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APPROXIMATING BEST PROXIMITY POINT OF MULTI-VALUED MAPPINGS VIA MANN'S ITERATION AND ISHIKAWA'S ITERATION

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Abstract. In this present work, we define the notion of proximally generalized nonexpansive and proximally quasi-contractions in case of multi-valued non-self mappings. Using projective operator, we propose Mann's iteration and Ishikawa's iteration associated with multi-valued non-self mappings in the setting of Banach space. Through the proposed iterative processes, we approximate best proximity point of multi-valued non-self mappings under certain assumptions.
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1. Introduction

Assume that M and N are nonempty subsets of a metric space (X, d). If $M \cap N = \emptyset$, a mapping Γ from M to N, does not have a solution for the fixed point equation $\Gamma(\xi) = \xi$. At this situation, it is desire to determine an approximate solution ξ such that the error $d(\xi, \Gamma\xi)$ is minimum.The purpose of theorems on best proximity point is to find sufficient conditions such that the minimization problem $\min_{\xi} d(\xi, \Gamma\xi)$, possess the existence of solution. In this sequel, many mathematicians proved the best proximity point theorems for different kind of contractions. We refer the reader for more existence theorems of best proximity point [8, 9, 20, 23, 24, 25].

The concept of multi-valued mappings plays an important role in many areas such as game theory, control theory, differential equations, economics and convex optimization. Initially, the theorems on fixed points for the mappings setvalued contractions and setvalued nonexpansive through the Hausdorff metric which was introduced by Markin in [16]. For detail analysis, we suggest [5, 6, 14, 17].

Later, the researchers want to find the approximate solution of fixed point equations for multi-valued mappings when it is non-self case. In [2], Abkar and Gabeleh derived the existence result for best proximity point for the case of multi-valued non-self mappings. For more research article, we refer [3, 12, 18] and references therein.

Here we state a result which was proved by Fan [11].

Theorem 1.1. *Let $M \neq \emptyset$, be a convex compact subset of a normed linear space X and $\Gamma : M \to X$ be continuous on M. Then there exists an element $\xi \in M$ such that $\|\xi - \Gamma\xi\| = dist(\Gamma\xi, M) := inf\{\|\Gamma\xi - c\| : c \in M\}$.*

In [26], Sastry et al. proved convergence results on fixed points in Hilbert space for various type of multi-valued mappings by using Mann and Ishikawa iterative process. In the similar manner, Panyanak [19], has proved the convergence results in Banach space setting. For more details on the convergence results we refer [1, 4, 5, 7, 8, 10, 6, 14, 17, 13, 21] and references therein.

In this research article, we define proximally generalized nonexpansive and proximally quasi-contractive multi-valued mappings and we construct the Mann and Ishikawa iteration in the setting of non-self multi-valued mappings and projective operator. Also, we prove the convergence theorems on best proximity point for multi-valued mappings through Ishikawa and Mann iterative process. For this, we define a new class of mappings which satisfies a condition, called Condition (I^*). This condition relaxes the assumption on compactness on domain. Some of our results extend the work of Shahzad and Zegeye [27].

2. Preliminaries

The following notions are used subsequently:
Let M, N be nonempty subsets of a given Banach space B.

$$
\begin{aligned}
&d(M,N) = dist(M,N) = inf\{\|\xi - \zeta\| : \xi \in M, \zeta \in N\};\\
&M_0 = \{\xi \in M : \|\xi - \zeta'\| = dist(M,N) \text{ for some } \zeta' \in N\};\\
&N_0 = \{\zeta \in N : \|\xi' - \zeta\| = dist(M,N) \text{ for some } \xi' \in M\};\\
&d(\xi,N) = inf\{\|\xi - \zeta\| : \zeta \in N\};\\
&CL(B) = \{U : U \text{ is closed in } B\}; \text{ and } 2^B = \{U : U \subseteq B\}.
\end{aligned}
$$

Assume that $\Gamma : M \to 2^N$. A point $\xi \in M$ is said to be best proximity point of Γ, if $d(\xi, \Gamma\xi) = d(M,N)$. The set of all best proximity point of Γ is denoted by $Bst(\Gamma)$.

A set $S \subset B$, is said to be proximinal if for $\xi \in B$, there exists $s \in S$ such that $d(\xi, s) = d(\xi, S) = inf\{\|\xi - \zeta\| : \zeta \in S\}$.
For example, every convex and closed subset of a uniformly convex Banach space is proximinal.
We consider the set

$$\wp(X) = \{M \subset X : M \neq \emptyset, \text{bounded, closed, convex}\}.$$

Let $H(M,N) = max\{\sup_{\xi \in M} d(\xi,N), \sup_{\zeta \in N} d(\zeta,M)\}$, Hausdorff distance on $\wp(X)$, where $M, N \in \wp(X)$.

From the well known iteration process such as Mann's and Ishikawa's iteration for self mappings, we give the Mann's type and Ishikawa's type sequence construction for non-self mappings as following:

Let B be a Banach space. Let (M, N) be a pair of convex subsets of B. Let the map $\Gamma : M \to 2^N$ be a multi-valued. Assume $\Gamma\xi_0 \subseteq N_0$ for every $\xi_0 \in M_0$. And we take $P_\Gamma(u) = \{v \in \Gamma u : \|u - v\| = d(u, \Gamma u)\}$. If Γu is convex, closed subset of a reflexive and strictly convex space, then $P_\Gamma(u)$ contains one element. Now we define following:

(I_1) The sequence of Mann's iteration, for $\xi_0 \in M_0$, we have $\Gamma\xi_0 \subseteq N_0$. Choose $\zeta_0 \in P_\Gamma\xi_0$, then there exists $u_0 \in M_0$ such that $\|u_0 - \zeta_0\| = d(M, N)$. Define $\xi_1 = (1 - \delta_0)\xi_0 + \delta_0 u_0$. By continuing this process, we define in general, for each $n \geq 0$, $\xi_n \in M_0$, we have $\Gamma\xi_n \subseteq N_0$. Choose $\zeta_n \in P_\Gamma\xi_n$, then there exist $u_n \in M_0$ such that $\|u_n - \zeta_n\| = d(M, N)$. Define $\xi_{n+1} = (1 - \delta_n)\xi_n + \delta_n u_n$, where $\{\delta_n\}$ satisfies $0 \leq \delta_n < 1$.

