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NEW DIRECTIONS IN FIXED POINT THEORY IN G-METRIC SPACES AND APPLICATIONS TO MAPPINGS CONTRACTING PERIMETERS OF TRIANGLES

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Abstract. We are concerned with the study of fixed points for mappings $T : X \to X$, where (X, G) is a G-metric space in the sense of Mustafa and Sims. After the publication of the paper [Journal of Nonlinear and Convex Analysis, 7(2) (2006) 289–297] by Mustafa and Sims, a great interest was devoted to the study of fixed points in G-metric spaces. In 2012, the first and third authors observed that several fixed point theorems established in G-metric spaces are immediate consequences of known fixed point theorems in standard metric spaces. This observation demotivated the investigation of fixed points in G-metric spaces. In this paper, we open new directions in fixed point theory in G-metric spaces. Namely, we establish new versions of the Banach, Kannan and Reich fixed point theorems in G-metric spaces. We point out that the approach used by the first and third authors [Fixed Point Theory Appl. 2012 (2012) 1–7] is inapplicable in the present study. We also provide some interesting applications related to mappings contracting perimeters of triangles.
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1. Introduction

The notion of G-metric spaces was introduced by Mustafa and Sims in [8]. After the publication of this work, a great attention was accorded to the study of fixed points in such spaces, see e.g. [1, 2, 5, 7, 9, 15, 17, 19] and the references therein. In [4] (see also [18]), Jleli and Samet remarked that several fixed point theorems in G-metric spaces can be deduced immediately from fixed point theorems in (standard) metric spaces. Namely, it was shown that, if $G : X \times X \times X \to \mathbb{R}^+$ is a G-metric on X, then the mapping $\delta : X \times X \to \mathbb{R}^+$ defined by $\delta(x, y) = \max\{G(x, x, y), G(y, x, x)\}$ is a metric on X. Moreover, if (X, G) is a complete G-metric space, then (X, δ) is a complete metric space. From this observation, it was proved that several contractions studied in complete G-metric spaces can be reduced to standard contractions in the complete metric space (X, δ). For instance, if (X, G) is a complete G-metric space and $T : X \to X$ is a mapping satisfying the inequality $G(Tx, Ty, Tz) \leq \lambda G(x, y, z)$ for all $x, y, z \in X$, where $\lambda \in (0, 1)$ is a constant, then (see [7]) T admits a unique fixed point. Taking $z = y$, the above inequality reduces to $G(Tx, Ty, Ty) \leq \lambda G(x, y, y)$ for all $x, y \in X$. Replacing (x, y, z) by (y, x, x), we obtain $G(Ty, Tx, Tx) \leq \lambda G(y, x, x)$ for all $x, y \in X$. Consequently, we get $\delta(Tx, Ty) \leq \lambda\delta(x, y)$ for all $x, y \in X$, and the Banach fixed point theorem (in metric spaces) applies. After the publications of the papers [4, 18], the attention paid to the study of fixed point theorems in G-metric spaces was attenuated.

In this paper, we open new directions in fixed point theory on G-metric spaces. Namely, we establish new extensions of Banach, Kannan and Reich fixed point theorems in G-metric spaces. We point out that the used approach in [4, 18] cannot be used to establish our obtained results. Moreover, motivated by the recent paper [11], we provide (among other applications) an application to the study of fixed points contracting perimeters of triangles.

The rest of the paper is organized as follow. In Section 2, we briefly recall some basic notions and properties related to G-metric spaces. Our main results are stated and proved in Section 3. Finally, some applications are provided in Section 4.

Throughout this paper, the following notations are used: $\mathbb{R}^+ = [0, \infty)$, X denotes a nonempty set, $|X|$ denotes the cardinal of X, and for a mapping $T : X \to X$, the set of fixed points of T is denoted by $\text{Fix}(T)$.

2. Preliminaries

Let us recall briefly some basic notions related to G-metric spaces. For more details, we refer to Mustafa and Sims [8]. Throughout this paper, X denotes a nonempty set and $\mathbb{R}^+ = [0, \infty)$.

Let $G : X \times X \times X \to \mathbb{R}^+$ be a given mapping. We say that G is a G-metric on X, if for all $x, y, z, w \in X$, we have

- (P_1) $G(x, y, z) = 0$ if and only if $x = y = z$;
- (P_2) If $x \neq y$, then $G(x, x, y) > 0$;
- (P_3) $G(x, y, z) = G(\sigma(x, y, z))$ for every permutation $\sigma : \{x, y, z\} \to \{x, y, z\}$;
- (P_4) If $y \neq z$, then $G(x, x, y) \leq G(x, y, z)$;
- (P_5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$.

If the above conditions are satisfied, then (X, G) is called a G-metric space.

Prototype examples of G-metrics are

$$G(x, y, z) = d(x, y) + d(y, z) + d(x, z), \quad x, y, z \in X \tag{2.1}$$

and

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}, \quad x, y, z \in X, \tag{2.2}$$

where d is a (standard) metric on X.

Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is G-convergent to x, if $\lim_{n,m\to\infty} G(x_n, x_m, x) = 0$. The following assertions are equivalent:

(A_1) $\{x_n\}$ is G-convergent to x;
(A_2) $\lim_{n\to\infty} G(x_n, x_n, x) = 0$;
(A_3) $\lim_{n\to\infty} G(x_n, x, x) = 0$.

Notice that, if $\{x_n\}$ is G-convergent to x and $\{x_n\}$ is G-convergent to y, then $x = y$. This can be easily seen by (P_5), (A_2) and (A_3).

We say that $\{x_n\}$ is G-Cauchy, if $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$.

If every G-Cauchy sequence in X is G-convergent to an element of X, then (X, G) is called a complete G-metric space.

The following two lemmas will be useful later.

