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NON-COMPACT TYPE OF EXISTENCE THEOREMS IN GENERALIZED BANACH ALGEBRAS AND APPLICATIONS

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Abstract. This paper is concerned with some fixed point theorems in generalized Banach algebras. Our fundamental results are detailed regarding in terms of the lack of compactness by using set-contractive operators with respect to some generalized measures of noncompactness. The obtained results are further applied to a certain coupled system for impulsive differential equations defined in the generalized Banach algebra $\mathcal{PC}([0,1],\mathbb{R}) \times \mathcal{PC}([0,1],\mathbb{R})$ of all couple of piecewise continuous functions on $[0,1]$.
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1. Introduction

Since the rise of the Banach and Schauder fixed point theorems [1, 16] presented by the Polish mathematicians S. Banach (1892-1945) and J. P. Schauder (1899-1943), the world has witnessed a big transition in mathematics due to its applications to ensure the existence (and uniqueness) for many processes in physics, optimisation, population dynamics, biology and medicine.

Moreover, several investigations have paid a great attention to combine these theorems, ones for the sum of two operators in Banach spaces, that is Krasnoselskii type of fixed point theorems [4, 11, 12], and others for the product of two operators in Banach algebras, this trend was initiated by B. C. Dhage [6, 7, 8, 13]. On the other hand, some authors focused on the generalization of Banach and Schauder fixed point theorems. One of this latter research directions involved the Russian mathematician

A. Perov work [19] in 1964 that shows the Banach fixed point theorem in complete vector-valued metrics spaces by replacing the contractive factor with a matrix M converging to zero, and A. Viorel [23], I-R. Petre and A. Petruşel [20] and A. Ouahab [18] which presented some versions of Schauder's and Krasnoselskii's fixed point theorems in these spaces.

Recently, some researchers have studied a class of differential equations which is known as impulsive differential equations, to describe evolution processes that are subject to short-term perturbations in their states which are negligible compared to the process duration. These instantaneously perturbations can be seen as impulses. For example, the action of a pendulum clock in physic, drug distribution for periodic treatment of some diseases in medicine and abrupt changes by biological parameters in population dynamics, for more details see [3, 14, 15, 21] and references mentioned therein. The study of coupled systems of impulsive differential equations has also took some attention.

The aim of this paper is to keep on exploring the mentioned issues for the generalized Banach algebras by using the technique of the generalized measure of noncompactness, to ensure some existence results to certain coupled systems of impulsive differential equations by dispensing the compactness condition to the involving operators defined in suitable generalized Banach algebras.

The content of this manuscript is structured as follows: In Section 2, we present the preliminary result necessary for the correct understanding of this manuscript. In Section 3, we define generalized Banach algebras and give some auxiliary lemmas, and we extend some existence results. In Section 4, we apply our results to substantiate the existence of a solution of the following coupled system of differential equations with just one impulse effect:

$$\begin{cases} \left(\dfrac{\varrho_1(t)-k_1(t,\varrho_1(t),\varrho_2(t))}{j_1(t,\varrho_1(t),\varrho_2(t))}\right)' = l_1(t,\varrho_1(t),\varrho_2(t)), & a.e.\ t\in[0,1] \\ \left(\dfrac{\varrho_2(t)-k_2(t,\varrho_1(t),\varrho_2(t))}{j_2(t,\varrho_1(t)\varrho_2(t))}\right)' = l_2(t,\varrho_1(t),\varrho_2(t)), & a.e.\ t\in[0,1] \\ \Delta\varrho_1(\theta) = I_1\left(\varrho_1(\theta^-),\varrho_2(\theta^-)\right), \quad \Delta\varrho_2(\theta) = I_2\left(\varrho_1(\theta^-),\varrho_2(\theta^-)\right), & \\ \varrho_1(0)=\varrho_1^0, \qquad \varrho_2(0)=\varrho_2^0. & \end{cases}$$

2. Preliminaries

Our purpose here is to recall some notations, definitions, and auxiliary results which will be used later to prove our results. We begin with defining on $\mathcal{M}_{m\times n}(\mathbb{R}_+)$ the partial order relation as follow: Let $M,\ N\in\mathcal{M}_{m\times n}(\mathbb{R}_+), m\geq 1$ and $n\geq 1$. Put $M=(M_{i,j})_{\substack{1\leq j\leq m\\ 1\leq i\leq n}}$ and $N=(N_{i,j})_{\substack{1\leq j\leq m\\ 1\leq i\leq n}}$. Then,

$$\begin{aligned} &M\preccurlyeq N \text{ if } N_{i,j}\geq M_{i,j} \quad \text{for all } j=1,\cdots,m,\ i=1,\cdots,n.\\ &M\prec N \text{ if } N_{i,j}> M_{i,j} \quad \text{for all } j=1,\cdots,m,\ i=1,\cdots,n.\end{aligned}$$

Let $\mathcal{A} = \prod_{i=1}^{n} \mathcal{A}_i$ be a bounded set in $\mathbb{R}^n$, we denote by the supremum bound (resp. the infimun bound) of $\mathcal{A}$ the vector

$$\widehat{\sup}\{\lambda : \lambda \in \mathcal{A}\} := \begin{pmatrix} \sup\{\lambda_1 : \lambda_1 \in \mathcal{A}_1\} \\ \vdots \\ \sup\{\lambda_n : \lambda_n \in \mathcal{A}_n\} \end{pmatrix},$$

$$\left(\text{resp. } \widehat{\inf}\{\lambda : \lambda \in \mathcal{A}\} := \begin{pmatrix} \inf\{\lambda_1 : \lambda_1 \in \mathcal{A}_1\} \\ \vdots \\ \inf\{\lambda_n : \lambda_n \in \mathcal{A}_n\} \end{pmatrix} \right).$$

Definition 2.1. Let $\mathcal{X}$ be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By a generalized norm on $\mathcal{X}$ we mean a map

$$\|.\|_G : \mathcal{X} \longrightarrow \mathbb{R}^n_+$$
$$\varrho \mapsto \|\varrho\|_G = \begin{pmatrix} \|\varrho\|_1 \\ \vdots \\ \|\varrho\|_n \end{pmatrix}$$

satisfying the following properties:

(*i*) For all $\varrho \in \mathcal{X}$; if $\|\varrho\|_G = 0_{\mathbb{R}^n_+}$, then $\varrho = 0_{\mathcal{X}}$,

(*ii*) $\|\lambda\varrho\|_G = |\lambda| \|\varrho\|_G$ for all $\varrho \in \mathcal{X}$ and $\lambda \in \mathbb{K}$, and

(*iii*) $\|\varrho + v\|_G \preccurlyeq \|\varrho\|_G + \|v\|_G$ for all $\varrho, v \in \mathcal{X}$.

The pair $(\mathcal{X}, \|.\|_G)$ is called a generalized normed space. If the generalized metric space generated by the map $\|.\|_G$ (i.e., $d_G(\varrho, v) = \|\varrho - v\|_G$) is complete, then the space $(\mathcal{X}, \|.\|_G)$ is called a generalized Banach space (in short, GBS).

Proposition 2.2. [9] *In a GBS in the sense of Perov, the notions of convergence sequence, continuity, open subset and closed subset are similar to those for usual Banach spaces.*

Throughout this paper and for $r = (r_1, \cdots, r_n) \in \mathbb{R}^n_+$, $\varrho_0 \in \mathcal{X}$, and $i = 1, \cdots, n$, we denote by:

$$B(\varrho_0, r) = \{\varrho \in \mathcal{X} : \|\varrho_0 - \varrho\|_G \prec r\} \ \ (\text{resp. } B_i(\varrho_0, r_i) = \{\varrho \in \mathcal{X} : \|\varrho_0 - \varrho\|_i < r_i\}),$$

for the open ball centered at ϱ_0 with radius r (resp. r_i), and by

$$\overline{B}(\varrho_0, r) = \{\varrho \in \mathcal{X} : \|\varrho_0 - \varrho\|_G \preccurlyeq r\} \ \ (\text{resp. } \overline{B_i}(\varrho_0, r_i) = \{\varrho \in \mathcal{X} : \|\varrho_0 - \varrho\|_i \leq r_i\}),$$

for the closed ball centered at ϱ_0 with radius r (resp. r_i). If $\varrho_0 = 0$ we simply write $B_r = B(0, r)$ and $\overline{B_r} = \overline{B}(0, r)$. Finally, we respectively denote by $\overline{\mathcal{K}}$ and $co(\mathcal{K})$ for the closure and the convex hull of an arbitrary subset $\mathcal{K}$ of $\mathcal{X}$.

Definition 2.3. Let $(\mathcal{X}, \|.\|_G)$ be a GBS and let $\mathcal{K}$ be a subset of $\mathcal{X}$. Then, $\mathcal{K}$ is said to be G-bounded, if there is a vector $V \in \mathbb{R}^n_+$ such that

$$\text{for all } \varrho \in \mathcal{K}, \qquad \|\varrho\|_G \preccurlyeq V,$$

and we write

$$\|\mathcal{K}\|_G := \widehat{\sup}\{\|\varrho\|_G : \varrho \in \mathcal{K}\} = \begin{pmatrix} \sup_{\varrho\in\mathcal{K}} \|\varrho\|_1 \\ \vdots \\ \sup_{\varrho\in\mathcal{K}} \|\varrho\|_n \end{pmatrix} \preccurlyeq V.$$

Definition 2.4. Let $(\mathcal{X}, \|.\|_G)$ be a GBS. A subset $\mathcal{K}$ of $\mathcal{X}$ is called G-compact if every open cover of $\mathcal{K}$ has a finite sub-cover. $\mathcal{K}$ is said relatively G-compact if its closure is G-compact.