(I_2) The sequence of Ishikawa's iteration, for $\xi_0 \in M_0$, we have $\Gamma\xi_0 \subseteq N_0$. Choose $\zeta_0 \in P_\Gamma\xi_0$, then there exist $v_0 \in M_0$ such that $\|v_0 - \zeta_0\| = d(M, N)$. Define $u_0 = (1 - \eta_0)\xi_0 + \eta_0 v_0$. Then $\Gamma u_0 \subseteq N_0$. Choose $z_0 \in P_\Gamma u_0$, then there exist $w_0 \in M_0$ such that $\|w_0 - z_0\| = d(M, N)$. Define $\xi_1 = (1 - \delta_0)\xi_0 + \delta_0 w_0$. By continuing this process, we define in general, for each $n \geq 0$, $\xi_n \in M_0$, we have $\Gamma\xi_n \subseteq N_0$. Choose $\zeta_n \in P_\Gamma\xi_n$, then there exist $v_n \in M_0$ such that $\|v_n - \zeta_n\| = d(M, N)$. Define $u_n = (1 - \eta_n)\xi_n + \eta_n v_n$. Then $\Gamma u_n \subseteq N_0$. Choose $z_n \in P_\Gamma u_n$, then there exist $w_n \in M_0$ such that $\|w_n - z_n\| = d(M, N)$. Define $\xi_{n+1} = (1 - \delta_n)\xi_n + \delta_n w_n$, where $\{\delta_n\}, \{\eta_n\}$ is a sequence such that $0 \leq \delta_n, \eta_n < 1$.

We extend the definitions in [26], for multi-valued mappings of the form $\Gamma : M \to 2^N$.

Definition 2.1. The multi-valued mapping $\Gamma : M \to 2^N$ is

(i) nonexpansive if $H(\Gamma\xi, \Gamma\zeta) \leq \|\xi - \zeta\|$ for every $\xi, \zeta \in M$.

(ii) quasi-nonexpansive if τ is best proximity point of Γ and satisfies

$$d(\Gamma\xi, \tau) \leq \|\xi - \tau\|$$

for every $\xi \in M$.

(iii) proximally generalized nonexpansive if

$$\begin{aligned} H(\Gamma\xi, \Gamma\zeta) \leq &\alpha\|\xi - \zeta\| + \beta\{d(\xi, \Gamma\xi) + d(\zeta, \Gamma\zeta) - 2d(M, N)\} \\ &+ \gamma\{d(\xi, \Gamma\zeta) + d(\zeta, \Gamma\xi) - 2d(M, N)\} \end{aligned}$$

for all $\xi, \zeta \in M$, where α, β, γ satify $\alpha + 2\beta + 2\gamma \leq 1$.

(iv) proximally quasi-contractive if $k \in [0, 1)$, with

$$\begin{aligned} H(\Gamma\xi, \Gamma\zeta) \leq &\kappa \max\{\|\xi - \zeta\|, d(\xi, \Gamma\xi) - d(M, N), d(\zeta, \Gamma\zeta) - d(M, N), \\ &d(\xi, \Gamma\zeta) - d(M, N), d(\zeta, \Gamma\xi) - d(M, N)\} \end{aligned}$$

for all $\xi, \zeta \in M$.

Definition 2.2. [2] If $M_0 \neq \emptyset$ then the pair (M, N) has P-property iff for any $\xi_1, \xi_2 \in M_0$ and $\zeta_1, \zeta_2 \in N_0$

$$\begin{cases} d(\xi_1, \zeta_1) = d(M, N) \\ d(\xi_2, \zeta_2) = d(M, N) \end{cases} \Rightarrow d(\xi_1, \xi_2) = d(\zeta_1, \zeta_2).$$

Lemma 2.3. [28] *Let B be a Banach space. Then B is uniformly convex space iff for every constant $\kappa > 0$, the norm $\|.\|^2$ of B is uniformly convex on B_κ, (closed ball with radius κ centered at the origin), there exists strictly increasing and continuous function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that*

$$\|(1-\delta)\xi + \delta\zeta\|^2 \leq (1-\delta)\|\xi\|^2 + \delta\|\zeta\|^2 - \delta(1-\delta)\phi(\|\xi - \zeta\|),$$

for all $\xi, \zeta \in B_\kappa, \delta \in [0,1]$.

Lemma 2.4. [15] *Let $\kappa \in [0,1]$. Then for $\xi, \zeta \in H$, where H is Hilbert space,*

$$\|(1-\kappa)\xi + \kappa\zeta\|^2 = (1-\kappa)\|\xi\|^2 + \kappa\|\zeta\|^2 - \kappa(1-\kappa)\|\xi - \zeta\|^2.$$

Lemma 2.5. [26] *Let $\{\delta_n\}, \{\eta_n\}$ be two sequences such that (i) $0 \leq \delta_n, \eta_n < 1$, (ii) $\eta_n \to 0$ and (iii) $\sum \delta_n \eta_n = \infty$. Let $\{\gamma_n\}$ be a non-negative real sequence such that $\sum \delta_n \eta_n (1-\eta_n)\gamma_n$ is bounded. Then $\{\gamma_n\}$ has a subsequence which converges to zero.*

Lemma 2.6. [22] *Let $\{\xi_n\}$ be a real sequence such that $\xi_{n+1} \leq \delta\xi_n + \eta_n$ where $\xi_n \geq 0, \eta_n \geq 0$ and $\lim_{n\to\infty} \eta_n = 0$, $0 \leq \delta < 1$. Then $\lim_{n\to\infty} \xi_n = 0$.*

In the next section, we prove our main results on multi-valued non-self mappings.

