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A CHARACTERIZATION OF CONSTRUCTIBLE NORMS FOR BOUNDED LIPSCHITZIAN MAPPINGS

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Abstract. Let $(X, \|\cdot\|)$ be a Banach space and C a nonempty subset of X. We will say that a norm for the Banach space of bounded Lipschitzian mappings $BLip(C, X)$ is $\|\cdot\|$-constructible if that depend only on the infinity norm $\|\cdot\|_\infty$ and the Lipschitz constant $K(\,\cdot\,, \|\cdot\|)$. In this work we characterize the $\|\cdot\|$-constructible norms such as those that does not separate $\|\cdot\|$-indistinguishable operators, and we characterize constructible norms $\|\cdot\|_0$ like those which are $\phi(\|\cdot\|_0)$-constructible where $\phi(\|\cdot\|_0)$ is the projection of $\|\cdot\|_0$ over the space X.
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1. Introduction

In 2008 P. K. Lin [14] proved that there exist a non reflexive Banach space with the FPP, by using a renorming $\|\cdot\|_L$ of the space ℓ_1 such that $(\ell_1, \|\cdot\|_L)$ has the FPP. His work solved one of the main open question in the Fixed Point Theory: The equivalence of reflexivity and the Fixed Point Property (FPP). After that, many authors started to studied the relationship between the FPP and renormings, see for example [6, 7, 8, 9, 10, 12, 13].

In [4, 2, 3, 15] the structure of nonexpansive and nonexpansive like mappings which are invariant under renormings began to be studied, in fact, the elements of this families was characterized. Subsequently in [1] the topological structure of families of nonexpansive bounded mappings was studied and used to ensure the existence of nonexpansive mappings by a meager approach.

In the present work we continue with the study of families of nonexpansive bounded mappings, but this time we seek to describe the norms that somehow always exist. We develop the intuition of the norms $\|\cdot\|_0$ for the space of Lipschitzian bounded functions T between arbitrary subsets C of Banach spaces $(X, \|\cdot\|)$, that can always be defined no matter the base space or norm. In general, the only things that are known about T are its infinity norm $\|T\|_\infty$ and its Lipschitz constant $K(T, \|\cdot\|)$. Thus it is natural to think that a $\|\cdot\|_0$ that can always be defined should depends only of these parameters.

This paper is divided in five sections. Section 1 is devoted to definitions and previous results. In Section 2 we prove a series of lemmas that related the possible values of the Lipschitz constant with the domain of definition of the operators and the norm of the space. In Section 3 a characterization of constructible norms was given as those that do not separate indistinguishable operators, and constructible norms were seen as the result of an action-like of the $\mathbb{R}^2$ norms over the space norms. In Section 4 we characterize the orbits of the action defined in Section 3 and we provided an intrinsic criterion to verify if an abstract norm is constructible, which consists in verifying if the norm is constructible with respect to its own projection, in other words, if the norms does not separate projection-indistinguishable operators. Finally Section 5 deals with examples.

We start with some notation and well known results. Let (X, d) and (C, ϕ) be two metric spaces. A function $T : (C, \phi) \to (X, d)$ is said to be Lipschitz if there exists $0 \leq r < \infty$ such that

$$d(Tx, Ty) \leq r\phi(x, y)$$

for each $x, y \in X$. So we define the family of bounded Lipchitzian functions from C to X as

$$BLip(C, X) = \left\{ T : C \to X \,|\, T \text{ is Lipschitz and } \sup_{x,y \in C} d(Tx, Ty) < \infty \right\}.$$

In particular we have the case in which (X, d) is a normed space $(X, \|\cdot\|)$ and C is a nonempty subset of X, then for each $T \in BLip(C, X)$ we denote by $K(T, \|\cdot\|)$ its Lipschitz constant with respect to $\|\cdot\|$, which is defined by

$$K(T, \|\cdot\|) = \sup \left\{ \frac{\|Tx - Ty\|}{\|x - y\|} \,\middle|\, x, y \in C, x \neq y \right\}.$$

Another important element is the infinity norm which is defined as

$$\|T\|_\infty = \sup_{x \in C} \|Tx\|.$$

and through

$$\|T\|_{1,\infty} = \sup_{x \in C} \|Tx\|_1,$$

in case of considering a specific norm $\|\cdot\|_1$ of X. With the presented notation it is clear that

$$BLip(C, X) = \{T : C \to X \,|\, K(T, \|\cdot\|) < \infty,\, \|T\|_\infty < \infty\}.$$

We denote by

$$\mathcal{N}(X) = \{\|\cdot\|' \,|\, \|\cdot\|' \text{ is equivalent to} \|\cdot\|\}$$

the set of equivalent norms for $(X, \|\cdot\|)$. In particular $\mathcal{N}(\mathbb{R}^2)$ is the set of norms for $\mathbb{R}^2$. The following two lemmas are in [1], however for a detailed exposition about Lipschitzian Algebras the recommended reference is [16].

Lemma 1.1. *Let $(X, \|\cdot\|)$ be a Banach space and $C \subset X$, $C \neq \emptyset$, then $(BLip(C,X), \|\cdot\|)$ is a Banach space with respect to the norm $\|T\| := \|T\|_\infty + K(T, \|\cdot\|)$ for each $T \in BLip(C,X)$.*

Therefore throughout this work we will consider $BLip(C,X)$ endowed with the 1-type norm. Thus $\mathcal{N}(BLip(C,X))$ is the set of equivalent norms to it. For example the infinity-type $\|T\|_\infty = \max\{\|T\|_\infty, K(T, \|\cdot\|)\}$ and the p-type $\|T\|_p = (\|T\|_\infty^p + K(T, \|\cdot\|)^p)^{\frac{1}{p}}$ are elements of $\mathcal{N}(BLip(C,X))$.

Lemma 1.2. *Let $X_1 = (X, \|\cdot\|_1)$ and $X_2 = (X, \|\cdot\|_2)$ be Banach spaces and C a nonempty subset of X. Then the following are equivalent:*

(1) *$\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.*
(2) *The sets $BLip(C,X_1)$ and $BLip(C,X_2)$ are equal, and the Banach spaces $BLip(C,X_1)$ and $BLip(C,X_2)$ are isomorphic.*

The previous lemma allows us to consider $BLip(C,X)$ as a linear space independently of the norm of the base space X. If $x, y \in X$, we denote by $[x,y]$ the convex hull of $\{x,y\}$.

$$[x,y] = \{\lambda x + (1-\lambda)y \mid 0 \leq \lambda \leq 1\}.$$

The following results and definitions can be found and are direct consequences of Propositions 1.2 and 1.4 in [5]. A metric space M is said to be an absolute 1-Lipschitz retract if for each metric space N containing M, there exists $T : N \to M$ Lipschitz with constant 1, such that T is the identity on M. The next lemma characterizes absolute 1-Lipschitz retracts.

Lemma 1.3. *A metric space X is an absolute 1-Lipschitz retract if and only if it is metrically convex and has the binary intersection property.*

In particular since $[x,y]$ is isometrically isomorphic to a closed bounded interval of $\mathbb{R}$. Then it is an absolute 1-Lipschitz retract. Hence we have the following result which ensures the existence of Lipschitz preserving constant extensions.

Lemma 1.4. *Let X be a normed space, $x, y \in X$ and $Z \subset Y$, $N \supset [x,y]$ be metric spaces then:*

(1) *Every bounded Lipschitzian function $f : Z \to [x,y]$ can be extended to a bounded Lipschitzian function $F : Y \to [x,y]$ with the same Lipschitz constant and infinity norm.*
(2) *Every bounded Lipschitzian function $f : [x,y] \to Z$ can be extended to a bounded Lipschitzian function $F : N \to Z$ with the same Lipschitz constant and infinity norm.*

Remark 1.5. The bounded conditions are redundant in Lemma 1.4 since $[x,y]$ is a bounded set. Then for any C, each function $f : C \to [x,y]$ necessarily has bounded image.