Lemma 2.1. *Let d be a metric on X and $G : X \times X \times X \to \mathbb{R}^+$ be the G-metric on X defined by* (2.1). *The following assertions are equivalent:*

(i) (X, d) *is a complete metric space;*
(ii) (X, G) *is a complete G-metric space.*

Proof. Assume first that (X, d) is a complete metric space. Let $\{x_n\} \subset X$ be a G-Cauchy sequence. For all n, m, we have

$$\begin{aligned} d(x_n, x_m) &\le 2d(x_n, x_m) \\ &= d(x_n, x_m) + d(x_m, x_m) + d(x_n, x_m) \\ &= G(x_n, x_m, x_m). \end{aligned}$$

Since $\{x_n\}$ is a G-Cauchy sequence, we have $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$, which implies by the above inequality that $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, that is, $\{x_n\}$ is a Cauchy sequence in the metric space (X, d). From the completeness of (X, d), we deduce the existence of $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0. \tag{2.3}$$

On the other hand, we have

$$\begin{aligned} G(x_n, x, x) &= d(x_n, x) + d(x, x) + d(x_n, x) \\ &= 2d(x_n, x), \end{aligned}$$

which implies by (2.3) that $\lim_{n\to\infty} G(x_n, x, x) = 0$, that is, $\{x_n\}$ is G-convergent to x. This shows that (X, G) is a complete G-metric space.

Assume now that (X, G) is a complete G-metric space. Let $\{x_n\} \subset X$ be a Cauchy sequence in the metric space (X, d). From the identity

$$G(x_n, x_m, x_m) = \frac{1}{2} d(x_n, x_m),$$

we deduce that $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$, that is, $\{x_n\}$ is G-Cauchy. Consequently, there exists $x \in X$ such that

$$\lim_{n\to\infty} G(x_n, x, x) = 0.$$

Then, from the identity

$$d(x_n, x) = \frac{1}{2}G(x_n, x, x),$$

we deduce that $\lim_{n\to\infty} d(x_n, x) = 0$, which shows that (X, d) is a complete metric space.

Lemma 2.2 (see [4]). *Let d be a metric on X and $G : X \times X \times X \to \mathbb{R}^+$ be the G-metric on X defined by (2.2). The following assertions are equivalent:*

(i) *(X, d) is a complete metric space;*
(ii) *(X, G) is a complete G-metric space.*

3. Main results

3.1. **Banach's fixed point theorem in G-metric spaces.** Our first main result is an extension of the Banach fixed point theorem to G-metric spaces.

Theorem 3.1. *Let (X, G) be a complete G-metric space with $|X| \geq 3$. Let $T : X \to X$ be a mapping satisfying the following conditions:*

(I) *For all $x \in X$, $T(Tx) \neq x$, provided $Tx \neq x$;*
(II) *There exists $\lambda \in (0, 1)$ such that for all pairwise distinct points $x, y, z \in X$, we have*

$$G(Tx, Ty, Tz) \leq \lambda G(x, y, z).$$

Then $\operatorname{Fix}(T) \neq \emptyset$ *and* $|\operatorname{Fix}(T)| \leq 2$.

Proof. Let us first prove that $\operatorname{Fix}(T) \neq \emptyset$. Let $u_0 \in X$ be fixed and consider the Picard sequence $\{u_n\} \subset X$ defined by

$$u_{n+1} = Tu_n, \quad n \geq 0.$$

If $u_n = u_{n+1}$ for some n, then $u_n \in \operatorname{Fix}(T)$ and the result is proved. So, we may suppose that

$$u_n \neq u_{n+1}, \quad n \geq 0,$$

which implies by (I) that $u_n \neq T(Tu_n)$ $(= u_{n+2})$. Consequently, u_n, u_{n+1} and u_{n+2} are pairwise distinct points for every $n \geq 0$. Then, making use of (II), we obtain

$$\begin{aligned}
&G(u_1, u_2, u_3) = G(Tu_0, Tu_1, Tu_2) \leq \lambda G(u_0, u_1, u_2),\\
&G(u_2, u_3, u_4) = G(Tu_1, Tu_2, Tu_3) \leq \lambda G(u_1, u_2, u_3) \leq \lambda^2 G(u_0, u_1, u_2),\\
&\vdots\\
&G(u_n, u_{n+1}, u_{n+2}) = G(Tu_{n-1}, Tu_n, Tu_{n+1}) \leq \lambda G(u_{n-1}, u_n, u_{n+1})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \leq \lambda^n G(u_0, u_1, u_2),
\end{aligned}$$

that is,

$$G(u_n, u_{n+1}, u_{n+2}) \leq \tau_0 \lambda^n, \quad n \geq 0, \tag{3.1}$$

where $\tau_0 = G(u_0, u_1, u_2) > 0$ (by (P_1)). We now show that $\{u_n\}$ is G-Cauchy. First, by (P_3), (P_4) and using that u_n, u_{n+1} and u_{n+2} are pairwise distinct points, we obtain

$$\begin{aligned} G(u_n, u_{n+1}, u_{n+1}) &= G(u_{n+1}, u_{n+1}, u_n) \\ &\leq G(u_{n+1}, u_n, u_{n+2}) \\ &= G(u_n, u_{n+1}, u_{n+2}), \end{aligned}$$

which implies by (3.1) that

$$G(u_n, u_{n+1}, u_{n+1}) \leq \tau_0 \lambda^n, \quad n \geq 0. \tag{3.2}$$

We now use (3.2) and (P_5) to obtain that for all $n < m$,

$$\begin{aligned} G(u_n, u_m, u_m) &\leq G(u_n, u_{n+1}, u_{n+1}) + G(u_{n+1}, u_m, u_m) \\ &\leq G(u_n, u_{n+1}, u_{n+1}) + G(u_{n+1}, u_{n+2}, u_{n+2}) + G(u_{n+2}, u_m, u_m) \\ &\vdots \\ &\leq G(u_n, u_{n+1}, u_{n+1}) + G(u_{n+1}, u_{n+2}, u_{n+2}) + \cdots + G(u_{m-1}, u_m, u_m) \\ &\leq \tau_0 \left(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right) \\ &= \frac{\tau_0}{1-\lambda} \lambda^n \left(1 - \lambda^{m-n}\right) \\ &\leq \frac{\tau_0}{1-\lambda} \lambda^n, \end{aligned}$$

which shows that (since $0 < \lambda < 1$)