3. Main results

In this section, first we extend the condition (I) in [27], to the case of non-self mappings.

Definition 3.1. The mapping $\Gamma : M \to \wp(N)$ is said to satisfy Condition (I^*) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0,\infty)$ such that $d(x, \Gamma x) - d(M,N) \geq f(d(x, Bst(\Gamma))$ for all $x \in M$.

Theorem 3.2. *Let M, N be two convex subsets of uniformly convex Banach space B. Suppose $\Gamma : M \to \wp(N)$ is a multi-valued mapping with $Bst(\Gamma) \neq \emptyset$ such that P_Γ is nonexpansive. Let $\{\xi_n\}$ be Ishikawa's type iteration defined by (I_2). Assume that Γ satisfies condition (I^*) and the sequences $\{\delta_n\}, \{\eta_n\}$ satisfy the following: (i) $0 \leq \delta_n, \eta_n < 1$, (ii) $\eta_n \to 0$ and (iii) $\sum \delta_n \eta_n = \infty$. Then the sequence $\{\xi_n\}$ converges to a best proximity point of Γ.*

Proof. Let τ be a best proximity point of Γ. By Lemma 2.3, we have

$$\begin{aligned} \|\xi_{n+1} - \tau\|^2 &= \|(1-\delta_n)\xi_n + \delta_n w_n - \tau\|^2 \\ &\leq (1-\delta_n)\|\xi_n - \tau\|^2 + \delta_n\|w_n - \tau\|^2 - \delta_n(1-\delta_n)\phi(\|\xi_n - w_n\|). \end{aligned}$$

Since $\Gamma\tau$ is proximinal, there exist $\tau^* \in \Gamma\tau$ such that $\|\tau - \tau^*\| = d(M,N)$. Because of τ is best proximity point of Γ, we get $\tau^* \in P_\Gamma(\tau)$. Now by P-property, we have

$$\|\xi_{n+1} - \tau\|^2 \leq (1-\delta_n)\|\xi_n - \tau\|^2 + \delta_n\|z_n - \tau^*\|^2 - \delta_n(1-\delta_n)\phi(\|\xi_n - w_n\|).$$

Since $P_\Gamma(u_n)$ is singleton implies $P_\Gamma(u_n) = \{z_n\}$. Then we have

$$\|z_n - \tau^*\| = d(P_\Gamma(u_n), \tau^*).$$

Therefore

$$\begin{aligned}\|\xi_{n+1}-\tau\|^2 &\leq (1-\delta_n)\|\xi_n-\tau\|^2+\delta_n d(P_\Gamma(u_n),\tau^*)^2\\ &\quad -\delta_n(1-\delta_n)\phi(\|\xi_n-w_n\|)\\ &\leq (1-\delta_n)\|\xi_n-\tau\|^2+\delta_n H^2(P_\Gamma u_n,P_\Gamma\tau)\\ &\quad -\delta_n(1-\delta_n)\phi(\|\xi_n-w_n\|)\\ &\leq (1-\delta_n)\|\xi_n-\tau\|^2+\delta_n\|u_n-\tau\|^2. \end{aligned} \tag{3.1}$$

Now

$$\begin{aligned}\|u_n-\tau\|^2 &= \|(1-\eta_n)\xi_n+\eta_n v_n-\tau\|^2\\ &\leq (1-\eta_n)\|\xi_n-\tau\|^2+\eta_n\|v_n-\tau\|^2-\eta_n(1-\eta_n)\phi(\|\xi_n-v_n\|)\\ &\leq (1-\eta_n)\|\xi_n-\tau\|^2+\eta_n\|\zeta_n-\tau^*\|^2-\eta_n(1-\eta_n)\phi(\|\xi_n-v_n\|).\end{aligned}$$

Since $P_\Gamma(\xi_n)$ is singleton implies $P_\Gamma(\xi_n)=\{\zeta_n\}$. Then we have

$$\|\zeta_n-\tau^*\|=d(P_\Gamma(\xi_n),\tau^*).$$

Therefore

$$\begin{aligned}\|u_n-\tau\|^2 &\leq (1-\delta_n)\|\xi_n-\tau\|^2+\delta_n d(P_\Gamma(\xi_n),\tau^*)^2-\delta_n(1-\delta_n)\phi(\|\xi_n-v_n\|)\\ &\leq (1-\eta_n)\|\xi_n-\tau\|^2+\eta_n H^2(P_\Gamma\xi_n,P_\Gamma\tau)-\eta_n(1-\eta_n)\phi(\|\xi_n-v_n\|)\\ &\leq \|\xi_n-\tau\|^2-\eta_n(1-\eta_n)\phi(\|\xi_n-v_n\|).\end{aligned} \tag{3.2}$$

Therefore (3.1) becomes,

$$\|\xi_{n+1}-\tau\|^2 \leq \|\xi_n-\tau\|^2-\delta_n\eta_n(1-\eta_n)\phi(\|\xi_n-v_n\|) \tag{3.3}$$

$$\delta_n\eta_n(1-\eta_n)\phi(\|\xi_n-v_n\|) \leq \|\xi_n-\tau\|^2-\|\xi_{n+1}-\tau\|^2.$$

Then

$$\sum_{n=1}^{\infty}\delta_n\eta_n(1-\eta_n)\phi(\|\xi_n-v_n\|)\leq\|\xi_1-\tau\|^2<\infty.$$

By Lemma 2.5, there exist a subsequence $\{\xi_{n_k}-v_{n_k}\}$ such that $\phi(\|\xi_{n_k}-v_{n_k}\|)\to 0$ as $k\to\infty$ and hence $\|\xi_{n_k}-v_{n_k}\|\to 0$.
By triangle inequality, we have

$$\begin{aligned}\|\xi_{n_k}-\zeta_{n_k}\| &\leq \|\xi_{n_k}-v_{n_k}\|+\|v_{n_k}-\zeta_{n_k}\|\\ &\leq \|\xi_{n_k}-v_{n_k}\|+d(M,N)\\ \|\xi_{n_k}-\zeta_{n_k}\|-d(M,N) &\leq \|\xi_{n_k}-v_{n_k}\|.\end{aligned}$$