Besides the operator $f : [x,y] \to Z$ clearly has bounded image. Since the extension $F : N \to Z$ of f can be constructed using a Lipschitz retraction $\tau : N \to [x,y]$ and $F = f \circ \tau$. Then F is the composition of bounded functions and has bounded image.

The infinity norm preserving result is trivial for the case (1). For the case (2) it follows from the fact that $F(N) = (f \circ \tau)(N) = f([x,y])$. Thus $\|F\|_\infty = \|f\|_\infty$.

2. Bounding Lipschitz constants

In this section we study the possible values adopted by Lipschitz constants $K(T, \|\cdot\|)$ and infinity norms $\|T\|_\infty$ for $T \in BLip(C, A)$. As we mentioned in the introduction, our objective is to characterize the constructible norms which depends only of Lipschitz constants and infinity norms, it is in that sense that the results of this section are necessary to prove a triangle inequality and the geometric shape which are the hardest parts of the proof of Theorem 3.1. First we present a key result which ensures the existence of appropriate Lipzchitz extensions.

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a normed space, C and A nonempty subsets of X, such that C has at least two elements and A is convex. Then for each $x, y \in C$ with $x \neq y$ and $a, b \in A$ exists $T \in BLip(C, A)$ with $Tx = a$, $Ty = b$,*

$$K(T, \|\cdot\|) = \frac{\|a-b\|}{\|x-y\|}$$

and

$$\|T\|_\infty = \max\{\|a\|, \|b\|\}.$$

Proof. Let $x, y \in C$ with $x \neq y$ and $a, b \in A$. We define

$$s = \frac{\|a-b\|}{\|x-y\|},$$

and $f : [x, y] \to A$ by

$$f(\lambda x + (1-\lambda)y) = \lambda a + (1-\lambda)b$$

for each $0 \leq \lambda \leq 1$. The function f is well defined since $[x, y]$ is parameterized by $0 \leq \lambda \leq 1$ and A is convex. It is clear that $fx = a$ and $fy = b$. By the convexity of norm

$$\begin{aligned} \|f\|_\infty &= \sup_{0\leq\lambda\leq 1} \|\lambda a + (1-\lambda)b\| \\ &= \max\{\|a\|, \|b\|\}. \end{aligned} \tag{2.1}$$

We affirm that f has Lipschitz constant $K(f, \|\cdot\|) = s$.

By (2.1) and Lemma 1.4, there exist $S : C \cup [x, y] \to A$ extension of f with $K(S, \|\cdot\|) = K(f, \|\cdot\|) = s$ and $\|S\|_\infty = \max\{\|a\|, \|b\|\}$. Hence the operator $T = S|_C$ satisfies the desired conditions. □

The next theorem is the first of three results Theorem 2.2, Lemma 2.5 and Lemma 2.6 that bounds the possible values taken by Lipschitz constants of a general family of bounded Lipschitzian operators $BLip(C, A)$ and it is in some sense a dual result of Proposition 2.1 in [11].

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a normed space, C and A nonempty subsets of X, such that C has at least two elements, then*

$$\sup_{T\in BLip(C,A)} K(T, \|\cdot\|) \leq \sup\left\{\frac{diam(A)}{\|x-y\|} : x, y \in C, x \neq y\right\}.$$

If additionally A is convex, then

$$\sup_{T\in BLip(C,A)} K(T,\|\cdot\|) = \sup\left\{\frac{diam(A)}{\|x-y\|} : x,y\in C, x\neq y\right\},$$

and if A has at least two distinct elements, then for each s with

$$0\leq s<\sup\left\{\frac{diam(A)}{\|x-y\|} : x,y\in C, x\neq y\right\},$$

there exist $T\in BLip(C,A)$ such that $K(T,\|\cdot\|)=s$.

Proof. Let $T\in BLip(C,A)$. Then

$$\begin{aligned}K(T,\|\cdot\|) &= \sup\left\{\frac{\|Tx-Ty\|}{\|x-y\|} : x,y\in C, x\neq y\right\}\\ &\leq \sup\left\{\frac{diam(A)}{\|x-y\|} : x,y\in C, x\neq y\right\}.\end{aligned}$$

Thus

$$\sup_{T\in BLip(C,A)} K(T,\|\cdot\|) \leq \sup\left\{\frac{diam(A)}{\|x-y\|} : x,y\in C, x\neq y\right\}.$$

Now we suppose that A is convex and we call

$$r=\sup\left\{\frac{diam(A)}{\|x-y\|} : x,y\in C, x\neq y\right\}.$$

Then we have two possibilities:

(i) $r=\infty$,
(ii) $r<\infty$.

First we prove under Assumption (i). Then again we have two cases

(1) $diam(A)=\infty$,
(2) $diam(A)<\infty$.

If we have Case (1). Let $x,y\in C$ with $x\neq y$. Thus for each $n\in\mathbb{N}$ there exists $a_n,b_n\in A$ such that

$$\|a_n-b_n\|>n\|x-y\|.$$

By Lemma 2.1, for each $n\in\mathbb{N}$ there exists $T_n: C\to A$ such that $T_nx=a_n$, $T_ny=b_n$, $T_n\in BLip(C,A)$ and

$$K(T_n,\|\cdot\|)=\frac{\|a_n-b_n\|}{\|x-y\|}>n.$$

Thus

$$\begin{aligned}\sup_{S\in BLip(C,A)} K(S,\|\cdot\|) &\geq \sup_{n\in\mathbb{N}} K(T_n,\|\cdot\|)\\ &\geq \sup_{n\in\mathbb{N}} n \\ &= \infty = r.\end{aligned} \tag{2.2}$$

Now we assume Case (2). Then A has at least two distinct elements a,b, and $\inf\{\|x-y\| \mid x,y\in C, x\neq y\}=0$. Thus exist $(x_n),(y_n)$ sequences in C such that

$x_n \neq y_n$ and $\lim_{n\in\mathbb{N}} \|x_n - y_n\| = 0$, without lost of generality we may suppose that for each $n \in \mathbb{N}$,

$$n\|x_n - y_n\| < \|a - b\|.$$

By Lemma 2.1, for each $n \in \mathbb{N}$ there exists $T_n : C \to A$ such that $T_n x_n = a$, $T_n y_n = b$, $T_n \in BLip(C, A)$ and

$$K(T_n, \|\cdot\|) = \frac{\|a - b\|}{\|x_n - y_n\|} > n.$$

Then by (2.2) this case is proved.

Now we suppose Case (ii). Let $\varepsilon > 0$. Then exist $x, y \in C$ with $x \neq y$ such that

$$r - \varepsilon < \frac{diam(A)}{\|x - y\|} \leq r.$$

Thus exist $a, b \in A$ such that

$$r - \varepsilon < \frac{\|a - b\|}{\|x - y\|} \leq \frac{diam(A)}{\|x - y\|} \leq r.$$

By Lemma 2.1, there exists $T : C \to A$ such that $Tx = a$, $Ty = b$, $T \in BLip(C, A)$ and

$$K(T, \|\cdot\|) = \frac{\|a - b\|}{\|x - y\|} > r - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude this part of the proof.