$$\lim_{n,m \to \infty} G(u_n, u_m, u_m) = 0,$$

that is, $\{u_n\}$ is a G-Cauchy sequence. Now, the completeness of (X, G) yields the existence of an element $u^* \in X$ such that $\{u_n\}$ is G-convergent to u^*. On the other hand, if there exists k such that for all $n \geq k$, we have $u_n = u^*$, this contradicts the fact that u_n, u_{n+1} and u_{n+2} are pairwise distinct points for all n. Consequently, we can extract a subsequence $\{u_{n(k)}\}_k$ of $\{u_n\}$ such that $u_{n(k)} \neq u^*$ and $u_{n(k)+1} \neq u^*$ for all k. In order to simplify writing, we always denote the sequence $\{u_{n(k)}\}_k$ by $\{u_n\}$ with $u_n \neq u^*$ for all n. We now show that $u^* \in \mathrm{Fix}(T)$. Indeed, by (P_3) and (P_5), we have

$$\begin{aligned} G(u^*, u^*, Tu^*) &\leq G(u^*, u_n, u_n) + G(u_n, u^*, Tu^*) \\ &= G(u^*, u_n, u_n) + G(u^*, u_n, Tu^*) \\ &\leq G(u^*, u_n, u_n) + G(u^*, u_{n+1}, u_{n+1}) + G(u_{n+1}, u_n, Tu^*) \\ &= G(u^*, u_n, u_n) + G(u^*, u_{n+1}, u_{n+1}) + G(Tu_n, Tu_{n-1}, Tu^*), \end{aligned}$$

which implies by (II) and the fact that u_n, u_{n-1} and u^* are pairwise distinct points that

$$G(u^*, u^*, Tu^*) \leq G(u^*, u_n, u_n) + G(u^*, u_{n+1}, u_{n+1}) + \lambda G(u_n, u_{n-1}, u^*). \tag{3.3}$$

On the other hand, by the definition of the G-convergence and (A_2), we have

$$\lim_{n \to \infty} \left[G(u^*, u_n, u_n) + G(u^*, u_{n+1}, u_{n+1}) + \lambda G(u_n, u_{n-1}, u^*)\right] = 0,$$

which implies by (3.3) together with (P_1) that $G(u^*, u^*, Tu^*) = 0$ and $u^* = Tu^*$. This proves that $\mathrm{Fix}(T) \neq \emptyset$.

Suppose now that v_i, $i = 1, 2, 3$, are pairwise distinct fixed points of T. Then, making use of (P_1) and (II), we get

$$0 < G(v_1, v_2, v_3) = G(Tv_1, Tv_2, Tv_3) \leq \lambda G(v_1, v_2, v_3)$$

and we reach a contradiction with $0 < \lambda < 1$. This shows that $|\,\mathrm{Fix}(T)| \leq 2$. The proof of Theorem 3.1 is then completed.

Remark 3.2. We point out that the condition $x, y, z \in X$ are pairwise distinct in assumption (II) of Theorem 3.1 is essential. Otherwise, as we mentioned in Section 1, using the approach in [4], (II) reduces to

$$\delta(Tx, Ty) \leq \lambda \delta(x, y), \quad x, y \in X,$$

where $\delta : X \times X \to \mathbb{R}^+$ is the metric on X defined by

$$\delta(x, y) = \max\{G(x, y, y), G(y, x, x)\}, \quad x, y \in X, \tag{3.4}$$

and the existence of a (unique) fixed point of T will be an immediate consequence of the Banach fixed point theorem in metric spaces.

We provide below an example to illustrate our obtained result.

Example 3.3. Let $X = \{A, B, C\} \subset \mathbb{R}^2$, where

$$A = \left(\frac{7}{8}, \frac{\sqrt{15}}{8}\right), \quad B = (1, 0), \quad C = (0, 0).$$

We introduce the mappings $T_1, T_2 : X \to X$ defined by

$$T_1A = A, \quad T_1B = B, \quad T_1C = A$$

and

$$T_2A = B, \quad T_2B = A, \quad T_2C = A.$$

Consider now the G-metric on X given by

$$G(u, v, w) = \max\{\|u - v\|, \|v - w\|, \|u - w\|\}, \quad u, v, w \in X,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$.

Clearly, the mapping T_1 satisfies condition (I) of Theorem 3.1. Furthermore, we have

$$\begin{aligned} G(T_1A, T_1B, T_1C) &= G(A, B, A) \\ &= \|A - B\| \\ &= \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} G(A, B, C) &= \max\{\|A - B\|, \|B - C\|, \|A - C\|\} \\ &= \max\left\{\frac{1}{2}, 1, 1\right\} \\ &= 1, \end{aligned}$$

which shows that condition (II) of Theorem 3.1 holds for all $\lambda \in \left[\frac{1}{2}, 1\right)$. On the other hand, $\mathrm{Fix}(T_1) = \{A, B\}$, which confirms the obtained result given by Theorem 3.1.

Remark also that the mapping T_2 satisfies condition (II) of Theorem 3.1. This can be easily seen observing that $G(T_2A, T_2B, T_2C) = \|A - B\| = G(T_1A, T_1B, T_1C)$. On the other hand, we have $T_2A \neq A$ and $T_2(T_2A) = T_2(B) = A$, which shows that condition (I) of Theorem 3.1 is not satisfied. Furthermore, we have $\mathrm{Fix}(T) = \emptyset$, which shows that in the absence of condition (I), the result of Theorem 3.1 is not true.

3.2. **Kannan's fixed point theorem in G-metric spaces.** The Kannan fixed point theorem [6] can be stated as follows: Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the inequality

$$d(Tx, Ty) \leq \lambda \left[d(x, Tx) + d(y, Ty)\right]$$

for every $x, y \in X$, where $\lambda \in \left(0, \frac{1}{2}\right)$ is a constant. Then T admits a unique fixed point.

Our second main result is an extension of the above result from metric spaces to G-metric spaces.

Theorem 3.4. *Let (X, G) be a complete G-metric space with $|X| \geq 3$. Let $T : X \to X$ be a mapping satisfying the following conditions:*

(I) *For all $x \in X$, $T(Tx) \neq x$, provided $Tx \neq x$;*

(II) *There exists $\lambda \in \left(0, \frac{1}{3}\right)$ such that for all pairwise distinct points $x, y, z \in X$, we have*

$$G(Tx, Ty, Tz) \leq \lambda \left[G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)\right].$$

Then $\mathrm{Fix}(T) \neq \emptyset$ *and* $|\,\mathrm{Fix}(T)| \leq 2$.