Since $\zeta_{n_k}\in P_\Gamma\xi_{n_k}$ implies $\|\xi_{n_k}-\zeta_{n_k}\|=d(\xi_{n_k},\Gamma\xi_{n_k})$. Since $\|\xi_{n_k}-v_{n_k}\|\to 0$ implies $d(\xi_{n_k},\Gamma\xi_{n_k})-d(M,N)\to 0$. By condition (I^*), we get $d(\xi_{n_k},Bst(\Gamma))\to 0$. Therefore, there exists a subsequence of ξ_{n_k} (we assume same sequence by without loss of generality) such that $\|\xi_{n_k}-\tau_k\|<\frac{1}{2^k}$ for some $\tau_k\in Bst(\Gamma)$ for all k. From (3.3), we get

$$\|\xi_{n_{k+1}}-\tau_k\|\leq\|\xi_{n_k}-\tau_k\|<\frac{1}{2^k}.$$

Now we prove that $\{\tau_k\}$ is Cauchy sequence in M_0. For,

$$\begin{aligned}\|\tau_{k+1}-\tau_k\| &\leq \|\tau_{k+1}-\xi_{n_{k+1}}\|+\|\xi_{n_{k+1}}-\tau_k\|\\ &\leq \frac{1}{2^{k+1}}+\frac{1}{2^k}\\ &\leq \frac{1}{2^{k-1}}.\end{aligned}$$

This implies that $\{\tau_k\}$ is a Cauchy sequence in M_0 and thus converges to $\tau' \in M_0$. Now we have

$$\begin{aligned}d(\tau_k,\Gamma\tau') &\leq d(\tau_k,\Gamma\tau_k)+H(\Gamma\tau_k,\Gamma\tau')\\ &\leq d(M,N)+\|\tau_k-\tau'\|.\end{aligned}$$

As $k\to\infty$, we obtain $d(\tau',\Gamma\tau')=d(M,N)$. Then $\tau'\in Bst(\Gamma)$. Clearly, we have $\xi_{n_k}\to\tau'$.

Now replacing τ' in place of τ, we get, the sequence $\{\|\xi_n-\tau'\|\}$, is a nonincreasing by (3.3). Since $\|\xi_{n_k}-\tau'\|\to 0$ as $k\to\infty$, then it gives that $\|\xi_n-\tau'\|\to 0$. Hence the theorem follows. □

Remark 3.3. If we assume $M=N$, the above result reduces to the result proved by Shahzad and Zegeye [27].

To support our main result, we provide the following example:

Example 3.1. Let $X=\mathbb{R}^2$ with Euclidean norm. Assume $M=\{(0,x):x\in[0,1]\}, N=\{(1,x):x\in[0,1]\}$ and $\Gamma:M\to\mathcal{P}(N)$ defined by $\Gamma(0,x)=\{(1,y):y\in[0,x]\}$ for $x\in[0,1/2], \Gamma(0,x)=\{(1,1/2)\}$ for $x\in(1/2,1]$. Therefore, $Bst(\Gamma)=\{(0,x):x\in[0,1/2]\}$. Also, one can observe that

$$P_\Gamma(0,x)=\begin{cases}\{(1,x)\} & \text{if } x\in[0,1/2],\\ \{(1,1/2)\} & \text{if } x\in(1/2,1].\end{cases}$$

Now we verify P_Γ is nonexpansive. If $(0,x),(0,y)\in M$ with $x,y\in[0,1/2]$.

$$\begin{aligned}H(P_\Gamma(0,x),P_\Gamma(0,y)) &= H(\{(1,x)\},\{(1,y)\})\\ &= \| (1,x)-(1,y) \|\\ &= \| (0,x)-(0,y) \| .\end{aligned}$$

If $(0,x),(0,y)\in M$ with $x\in[0,1/2], y\in(1/2,1]$. Since $1/2<y\leq 1$,

$$\begin{aligned}H(P_\Gamma(0,x),P_\Gamma(0,y)) &= H(\{(1,x)\},\{(1,1/2)\})\\ &\leq \| (0,x)-(0,y) \| .\end{aligned}$$

If $(0,x),(0,y)\in M$ with $x,y\in(1/2,1]$.

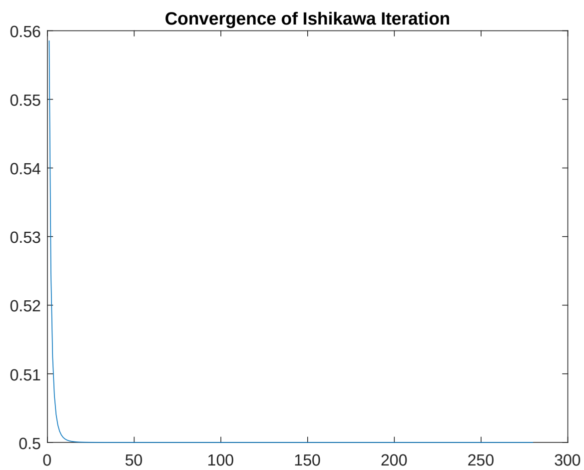
$$\begin{aligned}H(P_\Gamma(0,x),P_\Gamma(0,y)) &= H(\{(1,1/2)\},\{(1,1/2)\})=0\\ &\leq \| (0,x)-(0,y) \| .\end{aligned}$$

We choose the function $f:[0,\infty)\to[0,\infty)$ defined by $f(x)=x/3$. Then one can easily verify that Γ satisfies condition (I^*).
Let $\delta_n=\eta_n=\frac{1}{\sqrt{n+1}}, n\in\mathbb{N}$. If $\xi_1=(0,x)\in M$ with $x\in[0,1/2]$ then one can

observe that $\xi_n = (0,x)$ for all $n \in \mathbb{N}$. So the Ishikawa sequence $\{\xi_n\}$ converges to $(0,x) \in Bst(\Gamma)$. If $\xi_1 = (0,x) \in M$ with $x \in (1/2,1]$ the the Ishikawa sequence becomes $\xi_{n+1} = (1-\frac{1}{\sqrt{n+1}})(0,x)+\frac{1}{\sqrt{n+1}}(0,1/2)$. Using Matlab coding, we obtain the following convergence table for approaching the best proximity point of Γ.