We denote by $\mathcal{N}_G(\mathcal{X})$ the family of all relatively G-compact sets of $\mathcal{X}$.

Presently, we present a useful definition of generalized measures of noncompactness for generalized Banach spaces, which is similar that was introduced in 1980 by J. Banaś and K. Goebel [2].

Definition 2.5. Let $(\mathcal{X}, \|.\|_G)$ be a GBS and let $\mathcal{B}_G(\mathcal{X})$ be the family of G-bounded subsets of $\mathcal{X}$. A map

$$\mu_G : \mathcal{B}_G(\mathcal{X}) \longrightarrow [0, +\infty)^n$$
$$\mathcal{A} \mapsto \mu_G(\mathcal{A}) = \begin{pmatrix} \mu_1(\mathcal{A}) \\ \vdots \\ \mu_n(\mathcal{A}) \end{pmatrix}$$

is called a generalized measure of noncompactness (for short G-MNC) defined on $\mathcal{X}$ if it satisfies the following conditions:

(i) The family $\operatorname{Ker}\mu_G(\mathcal{X}) = \{\mathcal{A} \in \mathcal{B}_G(\mathcal{X}) : \mu_G(\mathcal{A}) = 0\}$ is nonempty and $\operatorname{Ker}\mu_G(\mathcal{X}) \subset \mathcal{N}_G(\mathcal{X})$.
(ii) Monotonicity: $\mathcal{A}_1 \subseteq \mathcal{A}_2 \Rightarrow \mu_G(\mathcal{A}_1) \preccurlyeq \mu_G(\mathcal{A}_2)$, for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{B}_G(\mathcal{X})$.
(iii) Invariance under closure and convex hull: $\mu_G(\mathcal{A}) = \mu_G(\overline{\mathcal{A}}) = \mu_G(co(\mathcal{A}))$, for all $\mathcal{A} \in \mathcal{B}_G(\mathcal{X})$.
(iv) Convexity: $\mu_G(\lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2) \preccurlyeq \lambda\mu_G(\mathcal{A}_1) + (1-\lambda)\mu_G(\mathcal{A}_2)$, for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{B}_G(\mathcal{X})$ and $\lambda \in [0,1]$.
(v) Generalized Cantor intersection property: If $(\mathcal{A}_m)_{m\geq 1}$ is a sequence of nonempty, closed subsets of $\mathcal{X}$ such that $\mathcal{A}_1$ is G-bounded and $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \ldots \supseteq \mathcal{A}_m \ldots$, and $\lim_{m\to+\infty} \mu_G(\mathcal{A}_m) = 0_{\mathbb{R}^n_+}$, then the set $\mathcal{A}_\infty := \bigcap_{m=1}^{\infty} \mathcal{A}_m$ is nonempty and is G-compact.

Example 2.6. A typical example of a G-MNC (in the sense of Definition 2.5) is the generalized diameter defined for each $\mathcal{A} \in \mathcal{B}_G(\mathcal{X})$ by:

$$\delta_G(\mathcal{A}) = \begin{cases} 0_{\mathbb{R}^n}, \text{ if } \mathcal{A} \text{ is empty} \\ \\ \widehat{\sup}\{\|\varrho - v\|_G : \varrho, v \in \mathcal{A}\}, \text{ else} \end{cases}$$
$$= \begin{pmatrix} \begin{cases} 0, \text{ if } \mathcal{A} \text{ is empty} \\ \sup\{\|\varrho - v\|_1 : \varrho, v \in \mathcal{A}\}, \text{ else} \end{cases} \\ \vdots \\ \begin{cases} 0, \text{ if } \mathcal{A} \text{ is empty} \\ \sup\{\|\varrho - v\|_n : \varrho, v \in \mathcal{A}\}, \text{ else} \end{cases} \end{pmatrix}$$
$$= \begin{pmatrix} \delta_1(\mathcal{A}) \\ \vdots \\ \delta_n(\mathcal{A}) \end{pmatrix}.$$

A G-MNC is called:
(vi) Sub-additive if $\mu_G(\mathcal{A}_1 + \mathcal{A}_2) \leq \mu_G(\mathcal{A}_1) + \mu_G(\mathcal{A}_2)$, for all $\mathcal{A}_1,\ \mathcal{A}_2 \in \mathcal{B}_G(\mathcal{X})$.
(vii) Regular if $\operatorname{Ker} \mu_G = \mathcal{N}_G(\mathcal{X})$.

Example 2.7. An example of a G-MNC which satisfies properties (vi) and (vii) is the generalized Hausdorff measure χ_G defined for all G-bounded subset $\mathcal{A} \subseteq \mathcal{X}$ by

$$\chi_G(\mathcal{A}) := \widehat{\inf}\left\{\varepsilon \in \mathbb{R}^n_+ : \mathcal{A} \text{ has a finite } \varepsilon \text{ -net }\right\},$$
$$= \widehat{\inf}\left\{\varepsilon \succ 0 : \mathcal{A} \subseteq \bigcup_{k=1}^{m} B(\varrho^k, r^k),\ \varrho^k \in \mathcal{X},\ r^k \prec \varepsilon,\ k = \overline{1,m}\ m \in \mathbb{N}\right\}$$
$$= \begin{pmatrix} \inf\left\{\varepsilon > 0 : \mathcal{A} \subseteq \bigcup_{k=1}^{m} B_1(\varrho^k, r^k_1),\ \varrho^k \in \mathcal{X},\ r^k_1 < \varepsilon_1,\ k = \overline{1,m},\ m \in \mathbb{N}\right\} \\ \vdots \\ \inf\left\{\varepsilon > 0 : \mathcal{A} \subseteq \bigcup_{k=1}^{m} B_n(\varrho^k, r^k_n),\ \varrho^k \in \mathcal{X},\ r^k_n < \varepsilon_n,\ k = \overline{1,m},\ m \in \mathbb{N}\right\} \end{pmatrix}$$
$$= \begin{pmatrix} \chi_1(\mathcal{A}) \\ \vdots \\ \chi_n(\mathcal{A}) \end{pmatrix}.$$

Definition 2.8. A matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is said to be convergent to zero if

$$M^m \longrightarrow 0, \quad \text{as} \quad m \longrightarrow \infty.$$

Lemma 2.9. [22] *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. So the following affirmations are equivalent:*
(i) $M^m \longrightarrow 0$, as $m \longrightarrow \infty$.
(ii) The matrix $I - M$ is invertible, and $(I - M)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$.

Definition 2.10. Let $(\mathcal{X}, d_G)$ be a complete generalized metric space and S be an operator from $\mathcal{X}$ into itself. S is called G-Lipschitzian with matrix M if there is a square matrix of non-negative numbers such that

$$d_G(S(\varrho), S(v)) \preccurlyeq M d_G(\varrho, v), \quad \text{for all } \varrho,\ v \in \mathcal{X}\ .$$

If the matrix M converges to zero, then S is called a M-contraction.

Theorem 2.11. *(Perov, [19]) Let $(\mathcal{X}, d_G)$ be a complete generalized metric space and let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be a M-contraction operator. Then, S has a unique fixed point $\varrho^* \in \mathcal{X}$.*

Definition 2.12. Let $(\mathcal{X}, \|.\|_G)$ be a GBS and let μ_G be a G-MNC. A self-mapping $S : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be a M-set contractive with respect to μ_G if S maps G-bounded sets into G-bounded sets, and there exists a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}^+)$ such that

$$\mu_G(S(\mathcal{A})) \preccurlyeq M\mu_G(\mathcal{A}),$$

for every nonempty G-bounded subset $\mathcal{A}$ of $\mathcal{X}$. If the matrix M converges to zero then, we say that S satisfies the generalized Darbo condition.

Lemma 2.13. *Let $\mathcal{K}$ be a subset of a GBS $\mathcal{X}$. Assume that $S : \mathcal{K} \to \mathcal{X}$ is G-Lipschitzian with a matrix M. Then, for each G-bounded subset $\mathcal{A}$ of $\mathcal{K}$, we have*

$$\chi_G(S(\mathcal{A})) \preccurlyeq M\chi_G(\mathcal{A}).$$

Proof. Let $\mathcal{A}$ be a G-bounded subset of $\mathcal{X}$ and $r \succ \chi_G(\mathcal{A})$. There exist $0_{\mathbb{R}^n} \preccurlyeq r_0 \preccurlyeq r$ and a finite subset $\mathcal{Q}$ of $\mathcal{X}$ such that $\mathcal{A} \subseteq \mathcal{Q} + B_{r_0}$. Let $\varrho \in \mathcal{A}$, then there is a $v \in \mathcal{Q}$ such that $\|\varrho - v\|_G \preccurlyeq r_0$. Since S is G-Lipschitzian with the matrix M, then $\|S(\varrho) - S(v)\|_G \preccurlyeq M\|\varrho - v\|_G \preccurlyeq Mr_0$. It follows that

$$S(\mathcal{A}) \subseteq S(\mathcal{Q}) + MB_{r_0}. \tag{2.1}$$

Hence,

$$\chi_G(S(\mathcal{A})) \preccurlyeq Mr_0.$$

Letting $r_0 \to \chi_G(\mathcal{K})$, we get

$$\chi_G(S(\mathcal{A})) \preccurlyeq M\chi_G(\mathcal{A}).$$ ☐

Now, we recall the Schauder type theorem for generalized Banach spaces.