Proof. We first show that T admits at least one fixed point. Let $u_0 \in X$ be fixed and consider the Picard sequence $\{u_n\} \subset X$ defined by

$$u_{n+1} = Tu_n, \quad n \geq 0.$$

As in the proof of Theorem 3.1, we may suppose that $u_n \neq u_{n+1}$ for all $n \geq 0$, which implies by (I) that u_n, u_{n+1} and u_{n+2} are pairwise distinct points for every $n \geq 0$. Then, making use of (II), we obtain

$$\begin{aligned} G(u_1, u_2, u_3) &= G(Tu_0, Tu_1, Tu_2) \\ &\leq \lambda \left[G(u_0, u_1, u_1) + G(u_1, u_2, u_2) + G(u_2, u_3, u_3)\right], \end{aligned}$$

which yields

$$(1 - \lambda) G(u_1, u_2, u_3) \leq \lambda \left[G(u_0, u_1, u_1) + G(u_2, u_3, u_3)\right]. \tag{3.5}$$

We now use (P_3), (P_4), (P_5) and the fact that u_n, u_{n+1} and u_{n+2} are pairwise distinct points to get

$$G(u_0, u_1, u_1) = G(u_1, u_1, u_0) \leq G(u_1, u_0, u_2) = G(u_0, u_1, u_2) \tag{3.6}$$

and

$$G(u_2, u_3, u_3) = G(u_3, u_3, u_2) \leq G(u_3, u_1, u_2) = G(u_1, u_2, u_3). \tag{3.7}$$

Then, it follows from (3.5), (3.6) and (3.7) that

$$(1-2\lambda)G(u_1,u_2,u_3) \leq \lambda G(u_0,u_1,u_2),$$

which implies (since $1-2\lambda > 0$) that

$$G(u_1,u_2,u_3) \leq kG(u_0,u_1,u_2),$$

where $k = \frac{\lambda}{1-2\lambda}$. Notice that by $0 < \lambda < \frac{1}{3}$, we have $0 < k < 1$. Next, by induction, we obtain easily that

$$G(u_n,u_{n+1},u_{n+2}) \leq k^n \tau_0, \quad n \geq 0,$$

where $\tau_0 = G(u_0,u_1,u_2) > 0$. Proceeding as in the proof of Theorem 3.1, we obtain that $\{u_n\}$ is a G-Cauchy sequence. Due to the completeness of (X,G), there exists $u^* \in X$ such that $\{u_n\}$ is G-convergent to u^*. By the proof of Theorem 3.1, without restriction of the generality, we may suppose that $u_n \neq u^*$ for all n. We now show that $u^* \in \mathrm{Fix}(T)$. By (P_3) and (P_4), we have (see the proof of Theorem 3.1)

$$G(u^*,u^*,Tu^*) \leq G(u^*,u_n,u_n) + G(u^*,u_{n+1},u_{n+1}) + G(Tu_n,Tu_{n-1},Tu^*),$$

which implies by (II) and the fact that u_n, u_{n-1} and u^* are pairwise distinct points that

$$\begin{aligned} G(u^*,u^*,Tu^*) &\leq G(u^*,u_n,u_n) + G(u^*,u_{n+1},u_{n+1}) \\ &\quad + \lambda\left[G(u_n,u_{n+1},u_{n+1}) + G(u_{n-1},u_n,u_n) + G(u^*,Tu^*,Tu^*)\right] \\ &\leq G(u^*,u_n,u_n) + G(u^*,u_{n+1},u_{n+1}) \\ &\quad + \lambda\left[G(u_n,u_{n+1},u_{n+1}) + G(u_{n-1},u_n,u_n) + 2G(u^*,u^*,Tu^*)\right], \end{aligned}$$

that is,

$$\begin{aligned} (1-2\lambda)G(u^*,u^*,Tu^*) &\leq G(u^*,u_n,u_n) + G(u^*,u_{n+1},u_{n+1}) \\ &\quad + \lambda\left[G(u_n,u_{n+1},u_{n+1}) + G(u_{n-1},u_n,u_n)\right]. \end{aligned}$$

Since $1-2\lambda > 0$, passing to the limit as $n \to \infty$ in the above inequality, $G(u^*,u^*,Tu^*) = 0$, which yields $u^* = Tu^*$. This shows that $\mathrm{Fix}(T) \neq \emptyset$.

Suppose now that v_i, $i = 1,2,3$, are pairwise distinct fixed points of T. Then, making use of (II), we get

$$\begin{aligned} G(v_1,v_2,v_3) &= G(Tv_1,Tv_2,Tv_3) \\ &\leq \lambda\left[G(v_1,Tv_1,Tv_1) + G(v_2,Tv_2,Tv_2) + G(v_3,Tv_3,Tv_3)\right] \\ &= \lambda\left[G(v_1,v_1,v_1) + G(v_2,v_2,v_2) + G(v_3,v_3,v_3)\right]. \end{aligned} \tag{3.8}$$

On the other hand, we have by (P_4) that

$$\begin{aligned} G(v_1,v_1,v_1) &\leq G(v_1,v_1,v_2) \leq G(v_1,v_2,v_3) \\ G(v_2,v_2,v_2) &\leq G(v_2,v_2,v_3) \leq G(v_2,v_3,v_1) = G(v_1,v_2,v_3) \\ G(v_3,v_3,v_3) &\leq G(v_3,v_3,v_2) \leq G(v_3,v_2,v_1) = G(v_1,v_2,v_3). \end{aligned}$$

Then, by (3.8), we obtain

$$(1-3\lambda)G(v_1,v_2,v_3) \leq 0,$$

and we reach a contradiction with $0 < \lambda < \frac{1}{3}$ and $G(v_1, v_2, v_3) > 0$. This shows that $|\operatorname{Fix}(T)| \leq 2$ and the proof of Theorem 3.4 is completed.

Remark 3.5. Notice that, if conditions (II) of Theorem 3.4 holds for every $x, y, z \in X$, then following the approach in [4], the existence of a (unique) fixed point of T can be deduced immediately from Ćirić's fixed point theorem in metric spaces [3]. Namely, in this case, taking $z = y$, (II) reduces to

$$G(Tx, Ty, Ty) \leq \lambda \left[G(x, Tx, Tx) + 2G(y, Ty, Ty)\right], \quad x, y \in X.$$

Replacing y by x and z by y, (II) reduces to

$$G(Tx, Tx, Ty) \leq \lambda \left[2G(x, Tx, Tx) + G(y, Ty, Ty)\right], \quad x, y \in X.$$

The above two inequalities yield

$$\delta(Tx, Ty) \leq k \max\left\{\delta(x, Tx), \delta(y, Ty)\right\},$$

where $k = 3\lambda$ and δ is the metric on X defined by (3.4). Since $k = 3\lambda \in (0, 1)$, then by Ćirić's fixed point theorem, T admits a (unique) fixed point.

