 0$, then there exists $\{u_{n_k}\}$ which satisfies $\lim_{k\to+\infty} \|u_{n_k}\| = 0$. It follows from $f(t,x) > 0$ that there exists $0 < d < 1$ such that

$$f(t,x) > N'x, \quad 0 \leq x \leq d,\ t \in [0,1],$$

where

$$N' > \left(C_0 \lambda_1 \max_{t\in[0,1]} \int_0^1 G(t,s) ds\right)^{-1}.$$

Then for k large enough, $\|u_{n_k}\| < d$ and $0 \leq u_{n_k}(t) < d,\ t \in [0,1]$. Similar to the proof of Lemma 2.5, we can obtain $u_{n_k} \in K$. From Lemma 2.4, we see that u_{n_k} is non-increasing. Therefore,

$$u_{n_k}(t) \geq u_{n_k}(1) = \varphi(u_{n_k}) \geq C_0 \|u_{n_k}\|, \quad t \in [0,1].$$

Explicit computations show that, for k large enough,

$$\begin{aligned}
\|u_{n_k}\| &= \varphi(u_{n_k}) + \lambda_{n_k} \max_{t\in[0,1]} \int_0^1 G(t,s) f(s, u_{n_k}(s)) ds \\
&> \lambda_1 \max_{t\in[0,1]} \int_0^1 G(t,s) N' u_{n_k}(s) ds \\
&\geq \|u_{n_k}\| \lambda_1 C_0 N' \max_{t\in[0,1]} \int_0^1 G(t,s) ds \\
&> \|u_{n_k}\|.
\end{aligned}$$

This is clearly a contradiction.

(ii) If $\beta = +\infty$, then there exists $\{u_{n_j}\}$ such that $\lim\limits_{j\to+\infty} \|u_{n_j}\| = +\infty$. From (H_2), there exists $R' > 1$ such that

$$f(t,x) > N'x, \quad x > C_0 R',\ t \in [0,1].$$

Similar to the proof of case (i), we get a contradiction. Thus $\alpha > 0$ and $\beta < +\infty$ as we wanted. It follows that

$$\varphi(u_n) \geq C_0\|u_n\| \geq C_0\alpha > 0,$$

and

$$\begin{aligned}
\beta &\geq u_n(0) \\
&= \varphi(u_n) + \lambda_n \int_0^1 G(0,s) f(s, u_n(s)) ds \\
&> \lambda_n \int_0^1 G(0,s) f(s, u_n(s)) ds \\
&\geq \lambda_n \int_0^1 G(0,s) f^m_{[0,1]\times[C_0\alpha,\beta]} ds \to \infty, \quad n \to +\infty,
\end{aligned}$$

which contradicts $\beta < +\infty$. Therefore, $\lambda^* < +\infty$.

Finally, we prove $(1.1)_\lambda$ has one positive solution when $\lambda = \lambda^*$. From the definition of λ^*, we know that there exists $\{(\lambda_m, u_m)\}$ such that $\lambda_1 < \lambda_2 < \cdots < \lambda_m < \lambda_{m+1} < \cdots$, $\lim_{m\to\infty} \lambda_m = \lambda^*$, and

$$u_m(t) = \varphi(u_m) + \lambda_m \int_0^1 G(t,s) f(s, u_m(s)) ds.$$

Similar to the previous proof, we can find that there exist $\alpha_1 > 0$ and $\beta_1 < +\infty$ such that $0 < \alpha_1 \leqslant \|u_m\| \leqslant \beta_1 < +\infty$, which implies $\{u_m\}$ is uniformly bounded. Since

$$|u'_m(t)| = \left|\lambda_m \int_0^1 G_t(t,s) f(s, u_m(s)) ds\right| \leq \lambda^* f^M_{[0,1]\times[0,\beta_1]} \int_0^1 |G_t(t,s)| ds < +\infty,$$

