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ANSWERS TO THE OPEN PROBLEM ON THE STABILITY OF THE GENERAL MIXED ADDITIVE AND QUADRATIC FUNCTIONAL EQUATIONS

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Abstract. The purpose of this paper is to answer Eskandani-Gavruta-Rassias-Zarghami open problem on the stability of the mixed additive and quadratic functional equations. In particular, we give an affirmative answer to the problem in case $\beta < a + b < 2\beta$ by fixed point method and two counterexamples in cases $a + b = \beta$ and $a + b = 2\beta$. The obtained results also extend the Eskandani-Gavruta-Rassias-Zarghami results on the stability of such functional equations in quasi-β-Banach spaces.

Key Words and Phrases: Hyers-Ulam stability, quasi-β-norm space, additive functional equation, quadratic functional equation.
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1. Introduction and preliminaries

Recently, authors have been interested in investigating the stability of functional equations in quasi-β-normed spaces. The quasi-β-normed space was defined in [10] as a generalization of a quasi-normed space [1, Definition 3]. Every quasi-normed space is a quasi-β-normed space with $\beta = 1$ and both of them are not continuous in general. However, a special case of quasi-β-norms, called (β, p)-norm, is continuous. In [5], Dung and Sintunavarat showed that every quasi-β-normed space is equivalent to a certain (β, p)-normed space.

Since the (β, p)-norm is continuous and every quasi-β-normed space is equivalent to a certain (β, p)-normed space, some authors prefer to investigate the stability in the (β, p)-normed space. In 2011, Eskandani *et al.* [6] formulated the general solution and investigated the stability of the following mixed additive and quadratic functional equation

$$f(\lambda x+y)+f(\lambda x-y)-f(x+y)-f(x-y)-(\lambda-1)[(\lambda+2)f(x)+\lambda f(-x)]=0 \quad (1.1)$$

where λ is a natural number with $\lambda \neq 1$ in (β, p)-Banach spaces. These results were based on Gavruta's idea [7] and Rassias-Kim's idea [10] concerning generalized control function φ. The authors also deduced [6, Corollary 3.15] on the stability of the functional equation (1.1) according to the upper bound, which is the mixed "product-sum" of powers of norms function. They also confirmed the analogue to Proposition 1.4 in the case $a+b>2\beta$, see [6, page 346]. However, the conclusion in the case $\beta \leq a+b \leq 2\beta$ is still open [6, page 346].

Fixed point theorems have been applied to investigate the stability of functional equations [2], [4], [11]. In 2018, Aydi *et al.* [2] introduced the generalized b-metric space and proved a fixed point theorem in this space. The generalized b-metric space is a generalization of the b-metric space [3] and the generalized metric space [9]. Later, Sintunavarat *et al.* [11] also gave the specific form of the fixed point theorem in b-metric spaces and proposed an approximation in the case of d not being continuous.

The purpose of this paper is to answer Eskandani-Gavruta-Rassias-Zarghami's open problem on the stability of the mixed additive and quadratic functional equations, see Question 1.5 below. In particular, we give an affirmative answer to the problem in case $\beta < a+b < 2\beta$ by fixed point method and two counterexamples in cases $a+b=\beta$ and $a+b=2\beta$. The obtained results also extend the Eskandani-Gavruta-Rassias-Zarghami's results in [6] on the stability of functional equation (1.1), in which λ is a complex number and $\lambda \neq 0, \lambda \neq 1$, in quasi-$\beta$-Banach spaces without assuming (p, β)-norm.

Now we recall notions and properties which are useful later.

Definition 1.1 ([2], page 1)**.** Let X be a nonempty set, $\kappa \geq 1$ and $d: X \to [0, \infty]$ be a function such that for all $x, y, z \in X$,

(1) $d(x,y)=0$ if and only if $x=y$.
(2) $d(x,y)=d(y,x)$.
(3) $d(x,y) \leq \kappa\big(d(x,z)+d(z,y)\big)$.

Then d is called a *generalized b-metric* on X and (X, d, κ) is called a *generalized b-metric space*. Without loss of generality, we can assume that κ is the smallest possible value.

The notions of convergent sequences, Cauchy sequences and complete generalized b-metric spaces are similar to those in metric spaces and b-metric spaces.

Definition 1.2 ([10], page 303)**.** Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $\kappa \geq 1$, $0 < \beta \leq 1$, and $\|.\|: X \to [0; \infty)$ be a function such that for all $x, y \in X$ and all $\lambda \in \mathbb{K}$,

(1) $\|x\|=0$ if and only if $x=0$.
(2) $\|\lambda x\|=|\lambda|^{\beta}\|x\|$.
(3) $\|x+y\| \leq \kappa\big(\|x\|+\|y\|\big)$.

Then we have

(1) $\|.\|$ is called a *quasi-β-norm* on X and $(X, \|.\|, \kappa)$ is called a *quasi-β-normed space*. Without loss of generality, we can assume that κ is the smallest possible value.
(2) $\|.\|$ is called a *(β, p)-norm* on X and $(X, \|.\|, \kappa)$ is called a *(β, p)-normed space* if there exists $0 < p \leq 1$ such that for all $x, y \in X$,
$$\|x + y\|^p \leq \|x\|^p + \|y\|^p.$$
(3) The sequence $\{x_n\}$ is called *convergent* to x, written $\lim_{n\to\infty} x_n = x$, if
$$\lim_{n\to\infty} \|x_n - x\| = 0.$$
(4) The sequence $\{x_n\}$ is called *Cauchy* if $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$.
(5) The quasi-β-normed space $(X, \|.\|, \kappa)$ is called *quasi-β-Banach* if each Cauchy sequence is a convergent sequence.
(6) The (β, p)-normed space $(X, \|.\|, \kappa)$ is called *(β, p)-Banach* if it is a quasi-β-Banach space.