TABLE 1. Convergence of Ishikawa iteration to best proximity point via Matlab coding with $\xi_1 = (0, 0.7) \in M$.

n	**Ishikawa Iteration**
1	(0, 0.558578643762691)
2	(0, 0.524758248017538)
3	(0, 0.512379124008769)
4	(0, 0.506843011451668)
⋮	⋮
262	(0, 0.500000000000001)
263	(0, 0.500000000000001)
264	**(0, 0.500000000000000)**



The figure shows convergence of Ishikawa iteration to the best proximity point by using the continuous data points from 0.7 to 300.

Corollary 3.1. *Let M,N be two convex subsets of uniformly convex Banach space B. Suppose $\Gamma : M \to 2^N$ is a multi-valued mapping with $Bst(\Gamma) \neq \emptyset$ such that P_Γ is nonexpansive. Assume $\Gamma(\xi)$ is closed convex in N for every $\xi \in M$. Let $\{\xi_n\}$ be Ishikawa's type iteration defined by (I_2). Assume that Γ satisfies condition (I^*) and the sequences $\{\delta_n\},\{\eta_n\}$ satisfy the following: (i) $0 \leq \delta_n, \eta_n < 1$, (ii) $\eta_n \to 0$ and (iii) $\sum \delta_n\eta_n = \infty$. Then the sequence $\{\xi_n\}$ converges to a best proximity point of Γ.*

Theorem 3.4. *Let M, N be two convex subsets of uniformly convex Banach space B. Suppose $\Gamma : M \to \wp(N)$ is a multi-valued mapping with $Bst(\Gamma) \neq \emptyset$ such that P_Γ is nonexpansive. Let $\{\xi_n\}$ be Mann's type iteration defined by (I_1). Assume that Γ satisfies condition (I^*) and the sequence $\{\delta_n\}$ satisfies the following: (i) $0 \leq \delta_n < 1$, (ii) $\sum \delta_n = \infty$. Then the sequence $\{\xi_n\}$ converges to a best proximity point of Γ.*

Proof. The result is similar to that of Theorem 3.2. □

Corollary 3.2. *Let M, N be two convex subsets of uniformly convex Banach space B. Suppose $\Gamma : M \to 2^N$ is a multi-valued mapping with $Bst(\Gamma) \neq \emptyset$ such that P_Γ is nonexpansive. Assume $\Gamma(\xi)$ is closed convex in N for every $\xi \in M$. Let $\{\xi_n\}$ be Mann's type iteration defined by (I_1). Assume that Γ satisfies condition (I^*) and the sequence $\{\delta_n\}$ satisfies the following: (i) $0 \leq \delta_n < 1$, (ii) $\sum \delta_n = \infty$. Then the sequence $\{\xi_n\}$ converges to a best proximity point of Γ.*

Theorem 3.5. *Let M, N be two convex subsets of uniformly convex Banach space B. Suppose $\Gamma : M \to \wp(N)$ is a multi-valued mapping with $Bst(\Gamma) \neq \emptyset$ such that P_Γ is proximally generalized nonexpansive. Let $\{\xi_n\}$ be Ishikawa's type iteration defined by (I_2). Assume that Γ satisfies condition (I^*) and the sequences $\{\delta_n\}, \{\eta_n\}$ satisfy the following: (i) $0 \leq \delta_n, \eta_n < 1$, (ii) $\eta_n \to 0$ and (iii) $\sum \delta_n \eta_n = \infty$. Then the sequence $\{\xi_n\}$ converges to a best proximity point of Γ.*

Proof. From the above theorem, we have

$$\begin{aligned} \|\xi_{n+1} - \tau\|^2 &\leq (1-\delta_n)\|\xi_n - \tau\|^2 + \delta_n H^2(P_\Gamma u_n, P_\Gamma \tau) \\ &\quad -\delta_n(1-\delta_n)\phi(\|\xi_n - w_n\|). \end{aligned} \tag{3.4}$$

Since P_Γ is a proximally generalized nonexpansive non-self map, we have

$$\begin{aligned} H(P_\Gamma u_n, P_\Gamma \tau) &\leq \alpha\|u_n - \tau\| + \beta\{d(u_n, P_\Gamma u_n) + d(\tau, P_\Gamma \tau) - 2d(M,N)\} \\ &\quad +\gamma\{d(\tau, P_\Gamma u_n) + d(u_n, P_\Gamma \tau) - 2d(M,N)\} \\ &\leq \alpha\|u_n - \tau\| + \beta\{\|u_n - \tau\| + d(\tau, P_\Gamma u_n) - d(M,N)\} \\ &\quad +\gamma\{d(\tau, P_\Gamma u_n) + \|u_n - \tau\| + d(\tau, P_\Gamma \tau) - 2d(M,N)\} \\ &\leq \alpha\|u_n - \tau\| + \beta\{\|u_n - \tau\| + d(\tau, P_\Gamma \tau) + H(P_\Gamma \tau, P_\Gamma u_n) \\ &\quad -d(M,N)\} + \gamma\{d(\tau, P_\Gamma \tau) + H(P_\Gamma \tau, P_\Gamma u_n) \\ &\quad +\|u_n - \tau\| - d(M,N)\} \\ &= (\alpha+\beta+\gamma)\|u_n - \tau\| + (\beta+\gamma)H(P_\Gamma \tau, P_\Gamma u_n). \end{aligned}$$

Hence

$$H(P_\Gamma u_n, P_\Gamma \tau) \leq \frac{(\alpha+\beta+\gamma)}{(1-(\beta+\gamma))}\|u_n - \tau\|.$$

Since $\frac{(\alpha+\beta+\gamma)}{(1-(\beta+\gamma))} \leq 1$, it gives $H(P_\Gamma u_n, P_\Gamma \tau) \leq \|u_n - \tau\|$. From (3.4), we get

$$\|\xi_{n+1} - \tau\|^2 \leq (1-\delta_n)\|\xi_n - \tau\|^2 + \delta_n\|u_n - \tau\|^2,$$

which is inequality (3.1). Similarly one can prove that the inequality (3.2),

$$\|u_n - \tau\|^2 = \|\xi_n - \tau\|^2 - \eta_n(1-\eta_n)\phi(\|\xi_n - v_n\|).$$

Now proceeding in the same way as in Theorem 3.2, the result follows. □

Remark 3.6. If we assume $M = N$, the above result reduces to the result proved by Sastry and Babu [26].