Example 3.6. Let $X = \{a, b, c\} \subset \mathbb{R}$, where

$$a = 0, \quad b = \frac{1}{5}, \quad c = 1.$$

We introduce the mappings $T_1, T_2 : X \to X$ defined by

$$T_1 a = a, \quad T_1 b = b, \quad T_1 c = a$$

and

$$T_2 a = b, \quad T_2 b = a, \quad T_2 c = a.$$

Consider now the G-metric on X given by

$$G(u, v, w) = \max\left\{|u - v|, |v - w|, |u - w|\right\}, \quad u, v, w \in X.$$

Clearly, the mapping T_1 satisfies condition (I) of Theorem 3.4. Furthermore, we have

$$G(T_1 a, T_1 b, T_1 c) = G(a, b, a) = |a - b| = \frac{1}{5}$$

and

$$\begin{aligned} G(a, T_1 a, T_1 a) + G(b, T_1 b, T_1 b) + G(c, T_1 c, T_1 c) &= G(a, a, a) + G(b, b, b) + G(c, a, a) \\ &= G(c, a, a) \\ &= |a - c| = 1, \end{aligned}$$

which shows that condition (II) of Theorem 3.4 holds for all $\lambda \in \left(0, \frac{1}{3}\right)$. On the other hand, $\operatorname{Fix}(T_1) = \{a, b\}$, which confirms the obtained result given by Theorem 3.4.

Remark also that the mapping T_2 satisfies condition (II) of Theorem 3.4. Namely, we have

$$G(T_2 a, T_2 b, T_2 c) = G(b, a, a) = |b - a| = \frac{1}{5}$$

and

$$\begin{aligned} G(a,T_2a,T_2a)+G(b,T_2b,T_2b)+G(c,T_2c,T_2c) &= G(a,b,b)+G(b,a,a)+G(c,a,a)\\ &= 2|a-b|+|c-a|\\ &= \frac{2}{5}+1=\frac{7}{5}, \end{aligned}$$

which shows that condition (II) of Theorem 3.4 holds for every $\lambda \in \left[\frac{1}{7},\frac{1}{3}\right)$. On the other hand, we have $T_2a \neq a$ and $T_2(T_2a)=T_2b=a$, which shows that condition (I) of Theorem 3.4 is not satisfied. Furthermore, we have $\mathrm{Fix}(T)=\emptyset$, which shows that in the absence of condition (I), the result of Theorem 3.4 is not true.

3.3. **The Reich fixed point theorem in G-metric spaces.** The following fixed point result was obtained by Reich [16]: Let (X,d) be a complete metric space and $T : X \to X$ be a mapping satisfying the inequality

$$d(Tx,Ty) \leq a_1 d(x,Tx)+a_2 d(y,Ty)+a_3 d(x,y)$$

for every $x,y \in X$, where a_i $(i=1,2,3)$ are nonnegative constants with

$$0<\sum_{i=1}^{3} a_i<1.$$

Then T admits a unique fixed point.

We now extend the Reich fixed point theorem to G-metric spaces.

Theorem 3.7. *Let (X,G) be a complete G-metric space with $|X| \geq 3$. Let $T : X \to X$ be a mapping satisfying the following conditions:*

(I) *For all $x \in X$, $T(Tx) \neq x$, provided $Tx \neq x$;*

(II) *There exist nonnegative constants a_i $(i=1,2,3,4)$ with $0<\sum_{i=1}^{4} a_i<1$ such that for all pairwise distinct points $x,y,z \in X$, we have*

$$G(Tx,Ty,Tz) \leq a_1G(x,Tx,Tx)+a_2G(y,Ty,Ty)+a_3G(z,Tz,Tz)+a_4G(x,y,z).$$

Then $\mathrm{Fix}(T) \neq \emptyset$ *and* $|\mathrm{Fix}(T)| \leq 2$.

Proof. Let us first show that T admits a fixed point. Let $u_0 \in X$ be fixed, but arbitrarily chosen and consider the Picard sequence $\{u_n\} \subset X$ defined by

$$u_{n+1}=Tu_n, \quad n \geq 0$$

.

As in the proof of Theorem 3.1, we can assume that $u_n \neq u_{n+1}$ for all $n \geq 0$, and thus, by (I) we obtain that u_n, u_{n+1}, u_{n+2} are pairwise distinct for all $n \geq 0$. Then, by (II), we have

$$\begin{aligned} G(u_1,u_2,u_3) &= G(Tu_0,Tu_1,Tu_2)\\ &\leq a_1G(u_0,Tu_0,Tu_0)+a_2G(u_1,Tu_1,Tu_1)+a_3G(u_2,Tu_2,Tu_2)\\ &\quad +a_4G(u_0,u_1,u_2) \end{aligned}$$

so

$$G(u_1,u_2,u_3) \leq a_1G(u_0,u_1,u_1)+a_2G(u_1,u_2,u_2)+a_3G(u_2,u_3,u_3)+a_4G(u_0,u_1,u_2). \tag{3.9}$$