Theorem 2.14. [23] *Let $(\mathcal{X}, \|.\|_G)$ be a GBS, $\mathcal{K}$ a closed, convex subset of $\mathcal{X}$, and let $S : \mathcal{K} \longrightarrow \mathcal{X}$ be a continuous operator such that $S(\mathcal{K})$ is relatively G-compact. Then, S has at least one fixed point in $\mathcal{K}$.*

The next theorem is an extension of both Theorem 2.14 and Darbo's fixed point Theorem [5] in generalized Banach spaces.

Theorem 2.15. [9] Let $\mathcal{X}$ be a GBS. Then, every nonempty G-bounded, closed, convex subset $\mathcal{K}$ of $\mathcal{X}$ has the fixed point property for continuous mappings satisfying the generalized Darbo condition.

3. Generalized Banach algebras

Definition 3.1. An algebra $\mathcal{X}$ is a vector space endowed with an internal composition law denoted by (.)

$$\begin{cases} (.) : \mathcal{X} \times \mathcal{X} & \rightarrow \mathcal{X} \\ \quad (\varrho, v) & \mapsto \varrho.v \end{cases}$$

which is associative and bilinear. We denote by $1_{\mathcal{X}}$ neutral multiplicative element with respect to (.).

Definition 3.2. A generalized normed algebra $\mathcal{X}$ is an algebra endowed with a generalized norm satisfying the following property

$$\text{for all } \varrho, v \in \mathcal{X} \quad \|\varrho \cdot v\|_G \leq \|\varrho\|_G \odot \|v\|_G,$$

where

$$\|\varrho \cdot v\|_G := \begin{pmatrix} \|\varrho \cdot v\|_1 \\ \\ \vdots \\ \\ \|\varrho \cdot v\|_n \end{pmatrix}$$

and

$$\|\varrho\|_G \odot \|v\|_G := \begin{pmatrix} \|\varrho\|_1 \|v\|_1 \\ \\ \vdots \\ \\ \|\varrho\|_n \|v\|_n \end{pmatrix}.$$

Note that $\odot$ defines the Hadamard product (also known as the element-wise) between two matrices M, N of the same dimension $m \times n$,

$$(M \odot N)_{i,k} = (M)_{i,k}(N)_{i,k}.$$

It should be noticed that the Hadamard product distributes over matrix addition (i.e, for all $m \times n$-matrices M, N, C , we have $M \odot (N + C) = (M \odot N) + (M \odot C)$). Unlike the usual matrix product, the Hadamard product is commutative: $M \odot N = N \odot M$ (for more details about Hadamard product see [10]).

Remark 3.3. Let $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$. Let us define the diagonal matrix D_v, by

$$(D_v)_{i,k} = \begin{cases} v_i & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, we can see that the Hadamard product of two vectors ϱ and v as a dot product of one vector by the corresponding diagonal matrix of the other vector i.e,:

$$v \odot \varrho = \varrho \odot v = \varrho D_v = v D_\varrho.$$

For the time being we present and prove a simple but useful lemma which we will utilise a few times in the sequel.

Lemma 3.4. *Let $\mathcal{K}$ and $\mathcal{K}'$ be two G-bounded subsets of a generalized Banach algebra (in short, GBA) $\mathcal{X}$. Then, we have*

$$\chi_G(\mathcal{K}.\mathcal{K}^{'}) \preccurlyeq \|\mathcal{K}^{'}\|_G \odot \chi_G(\mathcal{K}) + \|\mathcal{K}\|_G \odot \chi_G(\mathcal{K}') + \chi_G(\mathcal{K}) \odot \chi_G\left(\mathcal{K}^{'}\right).$$

Proof. Let $\mathcal{K}$ and $\mathcal{K}'$ be two G-bounded subsets of $\mathcal{X}$, and fix arbitrarily two vectors r, r' such $r \succ \chi_G(\mathcal{K})$ and $r' \succ \chi_G(\mathcal{K}')$. Then, there exist finite sets F and F' in $\mathcal{X}$ such that

$$\mathcal{K} \subseteq F + B_r$$
$$\mathcal{K}' \subseteq F' + B_{r'},$$

Furthermore, for $z \in \mathcal{K}.\mathcal{K}^{'}$, there is $\varrho \in \mathcal{K}$ and $v \in \mathcal{K}^{'}$ such that $z = \varrho \cdot v$. By using the previous inclusions we find $f \in F, f' \in F', x \in B_r$ and $y \in B_{r'}$ such that $\varrho = f + x$ and $v = f' + y$. Hence we get

$$\begin{aligned} z = \varrho \cdot v &= (f+x)\cdot(f'+y) = f\cdot f' + f\cdot y + x\cdot f' + x\cdot y \\ &= f'\cdot f + (\varrho - x)\cdot y + x\cdot(v-y) + x\cdot y \\ &= f\cdot f' + \varrho\cdot y + x\cdot v - x\cdot y \end{aligned}$$

This implies that

$$\mathcal{K}.\mathcal{K}' \subseteq F.F' + \mathcal{K}.B_{r'} + B_r.\mathcal{K}' + B_r.B_{r'} \subseteq F.F' + B_{\|\mathcal{K}\|_G \odot r' + \|\mathcal{K}'\|_G \odot r + r \odot r'}.$$

Since the product of two finite subsets is a finite set and by using the definition of the generalized Hausdorff measure of noncompactness χ_G, we obtain

$$\chi_G(\mathcal{K}.\mathcal{K}') \preccurlyeq \|\mathcal{K}\|_G \odot r' + \|\mathcal{K}^{'}\|_G \odot r + r \odot r'. \tag{3.1}$$

Hence, when $r \to \chi_G(\mathcal{K})$ and $r' \to \chi_G(\mathcal{K}')$, in Inequality (3.1) we deduce the desired result. □

We also need the following lemma.

Lemma 3.5. *Let $\mathcal{K}$ be a nonempty G-bounded closed subset of a GBA $\mathcal{X}$ and let $R, T : \mathcal{X} \to \mathcal{X}$ be two G-Lipschitzian mappings with matrices M and M' respectively. Assume that the matrix $M_* = (MD_{\|\mathcal{K}\|_G} + M')$ converges to zero, then $\left(\frac{I-T}{R}\right)^{-1}$: $\mathcal{K} \to \mathcal{X}$ exists and is continuous.*

Proof. Let us fix $v \in \mathcal{K}$. The map σ_v which is defined by

$$\begin{aligned} \sigma_v : &\mathcal{X} \longrightarrow \mathcal{X} \\ &\varrho \mapsto R(\varrho)\cdot v + T(\varrho) \end{aligned}$$

is M_*-contraction. Indeed, for all $\varrho_1, \varrho_2 \in \mathcal{X}$ we have:

$$\begin{aligned} \|\sigma_v(\varrho_1) - \sigma_v(\varrho_2)\|_G &\preccurlyeq \|R(\varrho_1) - R(\varrho_2)\|_G \odot \|v\|_G + \|T(\varrho_1) - T(\varrho_2)\|_G \\ &\preccurlyeq M\|\varrho_1 - \varrho_2\|_G D_{\|\mathcal{K}\|_G} + M'\|\varrho_1 - \varrho_2\|_G \\ &\preccurlyeq MD_{\|\mathcal{K}\|_G}\|\varrho_1 - \varrho_2\|_G + M'\|\varrho_1 - \varrho_2\|_G \\ &\preccurlyeq (MD_{\|\mathcal{K}\|_G} + M')\|\varrho_1 - \varrho_2\|_G. \end{aligned}$$

Now, the Perov fixed point theorem ensures the existence and uniqueness of a point $\varrho^* \in \mathcal{X}$ such that $\sigma_v(\varrho^*) = \varrho^*$, which means $v = \left(\frac{I-T}{R}\right)(\varrho^*)$. So, the operator $N := \left(\frac{I-T}{R}\right)^{-1} : \mathcal{K} \to \mathcal{X}$ is well defined. Now we will show that $N : \mathcal{K} \to \mathcal{X}$ is continuous. To see this, let $(\varrho_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}$ converging to a point ϱ. Since $\mathcal{K}$ is closed, then $\varrho \in \mathcal{K}$. First, it is easy to verify that that for each $y \in \mathcal{K}$ we have

$$N(y) = T(N(y)) + R(N(y)) \cdot y.$$

Hence,

$$\begin{aligned}\|N(\varrho_n) - N(\varrho)\|_G &\preccurlyeq \|T(N(\varrho_n)) - T(N(\varrho))\|_G + \|(R(N(\varrho_n))) \cdot \varrho_n - (R(N(\varrho))) \cdot \varrho\|_G \\ &\preccurlyeq \|T(N(\varrho_n)) - T(N(\varrho))\|_G + \|R(N(\varrho))\|_G \odot \|\varrho_n - \varrho\|_G \\ &+ \Theta(\varrho_n, \varrho) \\ &\preccurlyeq M' \|N(\varrho_n) - N(\varrho)\|_G + MD_{\|\mathcal{K}\|_G} \|N(\varrho_n) - N(\varrho)\|_G \\ &+ \Lambda(\varrho_n, \varrho),\end{aligned}$$

where

$$\Theta(\varrho_n, \varrho) = \|R(N(\varrho_n)) - R(N(\varrho))\|_G \odot \|\varrho_n\|_G$$

and

$$\Lambda(\varrho_n, \varrho) = \|R(N(\varrho))\|_G \odot \|\varrho_n - \varrho\|_G .$$

Now by Lemma 2.9, we get

$$\|N(\varrho_n) - N(\varrho)\|_G \preccurlyeq (I_{\mathcal{M}_{n\times n}} - M_*)^{-1} \|R(N(\varrho))\|_G \odot \|\varrho_n - \varrho\|_G \to 0 \text{ as } n \to \infty. \quad (3.2)$$

Thus the continuity of N on $\mathcal{K}$. □

Now, if $T = 0$, we have the following corollary.