it follows from Arzelà-Ascoli theorem that $\{u_m\}$ is relatively compact, and there is a subsequence of $\{u_m\}$, for convenience, still denoted by itself, such that $\lim\limits_{m\to\infty} u_m = u_{\lambda^*}$. Since

$$u_m(t) = \varphi(u_m) + \lambda_m \int_0^1 G(t,s) f(s, u_m(s)) ds > 0, \quad t \in [0,1],$$

then

$$u_{\lambda^*}(t) = \varphi(u_{\lambda^*}) + \lambda^* \int_0^1 G(t,s) f(s, u_{\lambda^*}(s)) ds \geq 0, \quad t \in [0,1],$$

and u_{λ^*} is a solution of $(1.1)_{\lambda^*}$. By the same argument as in the proof of Lemma 2.5, we obtain $u_{\lambda^*} \in K$. It follows from Lemma 2.4 that u_{λ^*} is non-increasing. Hence,

$$u_{\lambda^*}(t) \geq u_{\lambda^*}(1) = \varphi(u_{\lambda^*}) \geq C_0\|u_{\lambda^*}\| \geq C_0\alpha_1 > 0, \quad t \in [0,1].$$

Therefore, $(1.1)_\lambda$ has at least one positive solution when $\lambda = \lambda^*$. Similarly, $(1.1)_\lambda$ has at least one positive solution when $\lambda = \lambda_*$. The proof is complete.

Theorem 3.2 *Assume that* (H_1)-(H_2) *hold. Further assume that the following condition holds:*

(H_3) $f(t,x)$ *is non-decreasing with respect to* x *for* $t \in [0,1]$.

Then there exist $0 < \lambda_* \leq \lambda^*$ *such that*

(i) $(1.1)_\lambda$ *has at least two positive solutions in* K *if* $0 < \lambda < \lambda_*$;

(ii) $(1.1)_\lambda$ *has at least one positive solution in* K *if* $\lambda_* \leq \lambda \leq \lambda^*$;

(iii) $(1.1)_\lambda$ *has no positive solutions in* K *if* $\lambda > \lambda^*$.

Proof. Let S and S_1 be defined by (3.2) and (3.3), respectively. Set $\lambda_* = \sup S$ and $\lambda^* = \sup S_1$. By Theorem 3.1, (i) and (iii) are fulfilled, and $(1.1)_\lambda$ has at least one positive solution in K if $\lambda = \lambda_*$ and $\lambda = \lambda^*$.

It remains to prove that $(1.1)_\lambda$ has at least one positive solution in K if $\lambda_* < \lambda < \lambda^*$. Let $u_{\lambda^*}(t)$ be a positive solution of $(1.1)_{\lambda^*}$ and fix $\lambda \in (0, \lambda^*)$. We consider the following problem

$$\begin{cases} (a * u'')(t) + \lambda F(t, u(t)) = 0, \quad t \in (0,1), \\ u'(0) = 0, \quad u(1) = \varphi(u), \end{cases}$$

where $F(t,u(t)) = f(t, \tilde{u}(t))$, and $\tilde{u}(t) = \min\{u(t), u_{\lambda^*}(t)\}$ for $t \in [0,1]$. It is easy to check that (2.1) can be rewritten as

$$u(t) = \lambda \int_0^1 H(t,s) f(s, u(s)) ds,$$

where

$$H(t,s) = \frac{1}{1 - \varphi(\mathbf{1})} \int_0^1 G(\tau, s) d\alpha(\tau) + G(t,s), \quad t, s \in [0,1].$$