The next theorem shows that every quasi-β-normed space is equivalent to a certain (β, p)-normed space.

Theorem 1.3 ([5], Theorem 6)**.** *Let $(X, \|.\|, \kappa)$ be a quasi-β-normed space, $p = \log_{(2\kappa)^{\frac{1}{\beta}}} 2$ and*
$$|||x||| = \inf\Big\{\Big(\sum_{i=1}^{n} \|x_i\|^{\frac{p}{\beta}}\Big)^{\frac{\beta}{p}} : x = \sum_{i=1}^{n} x_i, x_i \in X, n \geq 1\Big\}$$
for all $x \in X$. Then $|||.|||$ is a quasi-β-norm on X satisfying
$$|||x + y|||^p \leq |||x|||^p + |||y|||^p$$
and
$$\frac{1}{2\kappa}\|x\| \leq |||x||| \leq \|x\|$$
for all $x, y \in X$. In particular, the quasi-β-norm $|||.|||$ is a (β, p)-norm, and if $\|.\|$ is a norm then $\beta = p = 1$ and $|||.||| = \|.\|$.

We must say that, by direct calculation, the value
$$\frac{\sqrt[p]{2}\kappa_Y^2}{4^\beta}\left[\frac{1}{\sqrt[p]{\lambda^{2p\beta} - \lambda^{a+b}}} - \frac{1}{\sqrt[p]{\lambda^{p\beta} - \lambda^{a+b}}}\right]\|x\|_X^{a+b}$$
in the approximation of [6, Theorem 3.14] is exactly as in (1.2) as follows. Note that through the paper we denote
$$D_\lambda f(x, y) = f(\lambda x+y)+f(\lambda x-y)-f(x+y)-f(x-y)-(\lambda-1)[(\lambda+2)f(x)+\lambda f(-x)].$$

Proposition 1.4 ([6], Corollary 3.15)**.** *Suppose that*
(1) *$(X, \|.\|_X)$ is a normed space, and $(Y, \|.\|_Y, \kappa_Y)$ is a (β, p)-Banach space.*
(2) *$f : X \to Y$ is a map, and there exist non-negative numbers a, b such that $a + b < \beta$ and for all $x, y \in X$,*
$$\|D_\lambda f(x, y)\| \leq \|x\|_X^a\|y\|_X^b + \|x\|_X^{a+b} + \|y\|_X^{a+b}.$$

Then there exist a unique additive map $A : X \to Y$ and a unique quadratic map $Q : X \to Y$ satisfying

$$\|f(x) - A(x) - Q(x)\| \le \frac{\sqrt[p]{2}\kappa_Y^2}{4^\beta}\left[\frac{1}{\sqrt[p]{\lambda^{2p\beta} - \lambda^{p(a+b)}}} + \frac{1}{\sqrt[p]{\lambda^{p\beta} - \lambda^{p(a+b)}}}\right]\|x\|_X^{a+b}. \quad (1.2)$$

Question 1.5 ([6], page 346). *Does Proposition 1.4 hold for $\beta \le a + b \le 2\beta$?*

We must say that, from the proof of [11, Theorem 2.2], the value L in [11, Theorem 2.2.(2).(b)] is exactly L^p as in (1.3) as follows.

Theorem 1.6 ([11], Theorem 2.2). *Suppose that*

(1) *(X, d, κ) is a complete generalized b-metric space.*
(2) *The mapping $T : X \to X$ satisfies for all $x, y \in X$ and some $L \in [0, 1)$,*

$$d(Tx, Ty) \le L.d(x, y).$$

Then for each $x \in X$, we have

(1) *Either $d(T^n x, T^{n+1}x) = \infty$ for all $n \in \mathbb{N} \cup \{0\}$, or*
(2) *There exists n_0 such that for all $n > n_0$, $p = \log_{2\kappa} 2$,*

$$d(T^n x, x^*) \le \left(\frac{4}{1 - L^p}\right)^{\frac{1}{p}} d(T^{n_0}x, T^{n_0+1}x) \quad (1.3)$$

where x^* is a fixed point of T and $x^* = \lim_{n\to\infty} T^n x$.

2. Main results

We first show some properties of solutions and the stability of functional equation (1.1) where λ is a complex number, $\lambda \ne 0$ and $\lambda \ne 1$.

Lemma 2.1. *Suppose that X, Y are two linear spaces and $f : X \to Y$ is a mapping satisfying (1.1) for all $x, y \in X$,*

(1) *If f is odd then f is additive and $f(x) = \frac{1}{\lambda^n} f(\lambda^n x)$ for all $x \in X, n \in \mathbb{N}$.*
(2) *If f is even then f is quadratic and $f(x) = \frac{1}{\lambda^{2n}} f(\lambda^n x)$ for all $x \in X, n \in \mathbb{N}$.*

Proof. (1). The additive property of f is proved as in [6, Lemma 2.2]. Replacing $y = 0$ in (1.1) and using the oddness of f, we have $f(\lambda x) = \lambda f(x)$. By induction on n, we get $f(x) = \frac{1}{\lambda^n} f(\lambda^n x)$.