Corollary 3.6. *Let $\mathcal{K}$ be a nonempty G-bounded closed subset of a GBA $\mathcal{X}$ and let $R : \mathcal{X} \to \mathcal{X}$ be a G-Lipschitzian mapping with matrix M. Assume that $\|\mathcal{K}\|_G \odot M$ converges to zero, then $\left(\frac{I}{R}\right)^{-1} : \mathcal{K} \to \mathcal{X}$ exists and is continuous.*

4. Fixed Point Result

Presently we are prepared to state and demonstrate our results.

Theorem 4.1. *Let $\mathcal{K}$ be a nonempty, G-bounded, closed and convex subset of a GBA $\mathcal{X}$, and assume that R and S are two operators from $\mathcal{K}$ into $\mathcal{X}$ such that*
(i) R and S are G-Lipschitzian with matrices M and M' respectively,
(ii) the matrix $M_* = MD_{\|S(\mathcal{K})\|_G} + M'D_{\|R(\mathcal{K})\|_G} + M'D_{(M\chi_G(\mathcal{K}))}$ converges to zero, and
(iii) $S(\varrho) \cdot R(\varrho) \in \mathcal{K}$ for all $\varrho \in \mathcal{K}$.
Then, there is $\varrho \in \mathcal{K}$ such that $R(\varrho) \cdot S(\varrho) = \varrho$.

Proof. Take $\mathcal{A}$ an arbitrary nonempty G-bounded subset of $\mathcal{K}$. By using our conditions and Lemmas 3.4 and 2.13, we get

$$\begin{aligned}
\chi_G\left(S(\mathcal{A})\cdot R(\mathcal{A})\right) &\preccurlyeq \|R(\mathcal{A})\|_G \odot \chi_G(S(\mathcal{A})) + \|S(\mathcal{A})\|_G \odot \chi_G(R(\mathcal{A})) \\
&+ \chi_G(R(\mathcal{A})) \odot \chi_G(S(\mathcal{A})) \\
&\preccurlyeq \|R(\mathcal{K})\|_G \odot (M'\chi_G(\mathcal{A})) + \|S(\mathcal{K})\|_G \odot (M\chi_G(\mathcal{A})) \\
&+ (M\chi_G(\mathcal{A})) \odot (M'\chi_G(\mathcal{A})) \\
&\preccurlyeq MD_{\|S(\mathcal{K})\|_G}\chi_G(\mathcal{A}) + M'D_{\|R(\mathcal{K})\|_G}\chi_G(\mathcal{A}) + M'D_{(M\chi_G(\mathcal{K}))}\chi_G(\mathcal{A}) \\
&\preccurlyeq \left[MD_{\|S(\mathcal{K})\|_G} + M'D_{\|R(\mathcal{K})\|_G} + M'D_{(M\chi_G(\mathcal{K}))}\right]\chi_G(\mathcal{A}) \\
&= M_*\chi_G(\mathcal{A}).
\end{aligned}$$

It follows that the operator RS satisfy the generalized Darbo condition. By Theorem 2.15, we get the desired result. □

Theorem 4.2. *Let $\mathcal{K}$ be a nonempty, G-bounded, closed, and convex subset of a GBA $\mathcal{X}$. Let $R, T : \mathcal{X} \to \mathcal{X}$, and $S : \mathcal{K} \to \mathcal{X}$ be three operators that fulfil the following conditions:*
(i) S is continuous and there exists a matrix Υ such that S is Υ-set contractive with respect to χ_G,
(ii) the operators R and T satisfy all the conditions of Lemma 3.5, and
(iii) $\varrho = R(\varrho)\cdot S(v) + T(\varrho)$, for all $v \in \mathcal{K}$ implies $\varrho \in \mathcal{K}$.
Then, the equation $\varrho = R(\varrho)\cdot S(\varrho) + T(\varrho)$ has at least one solution in $\mathcal{K}$ whenever the matrix $M_* = (I_{\mathcal{M}_{n\times n}} - MD_{\|S(\mathcal{K})\|_G} - M')^{-1}\cdot \Upsilon D_{(\|R(\mathcal{K})\|_G + M\chi_G(\mathcal{K}))}$ converges to zero.

Proof. Note that the equation $\varrho = R(\varrho)\cdot S(\varrho) + T(\varrho)$, has a solution for some $\varrho \in \mathcal{K}$ if, and only if, ϱ is a fixed point for the operator $N := \left(\frac{I-T}{R}\right)^{-1} \circ S$. Let $v \in \mathcal{K}$, from the assumption (ii), and Lemma 3.5, the operator $\left(\frac{I-T}{R}\right)^{-1}$ exists on $S(\mathcal{K})$, hence there is a unique $\varrho_v \in \mathcal{X}$ such that

$$\left(\frac{I-T}{R}\right)(\varrho_v) = S(v).$$

Consequently,

$$\varrho_v = R(\varrho_v)\cdot S(v) + T(\varrho_v).$$

By condition (iii), we get $\varrho_v \in \mathcal{K}$, and then N is well defined on $\mathcal{K}$. Again by using Lemma 3.5, we conclude that $\left(\frac{I-T}{R}\right)^{-1}$ is continuous. Using the fact that S is continuous, we infer that N is continuous. Now using the following equality

$$N = RN\cdot S + TN, \tag{4.1}$$

Let $\mathcal{A}$ be a G-bounded set in $\mathcal{K}$ then, by using the sub-additivity of χ_G and Lemma 3.4, we get

$$\begin{aligned}\chi_G\left(N(\mathcal{A})\right) &\preccurlyeq \chi_G\left(R\left(N(\mathcal{A})\right)S(\mathcal{A})\right) + \chi_G\left(T\left(N(\mathcal{A})\right)\right)\\ &\preccurlyeq \|S(\mathcal{K})\|_G \odot M\chi_G(N(\mathcal{A})) + \|R(\mathcal{K})\|_G \odot \chi_G\left(S(\mathcal{A})\right) + \Pi(\mathcal{A})\\ &\preccurlyeq (MD_{\|S(\mathcal{K})\|_G} + M')\chi_G(N(\mathcal{A})) + (\|R(\mathcal{K})\|_G + M\chi_G\left(\mathcal{K}\right)) \odot \Upsilon\chi_G(\mathcal{A})\\ &\preccurlyeq (MD_{\|S(\mathcal{K})\|_G} + M')\chi_G(N(\mathcal{A})) + \Upsilon D_{(\|R(\mathcal{K})\|_G + M\chi_G(\mathcal{K}))}\chi_G(\mathcal{A}),\end{aligned}$$

where

$$\Pi(\mathcal{A}) = \chi_G(S(\mathcal{A})) \odot \chi_G\left(R(N(\mathcal{A}))\right) + M'\chi_G\left(N(\mathcal{A})\right).$$

It follows that

$$\begin{aligned}\chi_G\left(N(\mathcal{A})\right) &\preccurlyeq (I_{\mathcal{M}_{n\times n}} - MD_{\|S(\mathcal{K})\|_G} - M')^{-1} \cdot \Upsilon D_{(\|R(\mathcal{K})\|_G + M\chi_G(\mathcal{K}))}\chi_G(\mathcal{A})\\ &\preccurlyeq M_*\chi_G(\mathcal{A}).\end{aligned}$$

Since M_* converges to zero then, N satisfies the generalized Darbo condition with respect to χ_G hence, Theorem 2.15 completes the proof. □

Now some consequences derive from Theorem 4.2. We first state the following result when $T = 0$.

Corollary 4.3. *Let $\mathcal{K}$ be a nonempty, G-bounded, closed and convex subset of a GBA $\mathcal{X}$. Let $R : \mathcal{X} \to \mathcal{X}$, and $S : \mathcal{K} \to \mathcal{X}$ be two operators that fulfil the following conditions:*
(i) S is continuous and there exists a matrix Υ such that S is Υ-set contractive with respect to χ_G,
(ii) R satisfy the conditions of Corollary 3.6, and
(iii) $\varrho = R(\varrho) \cdot S(v)$, for all $v \in \mathcal{K}$ implies $\varrho \in \mathcal{K}$.
If the matrix $M_* = (I_{\mathcal{M}_{n\times n}} - MD_{\|S(\mathcal{K})\|_G})^{-1} \cdot \Upsilon D_{(\|R(\mathcal{K})\|_G + M\chi_G(\mathcal{K}))}$ converges to zero, then the equation $\varrho = R(\varrho) \cdot S(\varrho)$ has at least one solution in $\mathcal{K}$.