Now we prove the last statement of the theorem. Let

$$0 \leq s < \sup\left\{\frac{diam(A)}{\|x - y\|} : x, y \in C, x \neq y\right\},$$

since the constant functions has Lipschitz constant equal to 0, then without lost of generality we may assume that $s > 0$. Thus by the same argument of previous paragraphs, there are $a, b \in A$ and $x, y \in C$ such that

$$s < \frac{\|a - b\|}{\|x - y\|} \leq \sup\left\{\frac{diam(A)}{\|x - y\|} : x, y \in C, x \neq y\right\}.$$

Then by the convexity of A there are $a', b' \in A$ such that

$$s = \frac{\|a' - b'\|}{\|x - y\|} < \frac{\|a - b\|}{\|x - y\|}.$$

By Lemma 2.1, there exists $T : C \to A$ such that $Tx = a'$, $Ty = b'$, $T \in BLip(C, A)$ and

$$K(T, \|\cdot\|) = \frac{\|a' - b'\|}{\|x - y\|} = s. \qquad \square$$

Remark 2.3. It is possible to generalize the previous theorem for the case of metric spaces, that is, by considering C and A two metric spaces not necessary subsets of a normed space X. To do this we need to impose some geometric-metric conditions in order to ensure the existence of retracts and has the equivalent result to Lemma 1.4.

The following lemma is similar to Lemma 2.1, except that in this the hypothesis implies a discrete metric structure in the domain of definition C, thus the statement and the proof use sequences instead of retracts.

Lemma 2.4. *Let* $(X, \|\cdot\|)$ *be a normed space,* $a, b \in X$ *and* $C \subset X$ *such that*

$$s = \inf\{\|x-y\| \,|\, x, y \in C, x \neq y\} > 0. \tag{2.3}$$

Then exist sequences (x_n) *and* (y_n) *in* C *with* $\lim_{n\in\mathbb{N}} \|x_n - y_n\| = s$*, and* $T \in BLip(C, X)$ *such that* $Tx_n = a$*,* $Ty_n = b$ *for each* $n \in \mathbb{N}$*,*

$$K(T, \|\cdot\|) = \frac{\|a-b\|}{s},$$

and

$$\|T\|_\infty = \max\{\|a\|, \|b\|\}.$$

Proof. By the definition of $s = \inf\{\|x-y\| \,|\, x, y \in C, x \neq y\}$, there exist sequences (x'_n) and (y'_n) in C such that

$$\lim_{n\in\mathbb{N}} \|x'_n - y'_n\| = s.$$

We affirm that can be constructed subsequences (x_n) of (x'_n) and (y_n) of (y'_n) such that $x_m \neq y_n$ for each $m, n \in \mathbb{N}$. In fact, Since $s > 0$, then

$$x'_n \neq y'_n \tag{2.4}$$

for each $n \in \mathbb{N}$. We have the following cases:

(i) Exist infinite indexes n_i such that x'_{n_i} or y'_{n_i} is constant in that index.
(ii) For each $n \in \mathbb{N}$ exist only a finite collection of indexes n_j such that $x'_n = x'_{n_j}$.

First we consider the Case (i). Without lost of generality we may suppose that exist infinite indexes n_i with $i \in \mathbb{N}$ such that $x'_{n_i} = x'_{n_j} = x$ for each $i, j \in \mathbb{N}$ and some $x \in C$. Then we define the subsequences $(x_i) = (x'_{n_i})$ and $(y_i) = (y'_{n_i})$. Hence by (2.4) for each $n, m \in \mathbb{N}$ we have that

$$x_m = x = x_n \neq y_n.$$

Now we suppose Case (ii). First we construct subsequences (x''_n) of (x'_n) and (y''_n) of (y'_n) such that

$$x''_m \neq x''_n \text{ and } y''_m \neq y''_n \text{ if } m \neq n. \tag{2.5}$$

We proceed inductively as follow. Let $J_0 = \mathbb{N}$, $n_0 = \min J_0$ and $x''_0 = x'_{n_0}$. We suppose J_l, n_l and x''_l constructed for $l < k$. Then

$$J_k = \{n \in J_{k-1} \,|\, \forall m \in J_{k-1}, m \geq n \text{ implies } x'_m \neq x''_i \text{ for } i < k\}.$$

By hypothesis (ii) the set $J_k \neq \emptyset$. We define $n_k = \min J_k$ and $x''_k = x'_{n_k}$. By construction the subsequence (x''_n) satisfies (2.5). We call $J = \{n_k \,|\, \forall k \in \mathbb{N}, n_k = \min J_k\}$, and proceed by induction as follow. $K_0 = J$, $m_0 = \min K_0$ and $y''_0 = y'_{n_0}$. We suppose K_l, m_l and y_l constructed for $l < k$. Then

$$K_k = \{n \in K_{k-1} \,|\, \forall m \in K_{k-1}, m \geq n \text{ implies } y'_m \neq y''_i \text{ for } i < k\}.$$

Since $K_k \neq \emptyset$, then we define $m_k = \min K_k$, $y''_k = y'_{m_k}$ and $K = \{m_k \,|\, \forall k \in \mathbb{N}, m_k = \min K_k\}$. Finally, abusing the notation, we will redefine (x''_n) by means of $x''_k = x'_{m_k}$ for each $k \in \mathbb{N}$ and $m_k \in K$. Then the subsequences (x''_n) and (y''_n) satisfies (2.5).

Now we construct subsequences (x_n) of (x'_n) and (y_n) of (y'_n) such that satisfies

$$x_m \neq x_n,\; y_m \neq y_n \text{ and } x_m \neq y_n \text{ for each } m, n \in \mathbb{N}. \tag{2.6}$$

From the construction we have just made about property (2.5), we can assume without loss of generality that (x'_n) and (y'_n) satisfies (2.5), and proceed by induction as follow. We define $x_0 = x'_0$, $y_0 = y'_0$,

$$J_1 = \{n \in \mathbb{N} \,|\, \forall m \in \mathbb{N}, m \geq n \text{ implies } x'_m \neq y_0\},$$

and

$$K_1 = \{n \in J_1 \,|\, \forall m \in J_1, m \geq n \text{ implies } y'_m \neq x_0\}.$$

By hypothesis (ii), the set K_1 is nonempty. Thus we call $x_1 = x'_{n_1}$ and $y_1 = y'_{n_1}$ with $n_1 = \min K_1$. We suppose x_l, y_l, J_l, K_l, and n_l constructed for $l < K$. Then

$$J_k = \{n \in K_{k-1} \,|\, \forall m \in K_{k-1}, m \geq n \text{ implies } x'_m \neq y_i \text{ for } i < k\}$$

and

$$K_k = \{n \in J_k \,|\, \forall m \in J_k, m \geq n \text{ implies } y'_m \neq x_i \text{ for } i < k\}.$$

Finally, we define $x_k = x'_{n_k}$ and $y_k = y'_{n_k}$ with $n_k = \min K_k$. By construction the subsequences (x_n) and (y_n) satisfies (2.6). Since without loss of generality we may suppose that (x'_n) and (y'_n) satisfies (2.6), then we define $T : C \to X$ by

$$Tz = \begin{cases} a, & \text{if } z \neq y'_n \text{ for all } n \in \mathbb{N} \\ b, & \text{if } z = y'_n \text{ for some } n \in \mathbb{N} \end{cases}$$

It is clear that for each $n \in \mathbb{N}$, $Tx'_n = a$ and $Ty'_n = b$. Hence for each $x', y' \in C$ we have that

$$\begin{aligned} \|Tx' - Ty'\| &\leq \|a - b\| \\ &= \frac{\|a-b\|}{s} s \\ &\leq \frac{\|a-b\|}{s} \|x' - y'\| \end{aligned}$$

Hence $K(T, \|\cdot\|) = \dfrac{\|a-b\|}{s}$, and $\|T\|_\infty = \max\{\|a\|, \|b\|\}$ which concludes the proof. □

The following lemma is very similar to Theorem 2.2, except that in this A is the ε-radius closed ball. Hence the infinity norm of the constructed operators is controlled.