(2). The quadratic property of f is proved as in [6, Lemma 2.1]. Replacing $y = 0$ in (1.1) and using the evenness of f, we have $f(\lambda x) = \lambda^2 f(x)$. By induction on n, we have $f(x) = \frac{1}{\lambda^{2n}} f(\lambda^n x)$. □

Next, we present a fixed point result which is an important tool to prove our stability results. It follows directly from [2, Theorem 3.1] and Theorem 1.6 by choosing $\varphi(t) = L.t$ for all $t \in [0, \infty)$.

Lemma 2.2. *Suppose that*

(1) *(X, d, κ) is a complete generalized b-metric space.*
(2) *$T : X \to X$ is a mapping such that for all $x, y \in X$ and some $L \in [0, 1)$,*

$$d(Tx, Ty) \le L.d(x, y).$$

(3) *There exist $n_0 \in \mathbb{N} \cup \{0\}$ and $x_0 \in X$ such that $d(T^{n_0+1}x_0, T^{n_0}x_0) < \infty$.*

Then we have

(1) *T has a unique fixed point x^* in the set $X^* = \{x \in X : d(T^{n_0}x_0, x) < \infty\}$.*
(2) $\lim_{n\to\infty} T^n x_0 = x^*$.
(3) *For each $x \in X^*$, $p = \log_{2\kappa} 2$,*

$$d(x, x^*) \le (\frac{4}{1 - L^p})^{\frac{1}{p}} d(x, Tx).$$

Now, we use the fixed point result in Lemma 2.2 to prove the stability of functional equation (1.1) in case the given approximate map f is an odd map.

Proposition 2.3. *Suppose that*

(1) *X is a linear space, and $(Y, \|.\|, \kappa)$ is a quasi-β-Banach space.*
(2) *$\varphi : X^2 \to [0, \infty)$ is a function such that for some $0 \le L < 1$ and for all $x, y \in X$,*

$$\varphi(\lambda x, \lambda y) \le L|\lambda|^\beta \varphi(x, y) \tag{2.1}$$

(3) *$f : X \to Y$ is an odd map such that for all $x, y \in X$,*

$$\|D_\lambda f(x, y)\| \le \varphi(x, y). \tag{2.2}$$

Then we have

(1) *There exists a unique odd map A such that*
  (a) *A is a solution of the functional equation (1.1).*
  (b) *For all $x \in X$, and $p = \log_{2\kappa} 2$,*

$$\|f(x) - A(x)\| \le \frac{4\kappa^2}{(2|\lambda|)^\beta \sqrt[p]{1 - L^p}} \varphi(x, 0). \tag{2.3}$$

(2) *A is an additive map defined by for all $x \in X$,*

$$A(x) = \lim_{n\to\infty} \frac{f(\lambda^n x)}{\lambda^n}.$$

Proof. Let $G = \{g : X \to Y\}$ and $d : G \times G \to [0, \infty]$ be defined by

$$d(g, h) = \inf\{\gamma \in [0, \infty] : \|g(x) - h(x)\| \le \gamma\varphi(x, 0), x \in X\}$$

for all $g, h \in G$, where $\inf \emptyset = \infty$. Then d is a generalized b-metric on G and (G, d, κ) is a complete generalized b-metric space. Let $T : G \to G$ be defined by

$$Tg(x) = \frac{g(\lambda x)}{\lambda},\ g \in G, x \in X. \tag{2.4}$$

Now, replacing $y = 0$ in (2.2) and using the oddness of f, we have

$$\|f(\lambda x) - \lambda f(x)\| \le \frac{1}{2^\beta} \varphi(x, 0). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\|Tf(x) - f(x)\| \leq \frac{1}{(2|\lambda|)^\beta}\varphi(x,0).$$

This implies that

$$d(Tf, f) \leq \frac{1}{(2|\lambda|)^\beta} < \infty. \tag{2.6}$$

Moreover, for all $g, h \in G$ and $x \in X$, by using definitions of T, d and (2.1) we have

$$\begin{aligned}\|Tg(x) - Th(x)\| &= \frac{1}{|\lambda|^\beta}\|g(\lambda x) - h(\lambda x)\| \\ &\leq \frac{1}{|\lambda|^\beta}d(g,h)\varphi(\lambda x, 0) \\ &\leq Ld(g,h)\varphi(x,0).\end{aligned}$$

This implies that $d(Tg, Th) \leq Ld(g, h)$. By Lemma 2.2, T has a unique fixed point A in the set $G^* = \{g \in G : d(f, g) < \infty\}$ where

$$\begin{aligned}A(x) &= \lim_{n\to\infty} T^n f(x) = \lim_{n\to\infty} \frac{T^{n-1}f(\lambda x)}{\lambda} \\ &= \lim_{n\to\infty} \frac{T^{n-2}f(\lambda^2 x)}{\lambda^2} = \cdots = \lim_{n\to\infty} \frac{f(\lambda^n x)}{\lambda^n}\end{aligned} \tag{2.7}$$

and

$$d(f, A) \leq \Big(\frac{4}{1 - L^p}\Big)^{\frac{1}{p}} d(f, Tf). \tag{2.8}$$

It follows from definition of d, (2.6) and (2.8), we obtain

$$\|f(x) - A(x)\| \leq d(f, A)\varphi(x, 0) \leq \frac{4\kappa^2}{(2|\lambda|)^\beta \sqrt[p]{1 - L^p}}\varphi(x, 0).$$