Corollary 4.4. [17] *Let $\mathcal{K}$ be a nonempty, G-bounded, closed, and convex subset of a GBA $\mathcal{X}$. Let $R : \mathcal{X} \to \mathcal{X}$, and $S : \mathcal{K} \to \mathcal{X}$ be two operators that fulfil the following conditions:*
(i) S is continuous operator and $S(\mathcal{K})$ is relatively G-compact,
(ii) R satisfies the conditions of Corollary 3.6, and
(iii) $\varrho = R(\varrho) \cdot S(v)$, for all $v \in \mathcal{K}$ implies $\varrho \in \mathcal{K}$.
Then, the equation $\varrho = R(\varrho) \cdot S(\varrho)$ has at least one solution in $\mathcal{K}$.

The next result discusses to case where $R = 1_{\mathcal{X}}$.

Corollary 4.5. *Let $\mathcal{K}$ be a nonempty, G-bounded, closed, and convex subset of a GBA $\mathcal{X}$. Let $T : \mathcal{X} \to \mathcal{X}$, and $S : \mathcal{K} \to \mathcal{X}$ be two operators that fulfil the following conditions:*
(i) S is continuous and there exists a matrix Υ such that S is Υ-set contractive with respect to χ_G,
(ii) the operator T is M-contraction, and
(iii) $\varrho = S(v) + T(\varrho)$, for all $v \in \mathcal{K}$ implies $\varrho \in \mathcal{K}$.

If the matrix $M_* = (I_{\mathcal{M}_{n\times n}} - M)^{-1} \cdot \Upsilon$ *converges to zero, then the equation* $\varrho = S(\varrho) + T(\varrho)$ *has at least one solution in* $\mathcal{K}$*.*

Theorem 4.6. *Let* $\mathcal{K}$ *be a nonempty, G-bounded, closed, and convex subset of a GBA* $\mathcal{X}$*. Let* $R, T : \mathcal{X} \to \mathcal{X}$*, and* $S : \mathcal{K} \to \mathcal{X}$ *be three operators that fulfil the following conditions:*
(i) S *is continuous and there exists a matrix* Υ *such that* S *is* Υ*-set contractive with respect to* χ_G*,*
(ii) the operators R *and* T *satisfy all conditions of Lemma 3.5,*
(iii) $R(\mathcal{K})$ *is a relatively G–compact set, and*
(iv) $\varrho = R(\varrho) \cdot S(v) + T(\varrho)$*, for all* $v \in \mathcal{K}$ *implies* $\varrho \in \mathcal{K}$*.*
Then, the equation $\varrho = R(\varrho) \cdot S(\varrho) + T(\varrho)$ *has a solution in* $\mathcal{K}$*, whenever the matrix* $M_* = (I_{\mathcal{M}_{n\times n}} - M')^{-1} \cdot \Upsilon \cdot D_{\|R(\mathcal{K})\|_G}$ *converges to zero.*

Proof. According to the proof of Theorem 4.2

$$N : \mathcal{K} \longrightarrow \mathcal{K}$$

$$\varrho \longrightarrow N(\varrho) = \left(\frac{I - T}{R}\right)^{-1} \circ S(\varrho)$$

is continuous. In order to achieve the proof, we will apply Theorem 2.15. Hence, we only have to prove that the operator N satisfies the generalized Darbo condition with respect to χ_G. Indeed, let $\mathcal{A} \in \mathcal{B}_G(\mathcal{X})$ then, by Equation (4.1), sub-additivity of χ_G and Lemma 3.4 we have

$$\begin{aligned} \chi_G\left(N(\mathcal{A})\right) &\preccurlyeq \chi_G\left(R\left(N(\mathcal{A})\right) S(\mathcal{A})\right) + \chi_G\left(T\left(N(\mathcal{A})\right)\right) \\ &\preccurlyeq \|S(\mathcal{K})\|_G \odot \chi_G(RN(\mathcal{A})) + \Omega(\mathcal{A}) \\ &\preccurlyeq \|R(\mathcal{K})\|_G \odot \Upsilon \chi_G\left(\mathcal{A}\right) + M' \chi_G\left(N\mathcal{A}\right), \end{aligned}$$

where

$$\Omega(\mathcal{A}) = \|R(\mathcal{K})\|_G \odot \chi_G\left(S(\mathcal{A})\right) + \chi_G(S(\mathcal{A})) \odot \chi_G\left(R(N\mathcal{A})\right) + M' \chi_G\left(N\mathcal{A}\right).$$

Consequently,

$$\begin{aligned} \chi_G\left(N\mathcal{A}\right) &\preccurlyeq (I_{\mathcal{M}_{n\times n}} - M')^{-1} \cdot \|R(\mathcal{K})\|_G \odot \Upsilon \chi_G(\mathcal{A}) \\ &\preccurlyeq M_* \chi_G(\mathcal{A}). \end{aligned}$$

□

5. Application

Let consider the following system of differential equations with just one impulse effect:

$$\begin{cases} \left(\dfrac{\varrho_1(t) - k_1(t, \varrho_1(t), \varrho_2(t))}{j_1(t, \varrho_1(t), \varrho_2(t))}\right)' = l_1(t, \varrho_1(t), \varrho_2(t)), \qquad a.e.\ t \in [0,1] \\ \left(\dfrac{\varrho_2(t) - k_2(t, \varrho_1(t), \varrho_2(t))}{j_2(t, \varrho_1(t)\varrho_2(t))}\right)' = l_2(t, \varrho_1(t), \varrho_2(t)), \qquad a.e.\ t \in [0,1] \\ \Delta\varrho_1\left(\theta\right) = I_1\left(\varrho_1(\theta^-), \varrho_2\left(\theta^-\right)\right), \quad \Delta\varrho_2\left(\theta\right) = I_2\left(\varrho_1(\theta^-), \varrho_2\left(\theta^-\right)\right), \\ \varrho_1(0) = \varrho_1^0, \qquad \varrho_2(0) = \varrho_2^0, \end{cases} \tag{5.1}$$

where $\theta \in]0,1[$, and for each $i=1,2$, $l_i, k_i \in \mathcal{C}([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $j_i \in \mathcal{C}([0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\backslash 0)$ and $I_i \in \mathcal{C}(\mathbb{R}\times\mathbb{R},\mathbb{R})$. We denote by $\varrho(\theta^+) = \lim_{a\to 0^+}\varrho(\theta+a)$ and $\varrho(\theta^-) = \lim_{a\to 0^+}\varrho(\theta-a)$. We will prove that the system (5.1) has a solution in the generalized Banach algebra $\mathcal{X} = \mathcal{PC}([0,1],\mathbb{R})\times\mathcal{PC}([0,1],\mathbb{R})$ of all couple of piecewise continuous functions on $[0,1]$ endowed with the generalized norm

$$\begin{aligned}\|.\|_G :&\mathcal{X}\longrightarrow \mathbb{R}^2_+\\ &\varrho=(\varrho_1,\varrho_2)\mapsto \|(\varrho_1,\varrho_2)\|_G = \begin{pmatrix}\|\varrho_1\|_{\mathcal{PC}}\\ \\ \|\varrho_2\|_{\mathcal{PC}}\end{pmatrix},\end{aligned}$$

where

$$\mathcal{PC}([0,1],\mathbb{R}) = \left\{\begin{array}{c}\varrho:[0,1]\to\mathbb{R},\ \varrho\in\mathcal{C}([0,1]\backslash\{\theta\},\mathbb{R}) \text{ such that}\\ \varrho(\theta^-) \text{ and } \varrho(\theta^+) \text{ exist and } \varrho(\theta^-)=\varrho(\theta)\end{array}\right\},$$

with the norm

$$\|\varrho\|_{\mathcal{PC}} = \sup_{t\in[0,1]}|\varrho(t)|.$$

Clearly, $(\mathcal{PC}([0,1],\mathbb{R}),\|\cdot\|_{\mathcal{PC}})$ is a Banach algebra.
First of all, let us state and prove the following lemma.