Lemma 2.5. *Let $(X, \|\cdot\|)$ be a normed space, C a subset of X with at least two elements, $\varepsilon \geq 0$ and*

$$s = \sup\left\{\frac{2\varepsilon}{\|x - y\|} : x, y \in C, x \neq y\right\}.$$

Then for each $0 \leq r < s$ there exists $T \in BLip(C, X)$ such that $\|T\|_\infty = \varepsilon$ and $K(T, \|\cdot\|) = r$.
Additionally, if $s < \infty$. Then the previous statement is valid for each $0 \leq r \leq s$.

Proof. This proof is similar as the proof of last statement of Theorem 2.2. Let

$$A = \overline{B(0, \varepsilon)} = \{x \in X \,|\, \|x\| \leq \varepsilon\}.$$

and $a \in A$ with $\|a\| = \varepsilon$. Then exist $x, y \in C$ and $b \in [-a, a]$ such that

$$\frac{\|a-b\|}{\|x-y\|} = r.$$

By Lemma 2.1, there exists $T : C \to A$ such that $Tx = a$, $Ty = b$,

$$K(T, \|\cdot\|) = \frac{\|a-b\|}{\|x-y\|} = r$$

and

$$\|T\|_\infty = \max\{\|a\|, \|b\|\} = \varepsilon.$$

Now we suppose that $s < \infty$. We define

$$s' = 2\varepsilon s^{-1} = \inf\{\|x-y\| \mid x, y \in C, x \neq y\} > 0.$$

Let $b' \in [-a, a]$ such that

$$\frac{\|a-b'\|}{s'} = r$$

By Lemma 2.4, there exist (x_n) and (y_n) sequences in C with

$$\lim_{n \in \mathbb{N}} \|x_n - y_n\| = s',$$

and $T \in BLip(C, X)$ such that $Tx_n = a$, $Ty_n = b'$ for each $n \in \mathbb{N}$,

$$K(T, \|\cdot\|) = \frac{\|a-b'\|}{s'} = r,$$

and

$$\|T\|_\infty = \max\{\|a\|, \|b'\|\} = \varepsilon$$

which satisfies the desired conditions. □

The following lemma allows us to decompose operators it two parts which in turn divides the infinite norm and Lipschitz constant. In other words, it provides a certain type of orthogonal decomposition of operators.

Lemma 2.6. *Let $(X, \|\cdot\|)$ be a normed space, C a subset of X with at least two elements. If for each $\varepsilon \geq 0$*

$$f(\varepsilon) = \sup\left\{\frac{2\varepsilon}{\|x-y\|} : x, y \in C, x \neq y\right\}$$

and $0 < \varepsilon_1, \varepsilon_2, r, s < \infty$ be such that $r \leq f(\varepsilon_1)$, $s \leq f(\varepsilon_2)$.
Then exist $R, S, T \in BLip(C, X)$ with $T = R + S$ such that it is satisfied: $\|R\|_\infty = \varepsilon_1$, $\|S\|_\infty = \varepsilon_2$,

$$\|T\|_\infty = \|R\|_\infty + \|S\|_\infty,$$

$K(R, \|\cdot\|) = r$, $K(S, \|\cdot\|) = s$, and

$$K(T, \|\cdot\|) = K(R, \|\cdot\|) + K(S, \|\cdot\|).$$

Proof. We have two possibilities

(1) $r < f(\varepsilon_1)$ and $s < f(\varepsilon_2)$.
(2) $r = f(\varepsilon_1)$ or $s = f(\varepsilon_2)$.

First we suppose Case (1), then exist $x, y \in C$ with $x \neq y$ such that

$$r < \frac{2\varepsilon_1}{\|x-y\|}$$

and

$$s < \frac{2\varepsilon_2}{\|x-y\|}.$$

Let $w \in X$ with $\|w\| = 1$. We can find $t_1, t_2 \in [-1, 1]$ such that

$$r = \frac{(1-t_1)\varepsilon_1}{\|x-y\|} = \frac{|\varepsilon_1 - t_1\varepsilon_1|}{\|x-y\|}$$

and

$$s = \frac{(1-t_2)\varepsilon_2}{\|x-y\|} = \frac{|\varepsilon_2 - t_2\varepsilon_2|}{\|x-y\|}.$$

By Lemma 2.1, there exist $R, S \in BLip(C, X)$ with

$$\begin{aligned} Rx &= \varepsilon_1 w, \ \ Ry = t_1\varepsilon_1 w, \\ Sx &= \varepsilon_2 w, \ \ Sy = t_2\varepsilon_2 w, \end{aligned} \tag{2.7}$$

infinity norms

$$\begin{aligned} \|R\|_\infty &= \max\{\|\varepsilon_1 w\|, \|t_1\varepsilon_1 w\|\} = \varepsilon_1, \\ \|S\|_\infty &= \max\{\|\varepsilon_2 w\|, \|t_2\varepsilon_2 w\|\} = \varepsilon_2. \end{aligned} \tag{2.8}$$

and Lipschitz constants

$$\begin{aligned} K(R, \|\cdot\|) &= \frac{\|\varepsilon_1 w - t_1\varepsilon_1 w\|}{\|x-y\|} = \frac{|\varepsilon_1 - t_1\varepsilon_1|}{\|x-y\|} = r, \\ K(S, \|\cdot\|) &= \frac{\|\varepsilon_2 w - t_2\varepsilon_2 w\|}{\|x-y\|} = \frac{|\varepsilon_2 - t_2\varepsilon_2|}{\|x-y\|} = s. \end{aligned} \tag{2.9}$$

We define $T = R + S$ and affirm that it satisfies the desired properties. In fact, since $\|\cdot\|_\infty$ and $K(\,\cdot\,, \|\cdot\|)$ are seminorms, then it is enough to prove that for special values $x', y' \in C$ the next equalities hold:

$$\|Tx'\| = \|R\|_\infty + \|S\|_\infty$$

and

$$\|Tx' - Ty'\| = (K(R, \|\cdot\|) + K(S, \|\cdot\|))\|x' - y'\|.$$

Hence by (2.7), (2.8) and (2.9) we have that

$$\begin{aligned} \|Tx\| &= \|(R+S)x\| \\ &= \|\varepsilon_1 w + \varepsilon_2 w\| \\ &= \varepsilon_1 + \varepsilon_2 \\ &= \|R\|_\infty + \|S\|_\infty \end{aligned}$$

and

$$\begin{aligned}\|Tx - Ty\| &= \|(R+S)x - (R+S)y\| \\ &= \|\varepsilon_1 w + \varepsilon_2 w - t_1\varepsilon_1 w - t_2\varepsilon_2 w\| \\ &= (\varepsilon_1 - t_1\varepsilon_1) + (\varepsilon_2 - t_2\varepsilon_2) \\ &= r\|x-y\| + s\|x-y\| \\ &= (K(R, \|\cdot\|) + K(S, \|\cdot\|))\|x-y\|.\end{aligned}$$

Now we suppose Case (2), this proof is a combination between the previous case and the proof of Lemma 2.5. From Case (2) and the lemma hypothesis $r, s < \infty$ it follows that

$$\sup\left\{\frac{1}{\|x-y\|} : x, y \in C, x \neq y\right\} < \infty.$$

We define

$$c = \inf\{\|x-y\| \mid x, y \in C, x \neq y\} > 0.$$

Let $w \in X$ such that $\|w\| = 1$ and $t_1, t_2 \in [-1, 1]$ with

$$r = \frac{|\varepsilon_1 - t_1\varepsilon_1|}{c} = \frac{\|\varepsilon_1 w - t_1\varepsilon_1 w\|}{c},$$

and

$$s = \frac{|\varepsilon_2 - t_2\varepsilon_2|}{c} = \frac{\|\varepsilon_2 w - t_2\varepsilon_2 w\|}{c}.$$