Corollary 3.3. *Let M, N be two convex subsets of uniformly convex Banach space B. Suppose $\Gamma : M \to 2^N$ is a multi-valued mapping with $Bst(\Gamma) \neq \emptyset$ such that P_Γ is proximally generalized nonexpansive. Assume $\Gamma(\xi)$ is closed convex in N for every $\xi \in M$. Let $\{\xi_n\}$ be Ishikawa's type iteration defined by (I_2). Assume that Γ satisfies condition (I^*) and the sequences $\{\delta_n\}, \{\eta_n\}$ satisfy the following: (i) $0 \leq \delta_n, \eta_n < 1$, (ii) $\eta_n \to 0$ and (iii) $\sum \delta_n \eta_n = \infty$. If M is compact, then the sequence $\{\xi_n\}$ converges to a best proximity point of Γ.*

Theorem 3.7. *Let M, N be two convex subsets of uniformly convex Banach space B. Suppose $\Gamma : M \to \wp(N)$ is a multi-valued mapping with $Bst(\Gamma) \neq \emptyset$ such that P_Γ is proximally generalized nonexpansive. Let $\{\xi_n\}$ be Mann's type iteration defined by (I_1). Assume that Γ satisfies condition (I^*) and the sequence $\{\delta_n\}$ satisfies the following: (i) $0 \leq \delta_n < 1$, (ii) $\sum \delta_n = \infty$. Then the sequence $\{\xi_n\}$ converges to a best proximity point of Γ.*

Corollary 3.4. *Let M, N be two convex subsets of uniformly convex Banach space B. Suppose $\Gamma : M \to 2^N$ is a multi-valued mapping with $Bst(\Gamma) \neq \emptyset$ such that P_Γ is proximally generalized nonexpansive. Assume $\Gamma(\xi)$ is closed convex in N for every $\xi \in M$. Let $\{\xi_n\}$ be Mann's type iteration defined by (I_1). Assume that Γ satisfies condition (I^*) and the sequence $\{\delta_n\}$ satisfies the following: (i) $0 \leq \delta_n < 1$, (ii) $\sum \delta_n = \infty$. Then the sequence $\{\xi_n\}$ converges to a best proximity point of Γ.*

Theorem 3.8. *Let M, N be two convex bounded subsets of Hilbert space X. Suppose $\Gamma : M \to \wp(N)$ is a multi-valued mapping with $Bst(\Gamma) \neq \emptyset$ such that P_Γ is proximally quasi-contractive. Let $\{\xi_n\}$ be Ishikawa's type iteration defined by (I_2). Assume the sequences $\{\delta_n\}, \{\eta_n\}$ satisfy the following: (i) $0 \leq \delta_n, \eta_n < 1$, (ii) $\eta_n \to 0$ as $n \to \infty$, with $c \leq \delta_n \leq 1 - \kappa^2$ for some $c > 0$. Then the sequence $\{\xi_n\}$ converges to a best proximity point of Γ.*

Proof. Let τ be a best proximity point of Γ. By Lemma 2.4, we have

$$\begin{aligned} \|\xi_{n+1} - \tau\|^2 &= \|(1-\delta_n)\xi_n + \delta_n w_n - \tau\|^2 \\ &= (1-\delta_n)\|\xi_n - \tau\|^2 + \delta_n\|w_n - \tau\|^2 - \delta_n(1-\delta_n)\|\xi_n - w_n\|^2. \end{aligned} \tag{3.5}$$

Since $\Gamma\tau$ is proximinal, there exist $\tau^* \in \Gamma\tau$ such that $\|\tau - \tau^*\| = d(M, N)$. Because of τ is best proximity point of Γ, we get $\tau^* \in P_\Gamma(\tau)$. Now by P-property, we have

$$\|w_n - \tau\| = \|z_n - \tau^*\|. \tag{3.6}$$

Since $P_\Gamma(u_n)$ is singleton implies $P_\Gamma(u_n) = \{z_n\}$. Then we have

$$\|z_n - \tau^*\| = d(P_\Gamma u_n, \tau^*).$$

Therefore, we obtain

$$\|w_n - \tau\| \leq d(P_\Gamma u_n, \tau^*) \leq H(P_\Gamma u_n, P_\Gamma \tau). \tag{3.7}$$

Now

$$\begin{aligned} H(P_\Gamma\tau, P_\Gamma u_n) &\leq \kappa \max\{\|\tau - u_n\|, d(\tau, P_\Gamma\tau) - d(M,N), \\ &\quad d(u_n, P_\Gamma u_n) - d(M,N), d(\tau, P_\Gamma u_n) - d(M,N), \\ &\quad d(u_n, P_\Gamma\tau) - d(M,N)\} \\ &\leq \kappa \max\{\|\tau - u_n\|, d(u_n, P_\Gamma u_n) - d(M,N), \\ &\quad d(\tau, P_\Gamma\tau) + H(P_\Gamma\tau, P_\Gamma u_n) - d(M,N), \\ &\quad \|u_n - \tau\| + d(\tau, P_\Gamma\tau) - d(M,N)\} \\ &= \kappa \max\{\|\tau - u_n\|, d(u_n, P_\Gamma u_n) - d(M,N), H(P_\Gamma\tau, P_\Gamma u_n), \\ &\quad \|u_n - \tau\|\}. \end{aligned}$$