Next, we show that A is additive. Using Theorem 1.3, (2.1), (2.2), (2.7) and the continuity of $|||.|||$, we get

$$\begin{aligned}|||D_\lambda A(x,y)||| &= |||\lim_{n\to\infty} \frac{1}{\lambda^n} D_\lambda f(\lambda^n x, \lambda^n y)||| \\ &= \lim_{n\to\infty} \frac{1}{|\lambda|^{n\beta}} |||D_\lambda f(\lambda^n x, \lambda^n y)||| \\ &\leq \lim_{n\to\infty} \frac{1}{|\lambda|^{n\beta}} \|D_\lambda f(\lambda^n x, \lambda^n y)\| \\ &\leq \lim_{n\to\infty} \frac{1}{|\lambda|^{n\beta}} \varphi(\lambda^n x, \lambda^n y) \\ &\leq \lim_{n\to\infty} L^n \varphi(x,y) = 0.\end{aligned}$$

Hence, $D_\lambda A(x, y) = 0$. This proves that A is a solution of (1.1). From (2.7) and the oddness of f, we have that A is also odd. It follows from Lemma 2.1.(1) that A is additive.

Finally, we prove the uniqueness of A. Suppose that $B : X \to Y$ is also an odd map satisfying the functional equation (1.1) and inequality (2.3). By Lemma 2.1.(1), we obtain

$$\frac{B(\lambda x)}{\lambda} = B(x)$$

for all $x \in X$. That is B is a fixed point of T. Since B satisfies inequality (2.3), we have

$$d(f, B) \le \frac{4\kappa^2}{(2|\lambda|)^\beta \sqrt[p]{1 - L^p}} < \infty.$$

This implies that B is also a fixed point of T in G^*. Since T has a unique fixed point A in G^*, $B = A$. □

The following result is formulated similarly to Proposition 2.3, in which the condition of φ is changed slightly.

Proposition 2.4. *Suppose that*

(1) *X is a linear space, and $(Y, \|.\|, \kappa)$ is a quasi-β-Banach space.*
(2) *$\varphi : X^2 \to [0, \infty)$ is a function such that for some $0 \le L < 1$ and for all $x, y \in X$,*

$$\varphi(\frac{x}{\lambda}, \frac{y}{\lambda}) \le \frac{L}{|\lambda|^\beta}\varphi(x, y)$$

(3) *$f : X \to Y$ is an odd map such that for all $x, y \in X$,*

$$\|D_\lambda f(x, y)\| \le \varphi(x, y).$$

Then we have

(1) *There exists a unique odd map A such that*
   (a) *A is a solution of the functional equation* (1.1).
   (b) *For all $x \in X$ and $p = \log_{2\kappa} 2$,*

$$\|f(x) - A(x)\| \le \frac{4\kappa^2}{(2|\lambda|)^\beta \sqrt[p]{1 - L^p}}\varphi(x, 0).$$

(2) *A is an additive map defined by for all $x \in X$,*

$$A(x) = \lim_{n\to\infty} \lambda^n f\big(\frac{x}{\lambda^n}\big).$$

We also use the fixed point result in Lemma 2.2 to prove the stability of the functional equation (1.1) in case the given approximate map f is an even map. The technique used to prove this is similar to that in Proposition 2.3.

Proposition 2.5. *Suppose that*

(1) *X is a linear space, and $(Y, \|.\|, \kappa)$ is a quasi-β-Banach space.*
(2) *$\varphi : X^2 \to [0, \infty)$ is a function such that for some $0 \le L < 1$ and for all $x, y \in X$,*

$$\varphi(\lambda x, \lambda y) \le L|\lambda|^{2\beta}\varphi(x, y)$$

(3) *$f : X \to Y$ is an even map such that for all $x, y \in X$,*

$$\|D_\lambda f(x, y)\| \le \varphi(x, y).$$

Then we have

(1) *There exists a unique even map Q such that*
 (a) *Q is a solution of the functional equation* (1.1).
 (b) *For all $x \in X$ and $p = \log_{2\kappa} 2$,*

$$\|f(x) - Q(x)\| \leq \frac{4\kappa^2}{(2|\lambda|^2)^\beta \sqrt[p]{1 - L^p}}\varphi(x, 0).$$

(2) *Q is an quadratic map defined by for all $x \in X$,*

$$Q(x) = \lim_{n\to\infty} \frac{f(\lambda^n x)}{\lambda^{2n}}.$$

The following result is formulated similarly to Proposition 2.4.

Proposition 2.6. *Suppose that*

(1) *X is a linear space, and $(Y, \|.\|, \kappa)$ is a quasi-β-Banach space.*
(2) *$\varphi : X^2 \to [0, \infty)$ is a function such that for some $0 \leq L < 1$ and for all $x, y \in X$,*

$$\varphi(\frac{x}{\lambda}, \frac{y}{\lambda}) \leq \frac{L}{|\lambda|^{2\beta}}\varphi(x, y)$$

(3) *$f : X \to Y$ is an odd map such that for all $x, y \in X$,*

$$\|D_\lambda f(x, y)\| \leq \varphi(x, y).$$

Then we have

(1) *There exists a unique even map Q such that*
 (a) *Q is a solution of the functional equation* (1.1).
 (b) *For all $x \in X$ and $p = \log_{2\kappa} 2$,*

$$\|f(x) - Q(x)\| \leq \frac{4\kappa^2}{(2|\lambda|^2)^\beta \sqrt[p]{1 - L^p}}\varphi(x, 0).$$

(2) *Q is an quadratic map defined by for all $x \in X$,*

$$Q(x) = \lim_{n\to\infty} \lambda^{2n} f(\frac{x}{\lambda^n}).$$

From the above propositions, we have the following result which answers Question 1.5 in the sense that the functional equation (1.1) is also stable in the case $\beta < a + b < 2\beta$.