By Lemma 2.4, there exist sequences (x_n) and (y_n) in C with

$$\lim_{n\in\mathbb{N}} \|x_n - y_n\| = c,$$

and $R, S \in BLip(C, X)$ such that for each $n \in \mathbb{N}$

$$\begin{aligned}Rx_n = \varepsilon_1 w,\ Ry_n = t_1\varepsilon_1 w, \\ Sx_n = \varepsilon_2 w,\ Sy_n = t_2\varepsilon_2 w,\end{aligned}$$

infinity norms

$$\begin{aligned}\|R\|_\infty &= \max\{\|\varepsilon_1 w\|, \|t_1\varepsilon_1 w\|\} = \varepsilon_1, \\ \|S\|_\infty &= \max\{\|\varepsilon_2 w\|, \|t_2\varepsilon_2 w\|\} = \varepsilon_2,\end{aligned}$$

and Lipschitz constants

$$\begin{aligned}K(R, \|\cdot\|) &= \frac{\|\varepsilon_1 w - t_1\varepsilon_1 w\|}{c} = r, \\ K(S, \|\cdot\|) &= \frac{\|\varepsilon_2 w - t_2\varepsilon_2 w\|}{c} = s.\end{aligned}$$

We define $T = R + S$. Then

$$\begin{aligned}\|T\|_\infty &\geq \|Tx_1\| \\ &= \|Rx_1 + Sx_1\| \\ &= \|\varepsilon_1 w + \varepsilon_2 w\| \\ &= \varepsilon_1 + \varepsilon_2 \\ &= \|R\|_\infty + \|S\|_\infty.\end{aligned}$$

Hence $\|T\|_\infty = \|R\|_\infty + \|S\|_\infty$. Finally we calculate the Lipschitz constant of T.

$$\begin{aligned}
K(T,\|\cdot\|) &= \sup\left\{ \frac{\|Tx-Ty\|}{\|x-y\|} \,\middle|\, x,y\in C, x\neq y\right\}\\
&\geq \sup\left\{ \frac{\|Tx_n-Ty_n\|}{\|x_n-y_n\|} \,\middle|\, n\in\mathbb{N}\right\}\\
&= \sup\left\{ \frac{\|\varepsilon_1 w+\varepsilon_2 w - t_1\varepsilon_1 w - t_2\varepsilon_2 w\|}{\|x_n-y_n\|} \,\middle|\, n\in\mathbb{N}\right\}\\
&= (|\varepsilon_1 - t_1\varepsilon_1| + |\varepsilon_2 - t_2\varepsilon_2|)\frac{1}{\inf\{\|x-y\| \,|\, x,y\in C, x\neq y\}}\\
&= \frac{\|\varepsilon_1 w - t_1\varepsilon_1 w\|}{c} + \frac{\|\varepsilon_2 w - t_2\varepsilon_2 w\|}{c}\\
&= K(R,\|\cdot\|) + K(S,\|\cdot\|).
\end{aligned}$$

Thus $K(T,\|\cdot\|) = K(R,\|\cdot\|) + K(S,\|\cdot\|)$. □

3. Characterizing constructible norms

This section consists only of one theorem in which we characterize the norms for the Banach space $BLip(C,X)$ that depends only of the infinity norm and the Lipschitz constant. In order to formalize, we will give some definitions and remarks.

Let X be a Banach space and C a nonempty subset of X. We will say that a norm $\|\cdot\|_0$ for $BLip(C,X)$ is constructible with respect to $\|\cdot\|\in\mathcal{N}(X)$ and $\rho:\mathbb{R}^2\to\mathbb{R}$ if

$$\|T\|_0 = \rho(\|T\|_\infty, K(T,\|\cdot\|)) \tag{3.1}$$

for each $T\in BLip(C,X)$, in which case we will say that $\|\cdot\|$ and ρ construct or induce $\|\cdot\|_0$, and we will adopt the notation $\rho\|\cdot\|$ to denote the induced norm which operates by

$$\rho\|T\| = \rho(\|T\|_\infty, K(T,\|\cdot\|))$$

for each $T\in BLip(C,X)$. It is clear that given ρ and $\|\cdot\|$ the induced norm $\rho\|\cdot\|$ is unique and well defined. If a norm $\|\cdot\|_0$ for $BLip(C,X)$ is constructible with respect to some $\|\cdot\|\in\mathcal{N}(X)$ and $\rho:\mathbb{R}^2\to\mathbb{R}$, we will simply say that $\|\cdot\|_0$ is constructible or $\|\cdot\|$-constructible in case the norm $\|\cdot\|$ is known. Since for each $\|\cdot\|\in\mathcal{N}(X)$ we have that $\|\cdot\|_\infty$ and $K(\,\cdot\,,\|\cdot\|)$ are respectively a norm and a seminorm for $BLip(C,X)$, then the function ρ in (3.1) necessarily satisfies the norm axioms in the sets

$$N\subset\{(\|T\|_\infty, K(T,\|\cdot\|)) \,|\, T\in BLip(C,X)\}\subset\mathbb{R}^2 \tag{3.2}$$

that allow us to verify the norm axioms, that is, as long as N is a cone. The structure of N, the extension of ρ from N to a norm in the whole space $\mathbb{R}^2$, and the fact that the constructed norms are always equivalent norms, $\rho\|\cdot\|\in\mathcal{N}(BLip(C,X))$, will be studied in deep in Theorem 3.1.

We will say that two operators $S,T\in BLip(C,X)$ are indistinguishable with respect to $\|\cdot\|\in\mathcal{N}(X)$ if they have the same infinity norm and Lipschitz constant, that is, $\|S\|_\infty = \|T\|_\infty$ and $K(S,\|\cdot\|) = K(T,\|\cdot\|)$. It is clear that if $S,T\in BLip(C,X)$ are $\|\cdot\|$-indistinguishable, then for any norm $\rho\|\cdot\|$ is satisfied $\rho\|S\| = \rho\|T\|$, in other words, $\|\cdot\|$-constructible norms cannot separate $\|\cdot\|$-indistinguishable operators. It is in this sense that the next theorem is the reciprocal and characterizes the constructible norms as the family of norms that cannot separate indistinguishable operators.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a normed space, C a nonempty subset of X and $\|\cdot\|_0$ be a norm in $BLip(C, X)$. Then the following statements are equivalent:*

(1) *For each $S, T \in BLip(C, X)$ that are $\|\cdot\|$-indistinguishable is satisfied $\|S\|_0 = \|T\|_0$.*
(2) *There exists $\rho \in \mathcal{N}(\mathbb{R}^2)$ such that $\|\cdot\|_0 = \rho\|\cdot\|$.*

Moreover, if $\|\cdot\|_0$ satisfies (1) or (2), then $\|\cdot\|_0 \in \mathcal{N}(BLip(C, X))$.

Proof. The implication (2) $\Rightarrow$ (1) is obvious.

Now we prove (1) implies (2). Let $\phi : BLip(C, X) \to \mathbb{R}^2$ defined for each $T \in BLip(C, X)$ as

$$\phi(T) = (\|T\|_\infty, K(T, \|\cdot\|))$$

and $\rho : Im(\phi) \subset \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\rho(\phi(T)) = \|T\|_0.$$

Hypothesis (1) ensures that ρ is well defined. It is not hard to check that ρ is non-negative, it is 0 only at $(0, 0)$ and it is absolute homogeneous where the domain of definition allows it.

We affirm that ρ satisfies the triangle inequality where the domain of definition allows it. We have two cases:

(i) C has at least two elements.
(ii) C is an unitary set $C = \{x\}$ for some $x \in X$.