Lemma 5.1. *Assume that for each $i=1,2$, $l_i\in\mathcal{C}([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $j_i\in\mathcal{C}([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R}\backslash\{0\})$, $j_i\left(0,\varrho_1^0,\varrho_2^0\right):=j_i^0$ and $k_i\left(0,\varrho_1^0,\varrho_2^0\right):=k_i^0$. Then $\varrho:=(\varrho_1,\varrho_2)$ is a solution of the system of the impulsive differential equation* (5.1) *if, and only if, ϱ is a solution of the following system of integral equations (SIE).*

$$\begin{pmatrix}\varrho_1(t)\\ \\ \varrho_2(t)\end{pmatrix} = \begin{pmatrix} j_1(t,\varrho_1(t),\varrho_2(t))\Big(\displaystyle\int_0^t l_1(s,\varrho_1(s),\varrho_2(s))ds + \frac{I_1(\varrho_1(\theta),\varrho_2(\theta))}{j_1(\theta,(\varrho_1(\theta),\varrho_2(\theta)))}\\ +\dfrac{\varrho_1^0-k_1^0}{j_1^0}\Big) + k_1(t,\varrho_1(t),\varrho_2(t)),\ a.e.\ t\in[0,1],\\ \\ j_2(t,\varrho_1(t),\varrho_2(t))\Big(\displaystyle\int_0^t l_2(s,\varrho_1(s),\varrho_2(s))ds + \frac{I_2(\varrho_1(\theta),\varrho_2(\theta))}{j_2(\theta,(\varrho_1(\theta),\varrho_2(\theta)))}\\ +\dfrac{\varrho_2^0-k_2^0}{j_2^0}\Big) + k_2(t,\varrho_1(t),\varrho_2(t)),\ a.e.\ t\in[0,1]. \end{pmatrix}. \tag{5.2}$$

Proof. Let $\rho=(\rho_1,\rho_2)$ be a solution of (SIE) (5.1) then, for each $i=1,2$ and $t\in[0,\theta]$, we transform the system of differential equations (5.1) into an equivalent integral equation

$$\frac{\varrho_i(t)-k_i(t,\varrho_1(t),\varrho_2(t))}{j_i(t,\varrho_1(t),\varrho_2(t))} = \int_0^t l_1(s,\varrho_1(s),\varrho_2(s))ds + \frac{\varrho_i^0-k_i^0}{j_i^0}. \tag{5.3}$$

Then,

$$\varrho_i(t) = j_i(t,\varrho_1(t),\varrho_2(t))\left(\int_0^t l_i(s,\varrho_1(s),\varrho_2(s))ds + \frac{\varrho_i^0-k_i^0}{j_i^0}\right) + k_i(t,\varrho_1(t),\varrho_2(t)).$$

In a similar way, if $t \in]\theta, 1]$, then for each $i = 1, 2$, we have:

$$\frac{\varrho_i(t) - k_i(t, \varrho_1(t), \varrho_2(t))}{j_i(t, \varrho_1(t), \varrho_2(t))} = \int_\theta^t l_1(s, \varrho_1(s), \varrho_2(s))ds + \frac{\varrho_i(\theta^+) - k_i(\theta, \varrho_1(\theta), \varrho_2(\theta))}{j_i(\theta, \varrho_1(\theta), \varrho_2(\theta))}. \tag{5.4}$$

But,

$$\varrho_i(\theta^+) = \varrho_i(\theta^-) + I_i\left(\varrho_1\left(\theta\right), \varrho_2\left(\theta\right)\right)$$

that means

$$\varrho_i(\theta^+) = j_i(\theta, \varrho_1(\theta), \varrho_2(\theta))\Gamma_i(\theta, \varrho_1, \varrho_2) + k_i(\theta, \varrho_1(\theta), \varrho_2(\theta)) + I_i\left(\varrho_1\left(\theta\right), \varrho_2\left(\theta\right)\right),$$

where

$$\Gamma_i(\theta, \varrho_1, \varrho_2) = \left(\int_0^\theta l_i(s, \varrho_1(s), \varrho_2(s))ds + \frac{\varrho_i^0 - k_i^0}{j_i^0}\right).$$

Turning to Equation (5.4)

$$\frac{\varrho_i(t) - k_i(t, \varrho_1(t), \varrho_2(t))}{j_i(t, \varrho_1(t), \varrho_2(t))} = \int_0^t l_i(s, \varrho_1(s), \varrho_2(s))ds + \frac{I_i(\varrho_1(\theta), \varrho_2(\theta))}{j_i(\theta, \varrho_1(\theta), \varrho_2(\theta))} + \frac{\varrho_i^0 - k_i^0}{j_i^0} \tag{5.5}$$

we have,

$$\varrho_i(t) = j_i(t, \varrho_1(t), \varrho_2(t))\left(\Gamma_i(t, \varrho_1, \varrho_2) + \frac{I_i(\varrho_1(\theta), \varrho_2(\theta))}{j_i(\theta, \varrho_1(\theta), \varrho_2(\theta))}\right) + k_i(t, \varrho_1(t), \varrho_2(t)).$$

On the other hand, for $i = 1, 2$ and $t \in [0, \theta],\ ]\theta, 1]$, suppose that $\varrho = (\varrho_1, \varrho_2)$ satisfies (5.2), so we get (5.5). Applying the derivative operator on both sides of (5.5) we find that ϱ satisfies the first and the second equality in (5.1). Again for the initial value conditions we substitute $t = 0$ in (5.2). With regard to impulsive conditions, we can obtain it by direct verification, and that completes the proof. □

We recall that problem (SIE) (5.2) can be written in the following form:

$$\varrho = S(\varrho)R(\varrho) + T(\varrho),$$

where

$$S(\varrho)(t) = \begin{pmatrix} S_1(\varrho)(t) \\ \\ S_2(\varrho)(t) \end{pmatrix} = \begin{pmatrix} j_1(t, \varrho_1(t), \varrho_2(t)) \\ \\ j_2(t, \varrho_1(t), \varrho_2(t)) \end{pmatrix},$$

and

$$\begin{aligned} R(\varrho)(t) &= \begin{pmatrix} R_1(\varrho)(t) \\ \\ R_2(\varrho)(t) \end{pmatrix} \\ &= \begin{pmatrix} \displaystyle\int_0^t l_1(s, \varrho_1(s), \varrho_2(s))ds + \frac{I_1(\varrho_1(\theta), \varrho_2(\theta))}{j_1(\theta, \varrho_1(\theta), \varrho_2(\theta))} + \frac{\varrho_1^0 - k_1^0}{j_1^0}, \\ \displaystyle\int_0^t l_2(s, \varrho_1(s), \varrho_2(s))ds + \frac{I_2(\varrho_1(\theta), \varrho_2(\theta))}{j_2(\theta, \varrho_1(\theta), \varrho_2(\theta))} + \frac{\varrho_2^0 - k_2^0}{j_2^0}, \end{pmatrix}, \end{aligned}$$

and,

$$T(\varrho)(t)=\begin{pmatrix}T_1(\varrho)(t)\\ \\ T_2(\varrho)(t)\end{pmatrix}=\begin{pmatrix}k_1(t,\varrho_1(t),\varrho_2(t))\\ \\ k_2(t,\varrho_1(t),\varrho_2(t))\end{pmatrix}.$$

The problem (5.1) will be discussed under the following assumptions:

($\mathcal{H}_1$) There exists a matrix $N=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\in\mathcal{M}_{2\times 2}\left(L^1([0,1])\right)$ such that for each $(t,\varrho_1,\varrho_2),(t,\bar{\varrho}_1,\bar{\varrho}_2)\in[0,1]\times\mathbb{R}\times\mathbb{R}$ and for $i\in\{1,2\}$, we have:

$$|l_i(t,\varrho_1,\varrho_2)-l_i(t,\bar{\varrho}_1,\bar{\varrho}_2)|\leq a_{i1}(t)\,|\varrho_1-\bar{\varrho}_1|+a_{i2}(t)\,|\varrho_2-\bar{\varrho}_2|\,.$$

($\mathcal{H}_2$) There exists a matrix $N'=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\in\mathcal{M}_{2\times 2}(\mathbb{R}_+)$ such that for each $(\varrho_1,\varrho_2),(\bar{\varrho}_1,\bar{\varrho}_2)\in[0,1]\times\mathbb{R}\times\mathbb{R}$ and for $i\in\{1,2\}$, we have:

$$|L_i(t,\varrho_1,\varrho_2)-L_i(t,\bar{\varrho}_1,\bar{\varrho}_2)|\leq b_{i1}\,|\varrho_1-\bar{\varrho}_1|+b_{i2}\,|\varrho_2-\bar{\varrho}_2|\,,$$

where

$$L_i:=\frac{I_i}{j_i}.$$

($\mathcal{H}_3$) There exists a matrix $M'=\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}\in\mathcal{M}_{2\times 2}(\mathbb{R}_+)$ such that for each $(t,\varrho_1,\varrho_2),(t,\bar{\varrho}_1,\bar{\varrho}_2)\in[0,1]\times\mathbb{R}\times\mathbb{R}$ and for $i\in\{1,2\}$, we have:

$$|k_i(t,\varrho_1,\varrho_2)-k_i(t,\bar{\varrho}_1,\bar{\varrho}_2)|\leq c_{i1}\,|\varrho_1-\bar{\varrho}_1|+c_{i2}\,|\varrho_2-\bar{\varrho}_2|\,.$$

($\mathcal{H}_4$) For $i\in\{1,2\}$, the functions j_i are linear on $\mathcal{X}$, and there exists a matrix $\Upsilon=\begin{pmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{pmatrix}\in\mathcal{M}_{2\times 2}(\mathcal{C}([0,1]),\mathbb{R}_+)$ such that for each $(t,\varrho_1,\varrho_2)\in[0,1]\times\mathbb{R}\times\mathbb{R}$:

$$|j_i(t,\varrho_1,\varrho_2)|\leq d_{i1}(t)|\varrho_1|+d_{i2}(t)\,|\varrho_2|\,,\ i=1,2.$$