By Lemma 2.3, $H(t,s)$ is continuous and non-increasing with respect to t. We define an operator by

$$(T_\lambda u)(t) = \lambda \int_0^1 H(t,s) F(s, u(s)) ds, \quad t \in [0,1].$$

By arguments similar to Lemma 2.5, we have $T_\lambda : K \to K$ is completely continuous. Choose sufficiently large $R_1 > \|u_{\lambda^*}\|$ such that

$$F(t,x) < R_1, \quad t \in [0,1], \ x \in [0, +\infty).$$

Take $R_0 = \max\left\{R_1, \lambda^* R_1 \max\limits_{t \in [0,1]} \int_0^1 H(t,s) ds\right\} + 1$. We suppose that $T_\lambda x \neq x$ for $x \in \partial B(0, R_0)$ (otherwise, the proof is finished). Suppose that there exist $u_0 \in \partial B(0, R_0)$ and $\mu_0 \geqslant 1$ such that $\mu_0 u_0 = T_\lambda u_0$. Thus

$$R_0 = \|u_0\| \leq \|\mu_0 u_0\| = \|T_\lambda u_0\| \leq \lambda^* R_1 \max_{t \in [0,1]} \int_0^1 H(t,s) ds < R_0,$$

which is a contradiction. Consequently, we deduce that

$$i_k(T_\lambda, B(0, R_0)) = 1.$$

In addition, since $\liminf_{x\to 0}\min_{t\in[0,1]}\frac{f(t,x)}{x}=+\infty$, we deduce $\liminf_{x\to 0}\min_{t\in[0,1]}\frac{F(t,x)}{x}=+\infty$. Then there exists $0<r_0<R_0$ such that

$$F(t.x)\geq M_0x,\quad t\in[0,1],\ x\in[0,r_0],$$

where $M_0>(\lambda C_0\max_{t\in[0,1]}\int_0^1 H(t,s)ds)^{-1}$. We suppose that there exist $u\in\partial B(0,r_0)$ and $\eta_0>0$ such that $u=T_\lambda u+\eta_0\mathbf{1}$. Since $G(t,s)$ is non-increasing with respect to t, we have

$$\begin{aligned}u(t)&=T_\lambda u(t)+\eta_0\\&\geq T_\lambda u(1)+\eta_0\\&=\varphi(T_\lambda u)+\eta_0\\&\geq C_0\|T_\lambda u\|+\eta_0\\&\geq C_0\max_{s\in[0,1]}(u(s)-\eta_0)+\eta_0\\&>C_0\|u\|,\quad t\in[0,1].\end{aligned}$$

Then

$$\begin{aligned}r_0&=\|u\|\\&=\|T_\lambda u+\eta_0\mathbf{1}\|\\&>\lambda\max_{t\in[0,1]}\int_0^1 H(t,s)F(s,u(s))ds\\&\geq\lambda\max_{t\in[0,1]}\int_0^1 H(t,s)M_0u(s)ds\\&\geq\lambda M_0\max_{t\in[0,1)}\int_0^1 H(t,s)C_0r_0ds\\&>r_0.\end{aligned}$$

This contradiction shows that

$$i_K(T_\lambda,B(0,r_0))=0.$$

By Lemma 2.6 we conclude that T_λ has a fixed point $u\in B(0,R_0)\backslash\overline{B(0,r_0)}$. According to the monotonicity of f, we see that $F(t,u(t))\leq f(t,u_{\lambda^*}(t))$ for $t\in[0,1]$. Then,

$$u(t)=T_\lambda u(t)\leq\lambda^*\int_0^1 H(t,s)f(s,u_{\lambda^*}(s))ds=u_{\lambda^*}(t),\quad t\in[0,1].$$

Therefore $u=T_\lambda u=F_\lambda u$, and F_λ has a fixed point $u\in B(0,R_0)\backslash\overline{B(0,r_0)}$. The proof is now complete.