Theorem 2.7. *Suppose that*

(1) *$(X, \|.\|_X)$ is a normed space, and $(Y, \|.\|_Y, \kappa)$ is a quasi-β-Banach space.*
(2) *$f : X \to Y$ is a map, and there exist $a, b \geq 0$ such that for all $x, y \in X$,*

$$\|D_\lambda f(x, y)\|_Y \leq \|x\|_X^a \|y\|_X^b + \|x\|_X^{a+b} + \|y\|_X^{a+b}. \tag{2.9}$$

Then we have

(1) *If either $|\lambda| > 1$ and $a + b < \beta$ or $|\lambda| < 1$ and $a + b > 2\beta$ then*
 (a) *There exist a unique odd map A and a unique even map Q such that*
  (i) *A and Q are solutions of the functional equation* (1.1).

(ii) *For all* $x \in X$ *and* $p = \log_{2\kappa} 2$,

$$\|f(x) - A(x) - Q(x)\|_Y \le \frac{8\kappa^4}{4^\beta}\left[\frac{1}{\sqrt[p]{|\lambda|^{p\beta} - |\lambda|^{p(a+b)}}} + \frac{1}{\sqrt[p]{|\lambda|^{2p\beta} - |\lambda|^{p(a+b)}}}\right]\|x\|_X^{a+b}.$$

(iii) *A is additive and Q is quadratic and defined by for all* $x \in X$,

$$A(x) = \lim_{n\to\infty} \frac{1}{2\lambda^n}\big(f(\lambda^n x) - f(-\lambda^n x)\big) \tag{2.10}$$

and

$$Q(x) = \lim_{n\to\infty} \frac{1}{2\lambda^{2n}}\big(f(\lambda^n x) + f(-\lambda^n x)\big). \tag{2.11}$$

(2) *If either* $|\lambda| > 1$ *and* $a + b > 2\beta$ *or* $|\lambda| < 1$ *and* $a + b < \beta$ *then*
  (a) *There exist a unique odd map A and a unique even map Q such that*
    (i) *A and Q are solutions of the functional equation* (1.1).
    (ii) *For all* $x \in X$ *and* $p = \log_{2\kappa} 2$,

$$\|f(x) - A(x) - Q(x)\|_Y \le \frac{8\kappa^4}{(4|\lambda|^2)^\beta}\left[\frac{|\lambda|^\beta}{\sqrt[p]{1 - |\lambda|^{p(\beta-a-b)}}} + \frac{1}{\sqrt[p]{1 - |\lambda|^{p(2\beta-a-b)}}}\right]\|x\|_X^{a+b}.$$

(iii) *A is additive and Q is quadratic and defined by for all* $x \in X$,

$$A(x) = \lim_{n\to\infty} \frac{\lambda^n}{2}\Big[f\Big(\frac{x}{\lambda^n}\Big) - f\Big(-\frac{x}{\lambda^n}\Big)\Big] \tag{2.12}$$

and

$$Q(x) = \lim_{n\to\infty} \frac{\lambda^{2n}}{2}\Big[f\Big(\frac{x}{\lambda^n}\Big) + f\Big(-\frac{x}{\lambda^n}\Big)\Big]. \tag{2.13}$$

(3) *If* $|\lambda| > 1$ *and* $\beta < a + b < 2\beta$ *then*
  (a) *There exist a unique odd map A and a unique even map Q such that*
    (i) *A and Q are solutions of the functional equation* (1.1).
    (ii) *For all* $x \in X$ *and* $p = \log_{2\kappa} 2$,

$$\|f(x) - A(x) - Q(x)\|_Y \le \frac{8\kappa^4}{(4|\lambda|^2)^\beta}\left[\frac{|\lambda|^\beta}{\sqrt[p]{1 - |\lambda|^{p(\beta-a-b)}}} + \frac{1}{\sqrt[p]{1 - |\lambda|^{p(a+b-2\beta)}}}\right]\|x\|_X^{a+b}.$$

(iii) *A is additive and Q is quadratic and defined by for all* $x \in X$,

$$A(x) = \lim_{n\to\infty} \frac{\lambda^n}{2}\Big[f\Big(\frac{x}{\lambda^n}\Big) - f\Big(-\frac{x}{\lambda^n}\Big)\Big]$$

and

$$Q(x) = \lim_{n\to\infty} \frac{1}{2\lambda^{2n}}\big(f(\lambda^n x) + f(-\lambda^n x)\big).$$

(4) *If* $|\lambda| < 1$ *and* $\beta < a + b < 2\beta$ *then*
  (a) *There exist a unique odd map A and a unique even map Q such that*
    (i) *A and Q are solutions of the functional equation* (1.1).