By (P_3) and (P_4), since u_n, u_{n+1}, u_{n+2} are pairwise distinct for all $n \geq 0$ we have

$$G(u_0, u_1, u_1) \leq G(u_0, u_1, u_2) \tag{3.10}$$

$$G(u_1, u_2, u_2) \leq G(u_0, u_1, u_2) \tag{3.11}$$

$$G(u_2, u_3, u_3) \leq G(u_1, u_2, u_3). \tag{3.12}$$

By (3.9), (3.10), (3.11) and (3.12) it follows that

$$(1 - a_3)G(u_1, u_2, u_3) \leq (a_1 + a_2 + a_4)G(u_0, u_1, u_2),$$

and since $1 - a_3 > 0$ we obtain

$$G(u_1, u_2, u_3) \leq \frac{a_1 + a_2 + a_4}{1 - a_3} G(u_0, u_1, u_2).$$

Let $k = \dfrac{a_1 + a_2 + a_4}{1 - a_3}$. Since $\sum_{i=1}^{4} a_i < 1$ notice that $k < 1$ and we have

$$G(u_1, u_2, u_3) \leq k\tau_0,$$

where $\tau_0 = G(u_0, u_1, u_2) > 0$. By induction, we get

$$G(u_n, u_{n+1}, u_{n+2}) \leq k^n \tau_0.$$

As in the proof of Theorem 3.1, we obtain that $\{u_n\}$ is a G-Cauchy sequence and by completeness of (X, G), there exists $u^* \in X$ such that $\{u_n\}$ is G-convergent to u^*. We will show that $u^* \in Fix(T)$. By (P_5), we have

$$\begin{aligned} G(u^*, Tu^*, Tu^*) &\leq G(u^*, u_n, u_n) + G(u_n, Tu^*, Tu^*) \\ &= G(u^*, u_n, u_n) + G(Tu_{n-1}, Tu^*, Tu^*), \end{aligned}$$

which implies, by (II) and the fact that u_n, u_{n-1} and u^* are pairwise distinct that

$$\begin{aligned} G(u^*, Tu^*, Tu^*) \leq\ & G(u^*, u_n, u_n) + a_1 G(u_{n-1}, Tu_{n-1}, Tu_{n-1}) \\ & + a_2 G(u^*, Tu^*, Tu^*) + a_3 G(u^*, Tu^*, Tu^*) + a_4 G(u_{n-1}, u^*, u^*), \end{aligned}$$

so we have

$$\begin{aligned} G(u^*, Tu^*, Tu^*) \leq\ & G(u^*, u_n, u_n) + a_1 G(u_{n-1}, u_n, u_n) \\ & + a_4 G(u_{n-1}, u^*, u^*). \end{aligned}$$

Since $1 - (a_2 + a_3) > 0$, passing to limit as $n \to \infty$ in the above inequality, we obtain $G(u^*, Tu^*, Tu^*) = 0$, which yields $u^* = Tu^*$, and thus $\mathrm{Fix}(T) \neq \emptyset$.

Now suppose that v_i, $i = 1, 2, 3$, are pairwise distinct points of T. Then, by (II) we have

$$\begin{aligned} G(v_1, v_2, v_3) &= G(Tv_1, Tv_2, Tv_3) \\ &\leq a_1 G(v_1, Tv_1, Tv_1) + a_2 G(v_2, Tv_2, Tv_2) + a_3 G(v_3, Tv_3, Tv_3) \\ &\quad + a_4 G(v_1, v_2, v_3) \\ &= a_1 G(v_1, v_1, v_1) + a_2 G(v_2, v_2, v_2) + a_3 G(v_3, v_3, v_3) + a_4 G(v_1, v_2, v_3), \end{aligned}$$

that is

$$\begin{aligned} (1 - a_4)G(v_1, v_2, v_3) &\leq a_1 G(v_1, v_1, v_1) + a_2 G(v_2, v_2, v_2) + a_3 G(v_3, v_3, v_3) \\ &= 0. \end{aligned}$$

Thus we obtain $a_4 \geq 1$ which is a contradiction. This shows that $|\operatorname{Fix}(T)| \leq 2$ and the proof of Theorem 3.7 is completed.

Remark 3.8. If condition (II) of Theorem 3.7 holds for every $x, y, z \in X$, then following the approach in [4], the existence of a (unique) fixed point of T can be deduced immediately from Ćirić's fixed point theorem in metric spaces [3] . Namely, taking $z = y$, (II) reduces to

$$\begin{aligned} G(Tx, Ty, Ty) &\leq a_1 G(x, Tx, Tx) + a_2 G(y, Ty, Ty) + a_3 G(y, Ty, Ty) + a_4 G(x, y, y) \\ &\leq a_1 \max\{G(x, Tx, Tx), G(x, x, Tx)\} \\ &\quad + (a_2 + a_3) \max\{G(y, Ty, Ty), G(y, y, Ty)\} \\ &\quad + a_4 \max\{G(x, y, y), G(y, x, x)\} \end{aligned}$$

Replacing y by x and z by y, (II) reduces to

$$\begin{aligned} G(Tx, Tx, Ty) &\leq a_1 G(x, Tx, Tx) + a_2 G(x, Tx, Tx) + a_3 G(y, Ty, Ty) + a_4 G(x, x, y) \\ &\leq (a_1 + a_2) \max\{G(x, Tx, Tx), G(x, x, Tx)\} \\ &\quad + a_3 \max\{G(y, Ty, Ty), G(y, y, Ty)\} \\ &\quad + a_4 \max\{G(x, y, y), G(y, x, x)\}. \end{aligned}$$

Then, it follows from the above two inequalities that

$$\delta(Tx, Ty) \leq A \max\{\delta(x, Tx), \delta(y, Ty), \delta(x, y)\}$$

for every $x, y \in X$, where $A = \sum_{i=1}^{4} a_i < 1$ and δ is the metric on X defined by (3.4), and Ćirić's fixed point theorem applies.

Example 3.9. Let $\lambda \in \left(0, \frac{1}{4}\right)$ be fixed. We consider the set $X_\lambda = \{a, b, c, d\} \subset \mathbb{R}$, where

$$a = 0, \quad b = \frac{2\lambda}{2\lambda - 1}, \quad c = 1, \quad d = 2.$$

We introduce the mapping $T : X \to X$ defined by

$$Ta = a, \quad Tb = b, \quad Tc = b, \quad Td = b.$$

Consider now the G-metric on X given by

$$G(u, v, w) = \max\{|u - v|, |v - w|, |u - w|\}, \quad u, v, w \in X.$$

Remark that $T(Tc) = Tb = b \neq c$ and $T(Td) = Tb = b \neq d$, which shows that condition (I) of Theorem 3.7 is satisfied.