Theorem 5.2. *Suppose that the assumptions* ($\mathcal{H}_1$) $-$ ($\mathcal{H}_4$) *are satisfied. Then, the system of integral equation (SIE)* (5.1) *has a solution in* $\mathcal{X}$ *if there is* $r\in\mathbb{R}^2$ *such that*

$$\begin{aligned}r\succcurlyeq&\left[I_{\mathcal{M}_{2\times 2}}-\begin{pmatrix}\|a_{11}\|_{L^1}+b_{11} & \|a_{12}\|_{L^1}+b_{12}\\ \\ \|a_{21}\|_{L^1}+b_{21} & \|a_{22}\|_{L^1}+b_{22}\end{pmatrix}D_V-M'\right]^{-1}\\ &\times\left[D_V\cdot\begin{pmatrix}\|l_1^*\|_\infty+|L_1^0|+\left|\dfrac{\varrho_1^0-k_1^0}{j_1^0}\right|\\ \|l_2^*\|_\infty+|L_2^0|+\left|\dfrac{\varrho_2^0-k_2^0}{j_2^0}\right|\end{pmatrix}+\begin{pmatrix}\|k_1^0\|_\infty\\ \\ \|k_2^0\|_\infty\end{pmatrix}\right],\end{aligned}\tag{5.6}$$

where

$$\|l_i^*\|_\infty=\sup_{t\in[0,1]}|l_i(t,0,0)|,\qquad \|k_i^*\|_\infty=\sup_{t\in[0,1]}|k_i(t,0,0)|,$$

and V *is vector defined by*

$$V=\begin{pmatrix}\|d_{11}\|_\infty & \|d_{12}\|_\infty\\ \|d_{21}\|_\infty & \|d_{22}\|_\infty\end{pmatrix}\begin{pmatrix}r_1\\ r_2\end{pmatrix}$$

and the matrices

$$M_1 = \begin{pmatrix} \|a_{11}\|_{L^1} + b_{11} & \|a_{12}\|_{L^1} + b_{12} \\ \\ \|a_{21}\|_{L^1} + b_{21} & \|a_{22}\|_{L^1} + b_{22} \end{pmatrix} D_V + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

$$M_2 = (I_{\mathcal{M}_{2\times 2}} - M_1)^{-1} \cdot \Upsilon D_{\left(\|R(\overline{B}_r)\|_G + M \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}\right)}$$

converge to zero.

Proof. Let $\mathcal{K}$ be the closed ball $\overline{B}_r$ of $\mathcal{X}$ centered at $0_{\mathcal{X}}$ with radius $r \succcurlyeq 0_{\mathbb{R}^2}$, where r satisfies the inequality (5.6). Our aim is to apply Theorem 4.2 to prove the existence of a fixed point for the operator $S \cdot R + T$ in $\mathcal{X}$. The proof will be broken up into several steps.

Claim 1: Showing that the operators defined in (5.2) are well-defined. Let $\varrho \in \mathcal{X}$ then, from our assumptions the maps $T(\varrho),\ S(\varrho) \in \mathcal{X}$. Now, we claim that the map $R(\varrho) \in \mathcal{X}$ for all $\varrho \in \mathcal{X}$. It suffices to prove that $R(\varrho)$ is continuous on $[0,1]$. To see that let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ converging to a point t in $[0,1]$. Then, for $i = 1, 2$ we have

$$\begin{aligned} |(R_i(\varrho))(t_n) - (R_i(\varrho))(t)| &= \left| \int_0^{t_n} l_i(s, \varrho_1(s), \varrho_2(s)) ds - \int_0^{t} l_i((s, \varrho_1(s), \varrho_2(s)) ds \right| \\ &\leq \int_t^{t_n} |l_i((s, \varrho_1(s), \varrho_2(s))|\, ds \\ &\leq |t_n - t| \int_0^1 |l_i((s, \varrho_1(s), \varrho_2(s))|\, ds \\ &\leq |t_n - t| \|l_i\|_{L^1([0,1])}, \end{aligned}$$

then, the operator $R_i(\rho) \in \mathcal{PC}([0,1])$ for each $i \in \{1,2\}$. Hence R is well defined.

Claim 2: Proving that S is continuous on $\mathcal{X}$ and set contractive with respect to χ_G. In fact, by using $(\mathcal{H}_4)$ and for each $(t, \varrho_1, \varrho_2), (t, \bar{\varrho}_1, \bar{\varrho}_2) \in [0,1] \times \mathbb{R} \times \mathbb{R}$ and for $i \in \{1,2\}$, we have:

$$\begin{aligned} |S_i(\varrho)(t) - S_i(\bar{\varrho})(t)| &= |j_i(t, \varrho_1(t), \varrho_2(t)) - j_i(t, \bar{\varrho}_1(t), \bar{\varrho}_2(t))| \\ &= |j_i(t, \varrho_1(t) - \bar{\varrho}_1(t), \varrho_2(t) - \bar{\varrho}_1(t))| \\ &\leq d_{i1}(t)|\varrho_i(t) - \bar{\varrho}_1(t)| + d_{i2}(t)\, |\varrho_2(t) - \bar{\varrho}_2(t)|, \end{aligned}$$

hence,

$$\|S_i(\varrho) - S_i(\bar{\varrho})\|_{\mathcal{PC}} \leq \|d_{i1}\|_\infty \|\varrho_i - \bar{\varrho}_1\|_{\mathcal{PC}} + \|d_{i2}\|_\infty\, \|\varrho_2 - \bar{\varrho}_2\|_{\mathcal{PC}}$$

thus,

$$\|S(\varrho) - S(\bar{\varrho})\|_G \preccurlyeq \begin{pmatrix} \|d_{11}\|_\infty & \|d_{12}\|_\infty \\ \|d_{21}\|_\infty & \|d_{22}\|_\infty \end{pmatrix} \|\varrho - \bar{\varrho}\|_G,$$

and then, S is G-Lipschitzian, hence S is continuous, and by using Lemma 2.13, S is Υ_*-set contractive with respect to χ_G, where

$$\Upsilon_* = \begin{pmatrix} \|d_{11}\|_\infty & \|d_{12}\|_\infty \\ \|d_{21}\|_\infty & \|d_{22}\|_\infty \end{pmatrix}.$$

Claim 3: Next, let us show that T is M'-Lipschitzian and there exists a matrix M such that R is M-Lipschitzian. From hypothesis $(\mathcal{H}_3)$ it is obvious that T is M'-Lipschitzian. Now, we shall prove our next claim, to do that let $,\bar{\varrho} \in \mathcal{X}$ and for $i = 1, 2$, we have

$$\begin{aligned} |R_i(\varrho)(t) - R_i(\bar{\varrho})(t)| &= |\Gamma_i(t, \varrho_1, \varrho_2) + L_i(\theta, \varrho_1(\theta), \varrho_2(\theta)) - \Gamma_i(t, \bar{\varrho}_1, \bar{\varrho}_2) - L_i(\theta, \bar{\varrho}_1(\theta), \bar{\varrho}_2(\theta))| \\ &\leq \int_0^t |l_i(s, \varrho_1(s), \varrho_2(s)) - l_i(s, \bar{\varrho}_1(s), \bar{\varrho}_2(s))| \, ds \\ &+ |L_i(\theta, \varrho_1(\theta), \varrho_2(\theta)) - L_i(\theta, \bar{\varrho}_1(\theta), \bar{\varrho}_2(\theta))| \\ &\leq \int_0^t a_{i1}(s) |\varrho_1(s) - \bar{\varrho}_1(s)| + a_{i2}(s) |\varrho_2(s) - \bar{\varrho}_2(s)| \, ds \\ &+ b_{i1} |\varrho_1(\theta) - \bar{\varrho}_1(\theta)| + b_{i2} |\varrho_2(\theta) - \bar{\varrho}_2(\theta)| \\ &\leq \|a_{i1}\|_{L^1} \|\varrho_1 - \bar{\varrho}_1\|_{\mathcal{PC}} + \|a_{i2}\|_{L^1} \|\varrho_2 - \bar{\varrho}_2\|_{\mathcal{PC}} + b_{i1} |\varrho_1(\theta) - \bar{\varrho}_1(\theta)| \\ &+ b_{i2} |\varrho_2(\theta) - \bar{\varrho}_2(\theta)| \end{aligned}$$

Taking supremum over t,

$$\begin{aligned} \|R_i(\varrho) - R_i(\bar{\varrho})\|_{\mathcal{PC}} &\leq \|a_{i1}\|_{L^1} \|\varrho_1 - \bar{\varrho}_1\|_{\mathcal{PC}} + \|a_{i2}\|_{L^1} \|\varrho_2 - \bar{\varrho}_2\|_{\mathcal{PC}} + b_{i1} \|\varrho_1 - \bar{\varrho}_1\|_{\mathcal{PC}} \\ &+ b_{i2} \|\varrho_2 - \bar{\varrho}_2\|_{\mathcal{PC}} \\ &\leq (\|a_{i1}\|_{L^1} + b_{i1}) \|\varrho_1 - \bar{\varrho}_1\|_{\mathcal{PC}} + (\|a_{i2}\|_{L^1} + b_{i2}) \|\varrho_2 - \bar{\varrho}_2\|_{\mathcal{PC}} \end{aligned},$$

thus

$$\|R(\varrho) - R(\bar{\varrho})\|_G \preccurlyeq M \|\varrho - \bar{\varrho}\|_G,$$

where

$$M = \begin{pmatrix} \|a_{11}\|_{L^1} + b_{11} & \|a_{12}\|_{L^1} + b_{12} \\ \\ \|a_{21}\|_{L^1} + b_{21} & \|a_{22}\|_{L^1} + b_{22} \end{pmatrix},$$

hence the second claim is verified.