(ii) *For all* $x \in X$ *and* $p = \log_{2\kappa} 2$,

$$\|f(x) - A(x) - Q(x)\|_Y \le \frac{8\kappa^4}{(4|\lambda|^2)^\beta}\left[\frac{|\lambda|^\beta}{\sqrt[p]{1-|\lambda|^{p(a+b-\beta)}}} + \frac{1}{\sqrt[p]{1-|\lambda|^{p(2\beta-a-b)}}}\right]\|x\|_X^{a+b}.$$

(iii) A *is additive and* Q *is quadratic and defined by for all* $x \in X$,

$$A(x) = \lim_{n\to\infty} \frac{1}{2\lambda^n}\big(f(\lambda^n x) - f(-\lambda^n x)\big)$$

and

$$Q(x) = \lim_{n\to\infty} \frac{\lambda^{2n}}{2}\Big[f\Big(\frac{x}{\lambda^n}\Big) + f\Big(-\frac{x}{\lambda^n}\Big)\Big].$$

Proof. For all $x \in X$, we have $f(x) = f_e(x) + f_o(x)$ where f_e is an even map, f_o is an odd map and $f_e, f_o : X \to Y$ are defined by for all $x \in X$,

$$f_e(x) = \frac{f(x) + f(-x)}{2},$$

$$f_o(x) = \frac{f(x) - f(-x)}{2}.$$

It follows from (2.9) that

$$\|D_\lambda f_e(x,y)\|_Y \le \Psi(x,y),$$
$$\|D_\lambda f_o(x,y)\|_Y \le \Psi(x,y)$$

where $\Psi(x,y) = \frac{2\kappa}{2^\beta}(\|x\|_X^a\|y\|_X^b + \|x\|_X^{a+b} + \|y\|_X^{a+b})$ for all $x, y \in X$.

(1) If either $|\lambda| > 1$ and $a + b < \beta$ or $|\lambda| < 1$ and $a + b > 2\beta$, then

$$L_1 = |\lambda|^{a+b-\beta}, L_2 = |\lambda|^{a+b-2\beta} \in [0,1).$$

For all $x, y \in X$,

$$\Psi(\lambda x, \lambda y) = |\lambda|^{a+b}\Psi(x,y) = L_1|\lambda|^\beta\Psi(x,y),$$
$$\Psi(\lambda x, \lambda y) = |\lambda|^{a+b}\Psi(x,y) = L_2|\lambda|^{2\beta}\Psi(x,y).$$

Hence, all the assumptions of Proposition 2.3, Proposition 2.5 are satisfied for Ψ, L_1, f_o, and $\Psi, L_2, , f_e$, respectively. Therefore, there exist a unique odd map A, and a unique even map Q satisfying functional equation (1.1) and

$$\|f_o(x) - A(x)\|_Y \le \frac{4\kappa^2}{(2|\lambda|)^\beta\sqrt[p]{1-L_1^p}}\Psi(x,0) = \frac{8\kappa^3\|x\|_X^{a+b}}{4^\beta\sqrt[p]{|\lambda|^{p\beta} - |\lambda|^{p(a+b)}}}. \tag{2.14}$$

$$\|f_e(x) - Q(x)\|_Y \le \frac{4\kappa^2}{(2|\lambda|^2)^\beta\sqrt[p]{1-L_2^p}}\Psi(x,0) = \frac{8\kappa^3\|x\|_X^{a+b}}{4^\beta\sqrt[p]{|\lambda|^{2p\beta} - |\lambda|^{p(a+b)}}}. \tag{2.15}$$

where A is additive and Q is quadratic and defined by (2.10) and (2.11).

It follows from (2.14) and (2.15) that

$$\begin{aligned}\|f(x) - A(x) - Q(x)\|_Y &\le \kappa(\|f_o(x) - A(x)\|_Y + \|f_e(x) - Q(x)\|_Y)\\ &\le \frac{8\kappa^4}{4^\beta}\left[\frac{1}{\sqrt[p]{|\lambda|^{p\beta} - |\lambda|^{p(a+b)}}} + \frac{1}{\sqrt[p]{|\lambda|^{2p\beta} - |\lambda|^{p(a+b)}}}\right]\|x\|_X^{a+b}.\end{aligned}$$

(2) If either $|\lambda| > 1$ and $a+b > 2\beta$ or $|\lambda| < 1$ and $a+b < \beta$ then

$$L_1^{-1} = |\lambda|^{\beta-a-b}, L_2^{-1} = |\lambda|^{2\beta-a-b} \in [0,1).$$

For all $x, y \in X$,

$$\Psi\Big(\frac{x}{\lambda}, \frac{y}{\lambda}\Big) = \frac{1}{|\lambda|^{a+b}}\Psi(x,y) = L_1^{-1}|\lambda|^{-\beta}\Psi(x,y)$$

$$\Psi\Big(\frac{x}{\lambda}, \frac{y}{\lambda}\Big) = \frac{1}{|\lambda|^{a+b}}\Psi(x,y) = L_2^{-1}|\lambda|^{-2\beta}\Psi(x,y)$$

Hence, all assumptions of Proposition 2.4 and Proposition 2.6 are satisfied for Ψ, L_1^{-1}, f_o, and Ψ, L_2^{-1}, f_e, respectively. Therefore, there exist a unique odd map A, and a unique even map Q satisfying functional equation (1.1) and

$$\|f_o(x) - A(x)\|_Y \le \frac{4\kappa^2}{(2|\lambda|)^\beta \sqrt[p]{1 - L_1^{-p}}}\Psi(x,0) = \frac{8\kappa^3\|x\|^{a+b}}{(4|\lambda|)^\beta \sqrt[p]{1 - |\lambda|^{p(\beta-a-b)}}}. \tag{2.16}$$

$$\|f_e(x) - Q(x)\|_Y \le \frac{4\kappa^2}{(2|\lambda|^2)^\beta \sqrt[p]{1 - L_2^{-p}}}\Psi(x,0) = \frac{8\kappa^3\|x\|^{a+b}}{(4|\lambda|^2)^\beta \sqrt[p]{1 - |\lambda|^{p(2\beta-a-b)}}}. \tag{2.17}$$

where A is additive and Q is quadratic, and defined by (2.12) and (2.13).