We now show that condition (II) of Theorem 3.7 is satisfied for all pairwise distinct points $x, y, z \in X$. By symmetry, we only study the cases

$$(x, y, z) \in \{(a, b, c), (a, b, d), (a, c, d), (b, c, d)\}.$$

- The case $(x, y, z) = (a, b, c)$: In this case, we have

$$G(Tx, Ty, Tz) = G(Ta, Tb, Tc) = G(a, b, b) = |a - b| = -b$$

and

$$\begin{aligned}&\lambda\left[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)+G(x,y,z)\right]\\&=\lambda\left[G(a,Ta,Ta)+G(b,Tb,Tb)+G(c,Tc,Tc)+G(a,b,c)\right]\\&=\lambda\left[G(a,a,a)+G(b,b,b)+G(c,b,b)+G(a,b,c)\right],\end{aligned}$$

that is,

$$\begin{aligned}&\lambda\left[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)+G(x,y,z)\right]\\&=\lambda\left[G(c,b,b)+G(a,b,c)\right]\\&=\lambda\left[|c-b|+\max\{|a-b|,|b-c|,|a-c|\}\right]\\&=\lambda\left[\frac{1}{1-2\lambda}+\max\left\{\frac{2\lambda}{1-2\lambda},\frac{1}{1-2\lambda},1\right\}\right]\\&=\lambda\left[\frac{1}{1-2\lambda}+\frac{1}{1-2\lambda}\right]\\&=\frac{2\lambda}{1-2\lambda}\\&=-b.\end{aligned}$$

Consequently, we obtain

$$G(Tx,Ty,Tz)=\lambda\left[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)+G(x,y,z)\right].$$

• The case $(x,y,z)=(a,b,d)$: In this case, we have

$$G(Tx,Ty,Tz)=G(Ta,Tb,Td)=G(a,b,b)=|a-b|=-b$$

and

$$\begin{aligned}&\lambda\left[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)+G(x,y,z)\right]\\&=\lambda\left[G(a,Ta,Ta)+G(b,Tb,Tb)+G(d,Td,Td)+G(a,b,d)\right]\\&=\lambda\left[G(a,a,a)+G(b,b,b)+G(d,b,b)+G(a,b,d)\right],\end{aligned}$$

that is,

$$\begin{aligned}&\lambda\left[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)+G(x,y,z)\right]\\&=\lambda\left[G(d,b,b)+G(a,b,d)\right]\\&=\lambda\left[|d-b|+\max\{|a-b|,|b-d|,|a-d|\}\right]\\&=\lambda\left[\frac{2-2\lambda}{1-2\lambda}+\max\left\{\frac{2\lambda}{1-2\lambda},\frac{2-2\lambda}{1-2\lambda},2\right\}\right]\\&=\lambda\left[\frac{2-2\lambda}{1-2\lambda}+\frac{2-2\lambda}{1-2\lambda}\right]\\&=\frac{4\lambda(1-\lambda)}{1-2\lambda}\\&=-2(1-\lambda)b.\end{aligned}$$

Since $-2(1-\lambda)b\le -b$, then it holds that

$$G(Tx,Ty,Tz)\le\lambda\left[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)+G(x,y,z)\right].$$

• The case $(x,y,z)=(a,c,d)$: In this case, we have

$$G(Tx,Ty,Tz)=G(Ta,Tc,Td)=G(a,b,b)=|a-b|=-b$$

and

$$\begin{aligned}&\lambda\left[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)+G(x,y,z)\right]\\&=\lambda\left[G(a,Ta,Ta)+G(c,Tc,Tc)+G(d,Td,Td)+G(a,c,d)\right]\\&=\lambda\left[G(a,a,a)+G(c,b,b)+G(d,b,b)+G(a,c,d)\right],\end{aligned}$$

that is,

$$\begin{aligned}&\lambda\left[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)+G(x,y,z)\right]\\&=\lambda\left[G(c,b,b)+G(d,b,b)+G(a,c,d)\right]\\&=\lambda\left[|c-b|+|d-b|+\max\{|a-c|,|c-d|,|a-d|\}\right]\\&=\lambda\left[\frac{1}{1-2\lambda}+\frac{2-2\lambda}{1-2\lambda}+\max\{1,1,2\}\right]\\&=\frac{\lambda(5-6\lambda)}{1-2\lambda}\\&=-\frac{(5-6\lambda)}{2}b.\end{aligned}$$

Since $-b\le-\frac{(5-6\lambda)}{2}b$, we also obtain

$$G(Tx,Ty,Tz)\le\lambda\left[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)+G(x,y,z)\right].$$

• The case $(x,y,z)=(b,c,d)$: In this case, we have

$$G(Tx,Ty,Tz)=G(Tb,Tc,Td)=G(b,b,b)=0,$$

which implies immediately that

$$G(Tx,Ty,Tz)\le\lambda\left[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)+G(x,y,z)\right].$$

Consequently, condition (II) of Theorem 3.7 holds for all pairwise distinct points $x,y,z\in X$ with $a_1=a_2=a_3=a_4=\lambda$. Notice that since $\lambda\in\left(0,\frac{1}{4}\right)$, then $\sum_{i=1}^4 a_i<1$. Furthermore, we have $\mathrm{Fix}(T)=\{a,b\}$, which confirms the result given by Theorem 3.7.

4. Applications

Some applications of our obtained results are provided in this section.

4.1. **Mappings contracting perimeters of triangles.** Petrov [11] introduced an interesting class of mappings $T:X\to X$, where (X,d) is a metric space, which can be characterized as mappings contracting perimeters of triangles.

Definition 4.1 (Petrov [11])**.** Let (X,d) be a metric space with $|X|\ge3$. We shall say that $T:X\to X$ is a mapping contracting perimeters of triangles on X, if there exists $\lambda\in(0,1)$ such that the inequality

$$d(Tx,Ty)+d(Ty,Tz)+d(Tz,Tx)\le\lambda[d(x,y)+d(y,z)+d(z,x)],\tag{4.1}$$

holds for all three pairwise distinct points $x,y,z\in X$.

The following result due to Petrov [11] is an immediate consequence of our Theorem 3.1.

Corollary 4.2. *Let (X,d), $|X| \geq 3$, be a complete metric space and let the mapping $T: X \to X$ satisfies the following two conditions:*

(I) *For all $x \in X$, $T(Tx) \neq x$, provided $Tx \neq x$;*
(II) *T is a mapping contracting perimeters of triangles on X.*

Then $\text{Fix}(T) \neq \emptyset$ *and* $|\text{Fix}(T)| \leq 2$.