Suppose $H(P_\Gamma\tau, P_\Gamma u_n)$ is maximum, then we have

$$H(P_\Gamma\tau, P_\Gamma u_n) \leq \kappa H(P_\Gamma\tau, P_\Gamma u_n) < H(P_\Gamma\tau, P_\Gamma u_n).$$

Therefore, we get

$$H(P_\Gamma\tau, P_\Gamma u_n) \leq \kappa \max\{\|\tau - u_n\|, d(u_n, P_\Gamma u_n) - d(M,N)\}.$$

Then

$$\begin{aligned} H^2(P_\Gamma\tau, P_\Gamma u_n) &\leq \kappa^2 \max\{\|\tau - u_n\|^2, [d(u_n, P_\Gamma u_n) - d(M,N)]^2\} \\ &\leq \kappa^2 \{\|\tau - u_n\|^2 + [d(u_n, P_\Gamma u_n) - d(M,N)]^2\}. \end{aligned} \tag{3.8}$$

Now by Lemma 2.4, we obtain

$$\begin{aligned} \|u_n - \tau\|^2 &= \|(1-\eta_n)\xi_n + \eta_n v_n - \tau\|^2 \\ &= (1-\eta_n)\|\xi_n - \tau\|^2 + \eta_n\|v_n - \tau\|^2 - \eta_n(1-\eta_n)\|\xi_n - v_n\|^2 \end{aligned}$$

and

$$\begin{aligned} [d(u_n, P_\Gamma u_n) - d(M,N)]^2 &\leq [\|u_n - z_n\| - d(M,N)]^2 \\ &\leq [\|u_n - w_n\| + \|w_n - z_n\| - d(M,N)]^2 \\ &= \|(1-\eta_n)\xi_n + \eta_n v_n - w_n\|^2 \\ &= (1-\eta_n)\|\xi_n - w_n\|^2 + \eta_n\|v_n - w_n\|^2 \\ &\quad -\eta_n(1-\eta_n)\|\xi_n - v_n\|^2. \end{aligned}$$

Therefore, (3.8) becomes

$$\begin{aligned} H^2(P_\Gamma\tau, P_\Gamma u_n) &\leq \kappa^2[(1-\eta_n)\|\xi_n - \tau\|^2 + \eta_n\|v_n - \tau\|^2 + (1-\eta_n)\|\xi_n - w_n\|^2 \\ &\quad +\eta_n\|v_n - w_n\|^2 - 2\eta_n(1-\eta_n)\|\xi_n - v_n\|^2]. \end{aligned}$$

So

$$\begin{aligned} \|w_n - \tau\|^2 &\leq \kappa^2[(1-\eta_n)\|\xi_n - \tau\|^2 + \eta_n\|v_n - \tau\|^2 + (1-\eta_n)\|\xi_n - w_n\|^2 \\ &\quad +\eta_n\|v_n - w_n\|^2 - 2\eta_n(1-\eta_n)\|\xi_n - v_n\|^2]. \end{aligned} \tag{3.9}$$

In the same way,

$$\|v_n - \tau\| = \|\zeta_n - \tau^*\|. \tag{3.10}$$

Since $P_\Gamma(\xi_n)$ is singleton implies $P_\Gamma(\xi_n) = \{\zeta_n\}$. Then we have

$$\|\zeta_n - \tau^*\| = d(P_\Gamma \xi_n, \tau^*).$$

Therefore, we obtain

$$\|v_n - \tau\| \leq d(P_\Gamma \xi_n, \tau^*) \leq H(P_\Gamma \xi_n, P_\Gamma \tau). \tag{3.11}$$

Now

$$\begin{aligned} H(P_\Gamma \tau, P_\Gamma \xi_n) &\leq \kappa \max\{\|\tau - \xi_n\|, d(\tau, P_\Gamma \tau) - d(M,N), d(\xi_n, P_\Gamma \xi_n) - d(M,N), \\ &\quad d(\tau, P_\Gamma \xi_n) - d(M,N), d(\xi_n, P_\Gamma \tau) - d(M,N)\} \\ &\leq \kappa \max\{\|\tau - \xi_n\|, d(\xi_n, P_\Gamma \xi_n) - d(M,N), \\ &\quad d(\tau, P_\Gamma \tau) + H(P_\Gamma \tau, P_\Gamma \xi_n) - d(M,N), \\ &\quad \|\xi_n - \tau\| + d(\tau, P_\Gamma \tau) - d(M,N)\} \\ &= \kappa \max\{\|\tau - \xi_n\|, d(\xi_n, P_\Gamma \xi_n) - d(M,N), H(P_\Gamma \tau, P_\Gamma \xi_n), \|\xi_n - \tau\|\}. \end{aligned} \tag{3.12}$$

Suppose $H(P_\Gamma \tau, P_\Gamma \xi_n)$ is maximum, then we have

$$H(P_\Gamma \tau, P_\Gamma \xi_n) \leq \kappa H(P_\Gamma \tau, P_\Gamma \xi_n) < H(P_\Gamma \tau, P_\Gamma \xi_n).$$

Therefore, we get

$$H(P_\Gamma \tau, P_\Gamma \xi_n) \leq \kappa \max\{\|\tau - \xi_n\|, d(\xi_n, P_\Gamma \xi_n) - d(M,N)\}.$$

Then

$$\begin{aligned} H^2(P_\Gamma \tau, P_\Gamma \xi_n) &\leq \kappa^2 \max\{\|\tau - \xi_n\|^2, [d(\xi_n, P_\Gamma \xi_n) - d(M,N)]^2\} \\ &\leq \kappa^2 \{\|\tau - \xi_n\|^2 + [d(\xi_n, P_\Gamma \xi_n) - d(M,N)]^2\}. \end{aligned} \tag{3.13}$$