It follows from (2.16) and (2.17) that

$$\begin{aligned}\|f(x) - A(x) - Q(x)\|_Y &\le \kappa(\|f_o(x) - A(x)\|_Y + \|f_e(x) - Q(x)\|_Y) \\ &\le \frac{8\kappa^4}{(4|\lambda|^2)^\beta}\left[\frac{|\lambda|^\beta}{\sqrt[p]{1 - |\lambda|^{p(\beta-a-b)}}} + \frac{1}{\sqrt[p]{1 - |\lambda|^{p(2\beta-a-b)}}}\right]\|x\|_X^{a+b}.\end{aligned}$$

(3) If $|\lambda| > 1$ and $\beta < a+b < 2\beta$ then by using the similar argument to the cases (1) and (2), we have all assumptions of Proposition 2.4 and Proposition 2.5 are satisfied for Ψ, L_1^{-1}, f_o, and Ψ, L_2, f_e, respectively. Therefore, there exist a unique odd map A, and a unique even map Q satisfying functional equation (1.1) and inequalities (2.16), (2.15), where A is additive and Q is quadratic and defined by (2.12) and (2.11). It follows from (2.15) and (2.16) that

$$\begin{aligned}\|f(x) - A(x) - Q(x)\|_Y &\le \kappa(\|f_o(x) - A(x)\|_Y + \|f_e(x) - Q(x)\|_Y) \\ &\le \frac{8\kappa^4}{(4|\lambda|^2)^\beta}\left[\frac{|\lambda|^\beta}{\sqrt[p]{1 - |\lambda|^{p(\beta-a-b)}}} + \frac{1}{\sqrt[p]{1 - |\lambda|^{p(a+b-2\beta)}}}\right]\|x\|_X^{a+b}.\end{aligned}$$

(4) If $|\lambda| < 1$ and $\beta < a+b < 2\beta$ then by using the similar argument to the cases (1) and (2), we have all assumptions of Proposition 2.3 and Proposition 2.6 are satisfied for Ψ, L_1, f_o, and Ψ, L_2^{-1}, f_e, respectively. Therefore, there exist a unique odd map A, and a unique even map Q satisfying functional equation (1.1) and inequalities (2.14), (2.17), where A is additive and Q is quadratic and defined by (2.10) and (2.13). It

follows from (2.14) and (2.17) that

$$\begin{aligned}\|f(x) - A(x) - Q(x)\|_Y &\le \kappa(\|f_o(x) - A(x)\|_Y + \|f_e(x) - Q(x)\|_Y) \\ &\le \frac{8\kappa^4}{(4|\lambda|^2)^\beta}\left[\frac{|\lambda|^\beta}{\sqrt[p]{1-|\lambda|^{p(a+b-\beta)}}} + \frac{1}{\sqrt[p]{1-|\lambda|^{p(2\beta-a-b)}}}\right]\|x\|_X^{a+b}.\end{aligned}$$

□

The following example shows that the functional equation (1.1) is not stable in the case $a + b = \beta$.

Example 2.8. Let $X = Y = \mathbb{R}$ with the absolute value norm, $\lambda > 1$,

$$\alpha = \frac{\lambda - 1}{2\lambda^3(\lambda^2+1)} > 0$$

and $\Psi, f : \mathbb{R} \to \mathbb{R}$ be given by

$$\Psi(x) = \begin{cases} \alpha x & \text{if } |x| < 1, \\ \alpha & \text{if } |x| \ge 1 \end{cases}$$

and for all $x \in \mathbb{R}$,

$$f(x) = \sum_{i=0}^{\infty} \frac{\Psi(\lambda^i x)}{\lambda^i}.$$

Then we have

(1) X is a normed space, and Y is a (β, p)-Banach space with $\beta = p = 1$.

(2) For all $a > 0$, $b > 0$ and $\beta = a + b$, f satisfies the inequality

$$|D_\lambda f(x,y)| \le |x| + |y| + |x|^a|y|^b. \tag{2.18}$$

(3) There do not exist $\delta > 0$, an additive map $A : \mathbb{R} \to \mathbb{R}$ and a quadratic map $Q : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$|f(x) - A(x) - Q(x)| \le \delta|x|.$$

Proof. (2) For all $x \in \mathbb{R}$, we have

$$|f(x)| \le \sum_{i=0}^{\infty} \frac{|\Psi(\lambda^i x)|}{\lambda^i} \le \sum_{i=0}^{\infty} \frac{\alpha}{\lambda^i} = \frac{\alpha\lambda}{\lambda - 1}. \tag{2.19}$$

If $x = y = 0$, then (2.18) holds.