First we suppose (i). Let $(\varepsilon_1, r), (\varepsilon_2, s) \in Im(\phi)$ such that

$$(\varepsilon_1, r) + (\varepsilon_2, s) \in Im(\phi).$$

For each $\varepsilon \geq 0$ we define

$$f(\varepsilon) = \sup\left\{\frac{2\varepsilon}{\|x - y\|} : x, y \in C, x \neq y\right\},$$

then applying Theorem 2.2 to $A = B(0, \varepsilon)$ we have that $r \leq f(\varepsilon_1)$ and $s \leq f(\varepsilon_2)$. It is clear that $0 \leq \varepsilon_1, \varepsilon_2, r, s < \infty$. Thus by Lemma 2.6 there are $R, S, T \in BLip(C, X)$ such that $T = R + S$, $\|R\|_\infty = \varepsilon_1$, $\|S\|_\infty = \varepsilon_2$,

$$\|T\|_\infty = \|R\|_\infty + \|S\|_\infty$$

and $K(R, \|\cdot\|) = r$, $K(S, \|\cdot\|) = s$,

$$K(T, \|\cdot\|) = K(R, \|\cdot\|) + K(S, \|\cdot\|).$$

Then

$$\begin{aligned}\rho((\varepsilon_1, r) + (\varepsilon_2, s)) &= \rho((\varepsilon_1 + \varepsilon_2, r + s)) \\ &= \rho(\phi(T)) \\ &= \|T\|_0 = \|R + S\|_0 \\ &\leq \|R\|_0 + \|S\|_0 \\ &= \rho(\phi(R)) + \rho(\phi(S)) \\ &= \rho((\varepsilon_1, r)) + \rho((\varepsilon_2, s)).\end{aligned}$$

For the case (ii) the proof is similar as the previous case, just by taking $Rx = \varepsilon_1 w$ and $Sx = \varepsilon_2 w$ for some $w \in X$ with $\|w\| = 1$, and $T = R + S$.

Finally, we note that to finish it is enough to prove that the domain of ρ has a geometric structure which guarantees the existence of a norm extension to the whole $\mathbb{R}^2$. We affirm that $Im(\phi)$ is a convex closed cone in the first quadrant of $\mathbb{R}^2$, such that contains the non negative real line or it is the first quadrant without the set $\{(0, b) \,|\, b > 0\}$, in fact, if C consists of only one element, then $Im(\phi)$ is just the non negative real axis and the proof is over.

We suppose that C has at least two elements. Let $\varepsilon \geq 0$, if $f(1) < \infty$, then by Lemma 2.5, for each $0 \leq s \leq f(\varepsilon)$ there exist $T \in BLip(C, X)$ with $\|T\|_\infty = \varepsilon$ and $K(T, \|\cdot\|) = s$, that is, for each $\varepsilon \geq 0$

$$\{(\varepsilon, s) \,|\, 0 \leq s \leq f(\varepsilon)\} \subset Im(\phi),$$

note that by Theorem 2.2 the bound $f(\varepsilon)$ is sharp, thus $Im(\phi)$ is the convex closed cone in the first quadrant determined by the directions $(1, 0)$ and $(1, f(1))$.

If $f(1) = \infty$, then by a similar argument as the previous case, for each $\varepsilon > 0$ we have

$$\{(\varepsilon, s) \,|\, s \geq 0\} \subset Im(\phi),$$

then $Im(\phi)$ is the first quadrant without the set $\{(0, b) \,|\, b > 0\}$.

Now we prove the last statement of the theorem. Since (1) is equivalent to (2), then exists $\rho \in \mathcal{N}(\mathbb{R}^2)$ such that for each $T \in BLip(C, X)$,

$$\|T\|_0 = \rho\|T\|.$$

Let us consider the norm $\|(x, y)\|_1 = |x| + |y|$ over $\mathbb{R}^2$. There are $l, u > 0$ such that for each $(x, y) \in \mathbb{R}^2$,

$$l\|(x, y)\|_1 \leq \rho(x, y) \leq u\|(x, y)\|_1$$

holds. Then for each $T \in BLip(C, X)$,

$$\begin{aligned}l(\|T\|_\infty + K(T, \|\cdot\|)) &= l\|(\|T\|_\infty, K(T, \|\cdot\|))\|_1 \\ &\leq \rho\|T\| \\ &\leq u\|(\|T\|_\infty, K(T, \|\cdot\|))\|_1 \\ &= u(\|T\|_\infty + K(T, \|\cdot\|)).\end{aligned}$$

□

Remark 3.2. Note that in the hypothesis of Theorem 3.1, the norm $\|\cdot\|_0$ is not necessarily equivalent to $\|\cdot\|$.

From Theorem 3.1, we have the following special well known cases of equivalent norms for $(BLip(C,X), \|\cdot\|)$:

$$\|T\|_1 = \|T\|_\infty + K(T, \|\cdot\|),$$
$$\|T\|_\infty = \max\{\|T\|_\infty, K(T, \|\cdot\|)\},$$

and for each $p > 1$

$$\|T\|_p = (\|T\|_\infty^p + K(T, \|\cdot\|)^p)^{\frac{1}{p}}.$$

As a consequence of Theorem 3.1 constructible norms $\|\cdot\|_0$ can be viewed as $\|\cdot\|_0 = \rho\|\cdot\|$ with $\rho \in \mathcal{N}(\mathbb{R}^2)$ and $\|\cdot\| \in \mathcal{N}(X)$. Therefore the constructibility can be considered as the action of the family $\mathcal{N}(\mathbb{R}^2)$ over the set $\mathcal{N}(X)$.

4. Separating constructible norms

In this section we will study how is the family of constructible norms that does not separate $\|\cdot\|$-indistinguishable operators, and characterize them as the orbit of the constructability action over the fixed norm $\|\cdot\|$. In general we will separate the orbits of the constructability action and we will give an intrinsic criterion to determine if an abstract norm is constructible.

We will start with some definitions. Let X be a Banach space and C a nonempty subset of X. We will denote the orbit of the constructability action over a fixed norm $\|\cdot\| \in \mathcal{N}(X)$ as

$$\mathcal{N}(\mathbb{R}^2)\|\cdot\| = \{\rho\|\cdot\| \mid \rho \in \mathcal{N}(\mathbb{R}^2)\},$$

and for $\rho \in \mathcal{N}(\mathbb{R}^2)$ the ρ-saturation of $\mathcal{N}(X)$ as

$$\rho\mathcal{N}(X) = \{\rho\|\cdot\| \mid \|\cdot\| \in \mathcal{N}(X)\}.$$

With this action-like notation, the family of constructible norms will be represented as

$$\begin{aligned}\mathcal{N}(\mathbb{R}^2)\mathcal{N}(X) &= \{\rho\|\cdot\| \mid \rho \in \mathcal{N}(\mathbb{R}^2), \|\cdot\| \in \mathcal{N}(X)\}\\ &= \bigcup_{\|\cdot\| \in \mathcal{N}(X)} \mathcal{N}(\mathbb{R}^2)\|\cdot\|\\ &= \bigcup_{\rho \in \mathcal{N}(\mathbb{R}^2)} \rho\mathcal{N}(X).\end{aligned}$$

Given norms $\|\cdot\|_1$ and $\|\cdot\|_2$ for X. We will say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are collinear in X if exists $c > 0$ such that for each $x \in X$,

$$\|x\|_1 = c\|x\|_2,$$

and will be denoted by $\|\cdot\|_1 \sim \|\cdot\|_2$. Collinear norms just rescale the infinity norm and preserve the Lipschitz constant, So if $R, S \in BLip(C,X)$ are $\|\cdot\|$-indistinguishable they are $\|\cdot\|'$-indistinguishable for every $\|\cdot\|' \sim \|\cdot\|$. Hence by Theorem 3.1 if a norm $\|\cdot\|_0 \in \mathcal{N}(BLip(C,X))$ does not separate $\|\cdot\|$-indistinguishable operators, then it is $\|\cdot\|'$-constructible for each $\|\cdot\|' \sim \|\cdot\|$. So exist $\rho, \rho' \in \mathcal{N}(\mathbb{R}^2)$ such that

$$\rho\|\cdot\| = \|\cdot\|_0 = \rho'\|\cdot\|'. \tag{4.1}$$

Thus the natural question arises in characterize all the ways in which a constructible norm can be represented through the constructibility action. As we mentioned, collinear norms provide different representantions for constructible norms, and these are in fact all possible, which will be proved in the following theorem.