Now consider

$$\begin{aligned} [d(\xi_n, P_\Gamma \xi_n) - d(M,N)]^2 &\leq [\|\xi_n - \zeta_n\| - d(M,N)]^2 \\ &\leq [\|\xi_n - v_n\| + \|v_n - \zeta_n\| - d(M,N)]^2 \\ &= \|\xi_n - v_n\|^2. \end{aligned}$$

Therefore, we obtain

$$\|v_n - \tau\|^2 \leq \kappa^2 [\|\xi_n - \tau\|^2 + \|\xi_n - v_n\|^2].$$

Then (3.9) becomes

$$\begin{aligned} \|w_n - \tau\|^2 &\leq \kappa^2 (1 - \eta_n) \|\xi_n - \tau\|^2 + \kappa^4 \eta_n \|\xi_n - \tau\|^2 + \kappa^4 \eta_n \|\xi_n - v_n\|^2 \\ &\quad - 2\kappa^2 \eta_n (1 - \eta_n) \|\xi_n - v_n\|^2 + \kappa^2 (1 - \eta_n) \|\xi_n - w_n\|^2 \\ &\quad + \kappa^2 \eta_n \|v_n - w_n\|^2 \\ &= \kappa^2 (1 - \eta_n + \kappa^2 \eta_n) \|\xi_n - \tau\|^2 - \kappa^2 \eta_n (2 - 2\eta_n - \kappa^2) \|\xi_n - v_n\|^2 \\ &\quad + \kappa^2 (1 - \eta_n) \|\xi_n - w_n\|^2 + \kappa^2 \eta_n \|v_n - w_n\|^2 \\ &\leq \kappa^2 \|\xi_n - \tau\|^2 - \kappa^2 \eta_n (2 - 2\eta_n - \kappa^2) \|\xi_n - v_n\|^2 \\ &\quad + \kappa^2 (1 - \eta_n) \|\xi_n - w_n\|^2 + \kappa^2 \eta_n \|v_n - w_n\|^2. \end{aligned}$$

Substituting above in (3.5), we obtain

$$\begin{aligned}\|\xi_{n+1}-\tau\|^2 &\leq [1-\delta_n(1-\kappa^2)]\|\xi_n-\tau\|^2-\kappa^2\delta_n\eta_n(2-2\eta_n-\kappa^2)\|\xi_n-v_n\|^2\\ &\quad+\kappa^2\delta_n\eta_n\|v_n-w_n\|^2-\delta_n(1-\delta_n-\kappa^2+\kappa^2\eta_n)\|\xi_n-w_n\|^2.\end{aligned}$$

Since $c \leq \delta_n \leq 1-\kappa^2$, we have $1-\delta_n(1-\kappa^2) \leq 1-c(1-\kappa^2) = \delta$ (say) and $0 < \delta < 1$. Since $\eta_n \to 0$, there exists N_1 such that $\eta_n \leq (2-\kappa^2)/2, \forall n \geq N_1$ so that $2-2\eta_n-\kappa^2 \geq 0, \forall n \geq N_1$. Also we have $1-\delta_n-\kappa^2+\kappa^2\eta_n \geq (1-\kappa^2)-(1-\kappa^2)+\kappa^2\eta_n \geq 0, \forall n$. Therefore, we obtain for sufficiently large n,

$$\begin{aligned}\|\xi_{n+1}-\tau\|^2 &\leq \delta\|\xi_n-\tau\|^2+\kappa^2(1-\kappa^2)\eta_n\|v_n-w_n\|^2\\ &\leq \delta\|\xi_n-\tau\|^2+\kappa^2(1-\kappa^2)\eta_n D^2,\end{aligned}$$

where D is the diameter of M_0. Now, by Lemma 2.6, the sequence $\xi_n \to \tau$, which completes the proof. □

Remark 3.9. If we assume $M = N$, the above result reduces to the result proved by Sastry and Babu [26].

Corollary 3.5. *Let M, N be two convex bounded subsets of Hilbert space X. Suppose $\Gamma : M \to 2^N$ is a multi-valued mapping with $Bst(\Gamma) \neq \emptyset$ such that P_Γ is proximally quasi-contractive. Assume $\Gamma(\xi)$ is closed convex in N for every $\xi \in M$. Let $\{\xi_n\}$ be Ishikawa's type iteration defined by (I_2). Assume the sequences $\{\delta_n\}, \{\eta_n\}$ satisfy the following: (i) $0 \leq \delta_n, \eta_n < 1$, (ii) $\eta_n \to 0$ as $n \to \infty$, with $c \leq \delta_n \leq 1-\kappa^2$ for some $c > 0$. Then the sequence $\{\xi_n\}$ converges to a best proximity point of Γ.*

Theorem 3.10. *Let M, N be two convex bounded subsets of Hilbert space X. Suppose $\Gamma : M \to \wp(N)$ is a multi-valued mapping with $Bst(\Gamma) \neq \emptyset$ such that P_Γ is proximally quasi-contractive. Let $\{\xi_n\}$ be Mann's type iteration defined by (I_1). Assume the sequence $\{\delta_n\}$ satisfies the following: (i) $0 \leq \delta_n < 1$, with $c \leq \delta_n \leq 1-\kappa^2$ for some $c > 0$. Then the sequence $\{\xi_n\}$ converges to a best proximity point of Γ.*

Corollary 3.6. *Let M, N be two convex bounded subsets of Hilbert space X. Suppose $\Gamma : M \to 2^N$ is a multi-valued mapping with $Bst(\Gamma) \neq \emptyset$ such that P_Γ is proximally quasi-contractive. Assume $\Gamma(\xi)$ is closed convex in N for every $\xi \in M$. Let $\{\xi_n\}$ be Mann's type iteration defined by (I_1). Assume the sequence $\{\delta_n\}$ satisfies the following: (i) $0 \leq \delta_n < 1$, with $c \leq \delta_n \leq 1-\kappa^2$ for some $c > 0$. Then the sequence $\{\xi_n\}$ converges to a best proximity point of Γ.*

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