Claim 4: Proving that $(\varrho = S(v)R(\varrho) + T(\varrho),\ v \in \overline{B}_r)$ implies $\varrho \in \overline{B}_r$

$$\begin{aligned} \|\varrho\|_G &= \|S(v)R(\varrho) + T(\varrho)\|_G \\ &= \begin{pmatrix} \|S_1(v)R_1(\varrho) + T_1(\varrho)\|_{\mathcal{PC}} \\ \\ \|S_2(v)R_2(\varrho) + T_2(\varrho)\|_{\mathcal{PC}} \end{pmatrix}, \end{aligned}$$

and, we have for $i = 1, 2$

$$\begin{aligned}
|\varrho_i(t)| &= |j_i(t, v_1(t), v_2(t)) \left(\Gamma_i(t, \varrho_1, \varrho_2) + L_i\left(\theta, \left(\varrho_1(\theta), \varrho_2(\theta)\right)\right)\right) + k_i(t, \varrho_1(t), \varrho_2(t))| \\
&\leq |j_i(t, v_1(t), v_2(t))| \left(\left| \int_0^t l_i(s, \varrho_1(s), \varrho_2(s)) ds \right| + |L_i\left(\theta, \left(\varrho_1(\theta), \varrho_2(\theta)\right)\right)| + \left| \frac{\varrho_i^0 - k_i^0}{j_i^0} \right| \right) \\
&\quad + |k_i(t, \varrho_1(t), \varrho_2(t))| \\
&\leq |j_i(t, v_1(t), v_2(t))| \Bigg(\int_0^t |l_i(s, \varrho_1(s), \varrho_2(s)) - l_i(s, 0, 0)| + |l_i(s, 0, 0)| ds + \left| \frac{\varrho_i^0 - k_i^0}{j_i^0} \right| \\
&\quad + |L_i\left(\theta, \varrho_1(\theta), \varrho_2(\theta)\right) - L_i\left(\theta, 0, 0\right)| + |L_i\left(\theta, 0, 0\right)| \Bigg) + |k_i(t, \varrho_1(t), \varrho_2(t)) - k_i(t, 0, 0)| \\
&\quad + |k_i(t, 0, 0)| \\
&\leq \Bigg(d_{i1}(t) |v_1(t)| + d_{i2}(t) |v_2(t)| \Bigg) \Bigg(\|a_{i1}\|_{L^1} \|\varrho_1\|_{\mathcal{PC}} + \|a_{i2}\|_{L^1} \|\varrho_2\|_{\mathcal{PC}} + \|l_i^*\|_\infty \\
&\quad + b_{i1} |\varrho_1(\theta)| + b_{i2} |\varrho_2(\theta)| + |L_i^\theta| + \left| \frac{\varrho_i^0 - k_i^0}{j_i^0} \right| \Bigg) + c_{i1} |\varrho_1(t)| + c_{i2} |\varrho_2(t)| + \|k_i^*\|_\infty \\
&\leq \Bigg(\|d_{i1}\|_\infty r_1 + \|d_{i2}\|_\infty r_2 \Bigg) \Bigg(\|a_{i1}\|_{L^1} \|\varrho_1\|_{\mathcal{PC}} + \|a_{i2}\|_{L^1} \|\varrho_2\|_{\mathcal{PC}} + \|l_i^*\|_\infty + b_{i1} |\varrho_1(\theta)| \\
&\quad + b_{i2} |\varrho_2(\theta)| + |L_i^\theta| + \left| \frac{\varrho_i^0 - k_i^0}{j_i^0} \right| \Bigg) + c_{i1} |\varrho_1(t)| + c_{i2} |\varrho_2(t)| + \|k_i^*\|_\infty.
\end{aligned}$$

So, by taking supremum over t, it follows that

$$\|\varrho\|_G \preccurlyeq \left[MD_V + M'\right] \begin{pmatrix} \|\varrho_1\|_{\mathcal{PC}} \\ \|\varrho_2\|_{\mathcal{PC}} \end{pmatrix} + D_V \cdot \begin{pmatrix} \|l_1^*\|_\infty + |L_1^\theta| + \left| \frac{\varrho_1^0 - k_1^0}{j_1^0} \right| \\ \|l_2^*\|_\infty + |L_2^\theta| + \left| \frac{\varrho_2^0 - k_2^0}{j_2^0} \right| \end{pmatrix} + \begin{pmatrix} \|k_1^*\|_\infty \\ \|k_2^*\|_\infty \end{pmatrix},$$

since $M_1 = MD_V + M'$ converges to zero and by using Lemma 2.9, we get

$$\begin{aligned}
\|\varrho\|_G &\preccurlyeq \left[I_{\mathcal{M}_{2\times 2}} - M_1\right]^{-1} \left[D_V \cdot \begin{pmatrix} \|l_1^*\|_\infty + |L_1^\theta| + \left| \frac{\varrho_1^0 - k_1^0}{j_1^0} \right| \\ \|l_2^*\|_\infty + |L_2^\theta| + \left| \frac{\varrho_2^0 - k_2^0}{j_2^0} \right| \end{pmatrix} + \begin{pmatrix} \|k_1^*\|_\infty \\ \|k_2^*\|_\infty \end{pmatrix} \right] \\
&\preccurlyeq \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = r.
\end{aligned}$$

To end the proof, we apply Theorem 4.2 and we get that (SIE) (5.2) has a solution in $\mathcal{K} \subset \mathcal{X}$. $\square$

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3**(1922), no. 1, 133–181.

[2] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, vol. 60 of Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 1980.

[3] M. Benchohra, J. Henderson, S. Ntouy, *Impulsive Differential Equations and Inclusions*, vol. 2, Hindawi Publishing Corporation New York, 2006.

[4] T. Burton, *A fixed-point theorem of Krasnoselskii*, Applied Math. Letters, **11**(1998), no. 1, 85–88.
[5] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, Rendiconti del Seminario Matematico della Università di Padova, **24**(1955), 84–92.
[6] B. C. Dhage, *On some variants of Schauder's fixed point principle and applications to nonlinear integral equations*, J. Math. Phys. Sci, **25**(1988), 603–611.
[7] B. C. Dhage, *Remarks on two fixed-point theorems involving the sum and the product of two operators*, Computers & Mathematics with Applications, **46**(12)(2003), 1779–1785.
[8] B. C. Dhage, *On a fixed point theorem in Banach algebras with applications*, Applied Mathematics Letters, **18**(2005), no. 3, 273–280.
[9] J. R. Graef, J. Henderson, A. Ouahab, *Topological Methods for Differential Equations and Inclusions*, CRC Press, 2018.
[10] R. Horn, C. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
[11] M. A. Krasnosel'skii, *Some problems of nonlinear analysis*, Uspekhi Matematicheskikh Nauk, **9**(1954), no. 3, 57–114.
[12] M. A. Krasnosel'skiı, *Two remarks on the method of successive approximations*, Uspekhi Matematicheskikh Nauk, **10**(1955), no. 1, 63.
[13] B. Krichen, B. Mefteh, D. O'Regan, *Fixed point theorems involving* 1*-set-weakly contractive operators in WC-Banach algebras*, Fixed Point Theory, **20**(2019), no. 2, 541–557.
[14] E. Kruger-Thimer, *Formal theory of drug dosage regiments*, J. Theoret. Biol., **13**(1966), 212-235.
[15] V. Lakshmikantham, P. S. Simeonov, et al., *Theory of Impulsive Differential Equations*, vol. 6, World Scientific, 1989.
[16] J. Leray, J. Schauder, *Topologie et Équations Fonctionnelles*, Annales Scientifiques de l'École Normale Supérieure, 1934.
[17] J. J. Nieto, A. Ouahab, R. Rodriguez-Lopez, *Fixed point theorems in generalized Banach algebras and applications*, Fixed Point Theory, **19**(2018), no. 2, 707–732.
[18] A. Ouahab, *Some Pervo's and Krasnoselskii type fixed point results and application*, Comm. Appl. Nonlinear Anal., **19**(2015), 623–642.
[19] A. Perov, *On the Cauchy problem for a system of ordinary differential equations*, (in Russian), Metod Res. Dif. Urav., Kiev, 1964.
[20] I.-R. Petre, A. Petruşel, *Krasnoselskii's theorem in generalized Banach spaces and application*, Electronic Journal of Qualitative Theory of Differential Equations, **85**(2012), 1–20.
[21] A. M. Samoilenko, N. Perestyuk, *Impulsive Differential Equations*, World Scientic, Singapore, 1995.
[22] R. S. Varga, *Matrix Iterative Analysis,* Springer Series in Computational Mathematics, 27, Springer-Verlag, New York, Berlin, Heidelberg, 2000.
[23] A. Viorel, *Contributions to the study of nonlinear evolution equations*, Ph.D. Thesis, 2011.

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