Proof. Consider the G-metric on X defined by (2.1). By Lemma 2.1, since (X,d) is a complete metric space, then (X,G) is a complete G-metric space. Furthermore, (4.1) is equivalent to

$$G(Tx,Ty,Tz) \leq \lambda G(x,y,z)$$

for all three pairwise distinct points $x, y, z \in X$. Then, applying Theorem 3.1, the desired result follows.

Further results related to mappings contracting perimeters of triangles can be found in [10,12,14].

4.2. **Further applications.** Further interesting consequences can be deduced from our obtained results. Let (X,d) be a metric space with $|X| \geq 3$. We introduce the class of mappings $T: X \to X$ satisfying the inequality

$$\max\{d(Tx,Ty), d(Ty,Tz), d(Tz,Tx)\} \leq \lambda \max\{d(x,y), d(y,z), d(x,z)\} \tag{4.2}$$

for all three pairwise distinct points $x, y, z \in X$, where $\lambda \in (0,1)$ is a constant.

Corollary 4.3. *Let (X,d), $|X| \geq 3$, be a complete metric space and let the mapping $T: X \to X$ satisfies the following two conditions:*

(I) *For all $x \in X$, $T(Tx) \neq x$, provided $Tx \neq x$;*
(II) (4.2) *holds for all three pairwise distinct points $x, y, z \in X$.*

Then $\text{Fix}(T) \neq \emptyset$ *and* $|\text{Fix}(T)| \leq 2$.

Proof. Consider the G-metric on X defined by (2.2). By Lemma 2.2, since (X,d) is a complete metric space, then (X,G) is a complete G-metric space. Furthermore, (4.2) is equivalent to

$$G(Tx,Ty,Tz) \leq \lambda G(x,y,z)$$

for all three pairwise distinct points $x, y, z \in X$. Then, applying Theorem 3.1, the desired result follows.

We now introduce the class of mappings $T: X \to X$ satisfying the inequality

$$d(Tx,Ty) + d(Ty,Tz) + d(Tx,Tz) \leq \lambda\left[d(x,Tx) + d(y,Ty) + d(z,Tz)\right] \tag{4.3}$$

for all three pairwise distinct points $x, y, z \in X$, where $\lambda \in \left(0, \frac{2}{3}\right)$ is a constant.

Corollary 4.4. *Let (X,d), $|X| \geq 3$, be a complete metric space and let the mapping $T: X \to X$ satisfies the following two conditions:*

(I) *For all $x \in X$, $T(Tx) \neq x$, provided $Tx \neq x$;*
(II) (4.3) *holds for all three pairwise distinct points $x, y, z \in X$.*

Then $\text{Fix}(T) \neq \emptyset$ *and* $|\text{Fix}(T)| \leq 2$.

Proof. Consider the G-metric defined by (2.1). By Lemma 2.1, since (X, d) is a complete metric space, then (X, G) is a complete G-metric space. Moreover, (4.3) is equivalent to

$$G(Tx, Ty, Tz) \leq \frac{\lambda}{2}[G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)]$$

for all three pairwise distinct points $x, y, z \in X$. Then, since $\lambda < \frac{2}{3}$, applying Theorem 3.4, the desired result follows.

Remark 4.5. Corollary 4.4 was previously established by Petrov and Bisht [13, Theorem 3.2].

Consider now the class of mappings $T : X \to X$ satisfying the inequality

$$\begin{aligned} &d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \\ &\leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(z, Tz) + a_4 \left(d(x, y) + d(y, z) + d(x, z)\right) \end{aligned} \tag{4.4}$$

for all three pairwise distinct points $x, y, z \in X$, where a_i $(i = 1, 2, 3, 4)$ are nonnegative constants with $0 < \sum_{i=1}^{4} a_i < 1$.

Corollary 4.6. *Let (X, d), $|X| \geq 3$, be a complete metric space and let the mapping $T : X \to X$ satisfies the following two conditions:*

(I) *For all $x \in X$, $T(Tx) \neq x$, provided $Tx \neq x$;*
(II) (4.4) *holds for all three pairwise distinct points $x, y, z \in X$.*

Then $\mathrm{Fix}(T) \neq \emptyset$ *and* $|\mathrm{Fix}(T)| \leq 2$.

Proof. Consider the G-metric defined by (2.1). By Lemma 2.1, since (X, d) is a complete metric space, then (X, G) is a complete G-metric space. Moreover, (4.4) is equivalent to

$$G(Tx, Ty, Tz) \leq \frac{a_1}{2} G(x, Tx, Tx) + \frac{a_2}{2} G(y, Ty, Ty) + \frac{a_3}{2} G(z, Tz, Tz) + a_4 G(x, y, z)$$

for all three pairwise distinct points $x, y, z \in X$. Then, since $\sum_{i=1}^{4} a_i < 1$, applying Theorem 3.7, the desired result follows.

We finally consider the class of mappings $T : X \to X$ satisfying the inequality

$$\begin{aligned} &\max\{d(Tx, Ty), d(Ty, Tz), d(Tx, Tz)\} \\ &\leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(z, Tz) + a_4 \max\{d(x, y), d(y, z), d(x, z)\} \end{aligned} \tag{4.5}$$

for all three pairwise distinct points $x, y, z \in X$, where a_i $(i = 1, 2, 3, 4)$ are nonnegative constants with $0 < \sum_{i=1}^{4} a_i < 1$.

Corollary 4.7. *Let (X, d), $|X| \geq 3$, be a complete metric space and let the mapping $T : X \to X$ satisfies the following two conditions:*

(I) *For all $x \in X$, $T(Tx) \neq x$, provided $Tx \neq x$;*
(II) (4.5) *holds for all three pairwise distinct points $x, y, z \in X$.*

Then $\mathrm{Fix}(T) \neq \emptyset$ *and* $|\mathrm{Fix}(T)| \leq 2$.

Proof. Consider the G-metric defined by (2.2). By Lemma 2.2, since (X, d) is a complete metric space, then (X, G) is a complete G-metric space. Moreover, (4.5) is equivalent to

$$G(Tx, Ty, Tz) \leq a_1 G(x, Tx, Tx) + a_2 G(y, Ty, Ty) + a_3 G(z, Tz, Tz) + a_4 G(x, y, z)$$

for all three pairwise distinct points $x, y, z \in X$. Then, applying Theorem 3.7, the desired result follows.

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