Theorem 4.1. *Let X be a normed space and $\|\cdot\|_1, \|\cdot\|_2 \in \mathcal{N}(X)$ then the following statements are equivalent:*

(1) $\|\cdot\|_1 \sim \|\cdot\|_2$.
(2) *There exist $\rho_1, \rho_2 \in \mathcal{N}(\mathbb{R}^2)$ such that $\rho_1\|\cdot\|_1 = \rho_2\|\cdot\|_2$.*
(3) *For each $\rho_1 \in \mathcal{N}(\mathbb{R}^2)$ exists $\rho_2 \in \mathcal{N}(\mathbb{R}^2)$ such that $\rho_2\|\cdot\|_2 = \rho_1\|\cdot\|_1$.*

Proof. It is clear that (3) implies (2). Now we prove that (2) implies (1). Let $\rho_1, \rho_2 \in \mathcal{N}(\mathbb{R}^2)$ such that $\rho_1\|\cdot\|_1 = \rho_2\|\cdot\|_2$, Then for each $x \in X$ we define $f_x : C \to X$ as $f_x(c) = x$ for each $c \in C$, Thus

$$\|f_x\|_\infty = \|x\| \text{ and } K(f_x, \|\cdot\|) = 0$$

for each $\|\cdot\| \in \mathcal{N}(X)$. Let $u \in X$. Then

$$\begin{aligned}
\rho_1(1,0)\|u\|_1 &= \rho_1(\|u\|_1, 0) \\
&= \rho_1(\|f_u\|_{1,\infty}, K(f_u, \|\cdot\|_1)) \\
&= \rho_1\|f_u\|_1 = \rho_2\|f_u\|_2 \\
&= \rho_2(\|u\|_2, 0) \\
&= \rho_2(1,0)\|u\|_2
\end{aligned}$$

Hence $\|\cdot\|_1 \sim \|\cdot\|_2$.

In order to prove (1) implies (3) we assume that $\|\cdot\|_1 \sim \|\cdot\|_2$. Then exists $c > 0$ such that $\|\cdot\|_2 = c\|\cdot\|_1$. Let $\rho_1 \in \mathcal{N}(\mathbb{R}^2)$ and ρ_2 defined by

$$\rho_2(a,b) = \rho_1(c^{-1}a, b).$$

Thus for each $T \in BLip(C,X)$ we have that

$$\|T\|_{2,\infty} = c\|T\|_{1,\infty} \text{ and } K(T, \|\cdot\|_2) = K(T, \|\cdot\|_1).$$

Hence

$$\begin{aligned}
\rho_2\|T\|_2 &= \rho_2(\|T\|_{2,\infty}, K(T, \|\cdot\|_2)) \\
&= \rho_1(c^{-1}\|T\|_{2,\infty}, K(T, \|\cdot\|_2)) \\
&= \rho_1(c^{-1}c\|T\|_{1,\infty}, K(T, \|\cdot\|_1)) \\
&= \rho_1(\|T\|_{1,\infty}, K(T, \|\cdot\|_1)) \\
&= \rho_1\|T\|_1
\end{aligned}$$

□

Remark 4.2. By Theorem 4.1 there are an infinity of different equivalent norms for $BLip(C,X)$ that can be constructed. To be precise, for each $\|\cdot\|_1, \|\cdot\|_2 \in \mathcal{N}(X)$ with $\|\cdot\|_1 \not\sim \|\cdot\|_2$ and $\rho_1, \rho_2 \in \mathbb{N}(\mathbb{R}^2)$ we have that

$$\rho_1\|\cdot\|_1 \neq \rho_2\|\cdot\|_2. \tag{4.2}$$

Also $\|\cdot\|$-constructible norms are only $\|\cdot\|'$-constructible for $\|\cdot\|' \sim \|\cdot\|$. So the only way to represent a given constructible norm modulus the collinearity relation is summarized in (4.1). Besides by (4.2) the orbits of the constructability action are in fact disjoint sets as in a standard group action. The above is summarized in the following corollary:

Corollary 4.3. *Let X be a normed space and $\|\cdot\|_1, \|\cdot\|_2 \in \mathcal{N}(X)$, then*

(1) $\mathcal{N}(\mathbb{R}^2)\|\cdot\|_1 = \mathcal{N}(\mathbb{R}^2)\|\cdot\|_2$ *if* $\|\cdot\|_1 \sim \|\cdot\|_2$.
(2) $\mathcal{N}(\mathbb{R}^2)\|\cdot\|_1 \bigcap \mathcal{N}(\mathbb{R}^2)\|\cdot\|_2 = \emptyset$ *if* $\|\cdot\|_1 \not\sim \|\cdot\|_2$.
(3) *If* $\|\cdot\|_0$ *is* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *constructible, then* $\|\cdot\|_1 \sim \|\cdot\|_2$.

Now we will focus on finding an intrinsic criterion to determine if a given norm is constructible, and as a consequence we will have another characterization of the family of constructible norms.

Let X be a Banach space and C a nonempty subset of X. For every $x \in X$ we will denote by f_x the constant function $f_x(c) = x$ with $c \in C$. For each $\|\cdot\| \in \mathcal{N}(BLip(C, X))$ we will define its projection $\phi(\|\cdot\|) \in \mathcal{N}(X)$ over X by

$$\phi(\|x\|) = \|f_x\| \tag{4.3}$$

for each $x \in X$. We have the following diagram which summarizes the constructability action and the projection over the base space:

$$\begin{array}{ccc} \mathcal{N}(BLip(C,X)) & \xrightarrow{\ \phi\ } & \mathcal{N}(X) \\ \uparrow & \swarrow_{\rho} & \\ \mathcal{N}(\mathbb{R}^2)\mathcal{N}(X) & & \end{array} \tag{4.4}$$

Given a norm $\|\cdot\| \in \mathcal{N}(BLip(C, X))$, we have three questions that arise naturally, the first one is the form of norms that are $\phi(\|\cdot\|)$-constructible, in other words, the orbit $\mathcal{N}(\mathbb{R}^2)\phi(\|\cdot\|)$. The second is when a norm its constructed by its own projection, in symbols, when we have that

$$\|\cdot\| \in \mathcal{N}(\mathbb{R}^2)\phi(\|\cdot\|). \tag{4.5}$$

The third is when the Diagram 4.4 commute. That is, when the constructability action and projection are inverse of each other. The answers to all three questions are related when the norm under consideration is constructible. In fact, in Theorem 4.6 we will prove that (4.5) is satisfied if and only if $\|\cdot\| \in \mathcal{N}(\mathbb{R}^2)\mathcal{N}(X)$, which is equivalent to the commutativity of (4.4), and characterize the orbits $\mathcal{N}(\mathbb{R}^2)\phi(\|\cdot\|)$. In order to prove Theorem 4.6, we will start by studying the properties of the projected norm of a constructible norm, which in fact is collinear to the norm of base space and therefore rescales the infinity norm and preserves the Lipschitz constant. The following lemma formalizes the previous statements.