If $0 < |x| + |y| < \frac{1}{\lambda}$, then there exists a positive integer number k such that

$$\frac{1}{\lambda^{k+2}} \le |x| + |y| < \frac{1}{\lambda^{k+1}}.$$

This implies that

$$\lambda^{k-1}x, \lambda^{k-1}(-x), \lambda^{k-1}(x+y), \lambda^{k-1}(x-y), \lambda^{k-1}(\lambda x+y), \lambda^{k-1}(\lambda x-y) \in (-1,1).$$

Hence, for each $i = 0, 1, \ldots, k-1$,

$$\lambda^i x, \lambda^i(-x), \lambda^i(x+y), \lambda^i(x-y), \lambda^i(\lambda x+y), \lambda^i(\lambda x-y) \in (-1,1).$$

Then, using the definitions of Ψ and $D_\lambda f$, we obtain

$$D_\lambda \Psi(\lambda^i x, \lambda^i y) = 0$$

for all $i = 0, 1, \ldots, k-1$. This follows that

$$\begin{aligned} |D_\lambda f(x,y)| &\leq \sum_{i=0}^{\infty} \frac{|D_\lambda \Psi(\lambda^i x, \lambda^i y)|}{\lambda^i} \\ &\leq \sum_{i=k}^{\infty} \frac{|D_\lambda \Psi(\lambda^i x, \lambda^i y)|}{\lambda^i} \\ &\leq \sum_{i=k}^{\infty} \frac{2\alpha(\lambda^2+1)}{\lambda^i} \\ &= \frac{2\alpha\lambda^{1-k}(\lambda^2+1)}{\lambda-1}. \end{aligned}$$

This implies that

$$\frac{|D_\lambda f(x,y)|}{|x|+|y|} \leq \frac{2\alpha\lambda^{1-k}(\lambda^2+1)}{\lambda-1}\lambda^{k+2} = \frac{2\alpha\lambda^3(\lambda^2+1)}{\lambda-1} = 1.$$

If $0 < |x| + |y| \geq \dfrac{1}{\lambda}$ then using (2.19), we obtain

$$|D_\lambda f(x,y)| \leq 2(\lambda^2+1)\frac{\alpha\lambda}{\lambda-1}.$$

This implies that

$$\frac{|D_\lambda f(x,y)|}{|x|+|y|} \leq \frac{2\alpha\lambda^2(\lambda^2+1)}{\lambda-1} \leq \frac{2\alpha\lambda^3(\lambda^2+1)}{\lambda-1} = 1.$$

Hence, we have

$$|D_\lambda f(x,y)| \leq |x| + |y| \leq |x| + |y| + |x|^a|y|^b.$$

Therefore, f satisfies (2.18).

(3) Suppose that there exist an additive map $A : \mathbb{R} \to \mathbb{R}$, a quadratic map $Q : \mathbb{R} \to \mathbb{R}$, and $\delta > 0$ such that for all $x \in \mathbb{R}$,

$$|f(x) - A(x) - Q(x)| \leq \delta|x|.$$

It follows from [8, Theorem 1] and [8, Corollary 2] that there exist $\theta_1, \theta_2 \in \mathbb{R}$ such that $A(x) = \theta_1 x, Q(x) = \theta_2 x^2$ for all $x \in \mathbb{R}$. This implies that

$$|f(x)| \leq \begin{cases} (|\theta_1| + |\theta_2| + \delta)|x| & \text{if } |x| < 1, \\ (|\theta_1| + |\theta_2| + \delta)|x|^2 & \text{if } |x| \geq 1. \end{cases} \tag{2.20}$$

Let m be large enough such that $m\alpha > |\theta_1| + |\theta_2| + \delta$. For $x \in \big(0, \frac{1}{\lambda^{m-1}}\big)$, we have

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} \frac{\Psi(\lambda^i x)}{\lambda^i} \\ &\geq \sum_{i=0}^{m-1} \frac{\alpha\lambda^i x}{\lambda^i} \\ &= m\alpha x \\ &> (|\theta_1| + |\theta_2| + \delta)x. \end{aligned}$$

This contradicts to (2.20). □

The following example shows that the functional equation (1.1) is not stable in the case $a + b = 2\beta$.

Example 2.9. Let $X = Y = \mathbb{R}$ with the absolute value norm, $\lambda > 1$,

$$\alpha = \frac{4\lambda^2 - 1}{32\lambda^4(\lambda^2 + 1)} > 0$$

and $\Psi, f : \mathbb{R} \to \mathbb{R}$ be given by

$$\Psi(x) = \begin{cases} \alpha x^2 & \text{if } |x| < 1, \\ \alpha & \text{if } |x| \geq 1 \end{cases}$$

and for all $x \in \mathbb{R}$,

$$f(x) = \sum_{i=0}^{\infty} \frac{\Psi(2^i\lambda^i x)}{(2\lambda)^{2i}}.$$

Then we have

(1) X is a normed space, and Y is a (β, p)-Banach space with $\beta = p = 1$.
(2) For all $a > 0$, $b > 0$ and $2\beta = a + b$, f satisfies the inequality

$$|D_\lambda f(x, y)| \leq |x|^2 + |y|^2 + |x|^a|y|^b.$$

(3) There do not exist $\delta > 0$, an additive map $A : \mathbb{R} \to \mathbb{R}$ and a quadratic map $Q : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A(x) - Q(x)| \leq \delta|x|^2.$$

Proof. Similar to the proof of Example 2.8. □

Remark 2.10. Theorem 2.7.(3)&(4), Example 2.8 and Example 2.9 are the answers to Question 1.5.

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