4. Example

Example 4.1 We consider the positive and non-increasing solutions for the following nonlocal boundary value problem

$$\begin{cases} ({}^C D_t^\alpha u)(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1,\ 1 < \alpha < 2, \\ u'(0) = 0, \quad u(1) = \varphi(u), \end{cases} \tag{4.1}$$

where $\varphi(u) = \int_0^1 u(s)\, d(\frac{1}{2}s)$, $f(t,x) = e^t(x+1)^2$, and $a(t) = g_{2-\alpha}(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}$. Then there exists $c = t^{\alpha-1}\Gamma(2-\alpha)(1 < \alpha < 2)$ such that (H_1) is satisfied. It is easy to verify that the conditions (H_2) and (H_3) hold. Therefore, it follows from Theorem 3.2 that there exist $0 < \lambda_* \le \lambda^*$ such that
(i) (4.1) has at least two positive solutions in K if $0 < \lambda < \lambda_*$;
(ii) (4.1) has at least one positive solution in K if $\lambda_* \le \lambda \le \lambda^*$;
(iii) (4.1) has no positive solutions in K if $\lambda > \lambda^*$.

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References

[1] C.S. Goodrich, *A one-dimensional Kirchhoff equation with generalized convolution coefficients*, J. Fixed Point Theory Appl., **23**(2021), 73, 23 pp.
[2] C.S. Goodrich, *A topological approach to nonlocal elliptic partial differential equations on an annulus*, Math. Nachr., **294**(2021), no. 2, 286-309.
[3] C.S. Goodrich, *Differential equations with multiple sign changing convolution coefficients*, Internat. J. Math., **32**(2021), no. 8, 2150057, 28 pp.
[4] C.S. Goodrich, *Nonlocal differential equations with concave coefficients of convolution type*, Nonlinear Anal., **211**(2021), 112437, 18 pp.
[5] C.S. Goodrich, *Nonlocal differential equations with convolution coefficients and applications to fractional calculus*, Adv. Nonlinear Stud., **21**(2021), no. 4, 767-787.
[6] C.S. Goodrich, *Topological analysis of doubly nonlocal boundary value problems*, J. Fixed Point Theory Appl., **23**(2021), 29, 24 pp.
[7] C.S. Goodrich, *An analysis of nonlocal difference equations with finite convolution coefficients*, J. Fixed Point Theory Appl., **24**(2022), no. 1, 19 pp.
[8] C.S. Goodrich, *Nonexistence and parameter range estimates for convolution differential equations*, Proc. Amer. Math. Soc. Ser. B, **9**(2022), 254-265.
[9] C.S. Goodrich, C. Lizama, *A transference principle for nonlocal operators using a convolutional approach: Fractional monotonicity and convexity*, Israel J. Math., **236**(2020), no. 2, 533-589.
[10] C.S. Goodrich, C. Lizama, *Positivity, monotonicity, and convexity for convolution operators*, Discrete Contin. Dyn. Syst., **40**(2020), no. 8, 4961-4983.
[11] C.S. Goodrich, C. Lizama, *Existence and monotonicity of nonlocal boundary value problems: The one-dimensional case*, Proc. Roy. Soc. Edinburgh Sect. A: Math., **152**(2022), no. 1, 1-27.
[12] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.
[13] J.W. He, C. Lizama, Y. Zhou, *The Cauchy problem for discrete time fractional evolution equations*, J. Comput. Appl. Math., **370**(2020), 112683, 15 pp.
[14] G. Infante, *Positive and increasing solutions of perturbed Hammerstein integral equations with derivative dependence*, Discrete Contin. Dyn. Syst. Ser. B, **25**(2020), no. 2, 691-699.

[15] R.-Y. Lai, Y.-H. Lin, *Inverse problems for fractional semilinear elliptic equations*, Nonlinear Anal., **216**(2022), 112699, 21 pp.

[16] T. Liimatainen, Y.H. Lin, M. Salo, T. Tyni, *Inverse problems for elliptic equations with fractional power type nonlinearities*, J. Differential Equations, **306**(2022), 189-219.

[17] F. Liu, H. Luo, G. Dai, *Global bifurcation and nodal solutions for homogeneous Kirchhoff type equations*, Electron. J. Qual. Theory Diff. Equ., 2020, no. 29, 13 pp.

[18] B. Yan, D. Wang, *The multiplicity of positive solutions for a class of nonlocal elliptic problem*, J. Math. Anal. Appl., **442**(2016), no. 1, 72-102.

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