Lemma 4.4. *Let X be a normed space, C a nonempty subset of X, $T \in BLip(C, X)$, $\rho \in \mathcal{N}(\mathbb{R}^2)$ and $\|\cdot\| \in \mathcal{N}(X)$, Then:*

(1) $\phi(\rho\|\cdot\|) \sim \|\cdot\|$.
(2) $\phi(\rho\|T\|)_\infty = \rho(1, 0)\|T\|_\infty$.

(3) $K(T, \phi(\rho\|\cdot\|)) = K(T, \|\cdot\|)$.

Proof. Let $x \in X$. Then

$$\begin{aligned}\phi(\rho\|x\|) &= \rho\|f_x\| \\ &= \rho(\|f_x\|_\infty, K(f_x, \|\cdot\|)) \\ &= \rho(\|x\|, 0) \\ &= \rho(1,0)\|x\|.\end{aligned}$$

Hence $\phi(\rho\|\cdot\|) \sim \|\cdot\|$ with proportionality constant $\rho(1,0)$. Statements (2) and (3) follows direct from collinearity since collinear norms only re-scale infinity norm and keep invariant Lipschitz constants. □

As an immediate consequence of the previous lemma we have that every constructible norm is constructed by its own projection. What has been said is summarized and proved in the following corollary:

Corollary 4.5. *Let X be a normed space, C a nonempty subset of X, $\rho\|\cdot\|$ a norm for $BLip(C,X)$ with $\rho \in \mathcal{N}(\mathbb{R}^2)$ and $\|\cdot\| \in \mathcal{N}(X)$, Then exists $\theta \in \mathcal{N}(\mathbb{R}^2)$ such that $\rho\|\cdot\| = \theta\phi(\rho\|\cdot\|)$.*

Proof. By Lemma 4.4 $\phi(\rho\|\cdot\|) \sim \|\cdot\|$, Then by (3) of Theorem 4.1 the proof is complete.

Additionally there is a proof using Theorem 3.1. Let $T, S \in BLip(C,X)$ that are $\phi(\|\cdot\|)$-indistinguishable, in symbols

$$\phi(\rho\|T\|)_\infty = \phi(\rho\|S\|)_\infty \text{ and } K(T, \phi(\rho\|\cdot\|)) = K(S, \phi(\rho\|\cdot\|)). \tag{4.6}$$

Then by Lemma 4.4

$$\begin{aligned}\rho(1,0)\|T\|_\infty &= \phi(\rho\|T\|)_\infty \\ &= \phi(\rho\|S\|)_\infty \\ &= \rho(1,0)\|S\|_\infty\end{aligned}$$

Thus $\|T\|_\infty = \|S\|_\infty$ and

$$\begin{aligned}K(T, \|\cdot\|) &= K(T, \phi(\rho\|\cdot\|)) \\ &= K(S, \phi(\rho\|\cdot\|)) \\ &= K(S, \|\cdot\|).\end{aligned}$$

Hence by Theorem 3.1

$$\rho\|T\| = \rho\|S\|. \tag{4.7}$$

That is, (4.6) implies (4.7), then by Theorem 3.1 follows the result. □

In the following theorem we characterizes the constructible norms in three ways. First, as the norms that belong to the orbit of their own projection. Second, as the norms that cannot separate the projection-indistinguishable operators. Third, as the norms that commute the diagram 4.4. The strength of the results to be presented lies in the fact that it provides an inherent criterion to verify if a norm is constructible, since the projection depends only on the norm to be studied.

Theorem 4.6. *Let X be a normed space, C a nonempty subset of X and $\|\cdot\| \in \mathcal{N}(BLip(C,X))$, Then the following statements are equivalent:*

(1) $\|\cdot\| \in \mathcal{N}(\mathbb{R}^2)\mathcal{N}(X)$.
(2) *For each $T, S \in BLip(C,X)$ that are $\phi(\|\cdot\|)$-indistinguishable is satisfied $\|T\| = \|S\|$.*
(3) $\|\cdot\| \in \mathcal{N}(\mathbb{R}^2)\phi(\|\cdot\|)$.
(4) *The Diagram 4.4 commutes.*

Proof. We suppose (1). Then by Corollary 4.5 exists $\theta \in \mathcal{N}(\mathbb{R}^2)$ such that $\|\cdot\| = \theta\phi(\|\cdot\|)$, Hence by Theorem 3.1 for each $T, S \in BLip(C,X)$ that are $\phi(\|\cdot\|)$-indistinguishable is satisfied $\|T\| = \|S\|$. So (1) implies (2).

Now we assume (2), Then by Theorem 3.1 there exists $\rho \in \mathcal{N}(\mathbb{R}^2)$ such that $\|\cdot\| = \rho\phi(\|\cdot\|)$. Hence $\|\cdot\| \in \mathcal{N}(\mathbb{R}^2)\phi(\|\cdot\|) \subset \mathcal{N}(\mathbb{R}^2)\mathcal{N}(X)$. Thus (2) implies (1) and (3).

Finally it is clear that (3) implies (1) and that (3) is equivalent to (4). $\square$

5. Examples

We will finish this work with two examples that relate the constructible norms with the geometry of the space. A norm $\rho \in \mathcal{N}(\mathbb{R}^2)$ is monotone if

$$x_1 \leq x_2 \text{ and } y_1 \leq y_2 \text{ implies } \rho(x_1, y_1) \leq \rho(x_2, y_2)$$

Example 5.1. Let X be a normed space and $C \subset X$ with $C \neq \emptyset$. If $\rho \in \mathcal{N}(\mathbb{R}^2)$ is monotone and convex, then for each $\|\cdot\| \in \mathcal{N}(X)$ the norm $\rho\|\cdot\| \in \mathcal{N}(BLip(C,X))$ is convex.

Let $T, S \in BLip(C,X)$ with $\rho\|T\| = \rho\|S\| = 1$, Then:

$$\begin{aligned}
\rho\|T+S\| &= \rho(\|T+S\|_\infty, K(T+S, \|\cdot\|)) \\
&\leq \rho(\|T\|_\infty + \|S\|_\infty, K(T, \|\cdot\|) + K(S, \|\cdot\|)) \\
&= \rho[(\|T\|_\infty, K(T, \|\cdot\|)) + (\|S\|_\infty, K(S, \|\infty\|))] \\
&< \rho\|T\| + \rho\|S\| \\
&= 2.
\end{aligned}$$

Thus $\rho\|\cdot\|$ is convex.

Example 5.2. Let X be a Banach space and C a nonempty subset of X. If $\rho \in \mathcal{N}(\mathbb{R}^2)$ is smooth and $\|\cdot\| \in \mathcal{N}(X)$, then $\rho\|\cdot\| \in \mathcal{N}(BLip(C,X)$ is smooth in the common points of differentiability of $\|\cdot\|_\infty$ and $K(\,\cdot\,, \|\cdot\|)$.

Let $T, S \in BLip(C,X)$ and $t \in \mathbb{R}$. Then we define

$$f(t) = \rho\|T + tS\|$$

Hence

$$f'(t)|_{t=0} = \lim_{\delta \to 0} \delta^{-1} \left(\rho\|T + \delta S\| - \rho\|T\|\right).$$

Additionally we take

$$g_1(t) = \|T + tS\|_\infty,$$

and

$$g_2(t) = K(\|T + tS\|, \|\cdot\|).$$

Then

$$f(t) = \rho(g_1(t), g_2(t)).$$

Thus

$$\left.\frac{\partial f}{\partial t}\right|_{t=0} = \left.\frac{\partial \rho}{\partial x}\right|_{(g_1(0),g_2(0))} \left.\frac{\partial g_1}{\partial t}\right|_{t=0} + \left.\frac{\partial \rho}{\partial y}\right|_{(g_1(0),g_2(0))} \left.\frac{\partial g_2}{\partial t}\right|_{t=0}.$$

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