Fixed Point Theory, 26(2025), No. 2, 553-562
DOI: 10.24193/fpt-ro.2025.2.13
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINT RESULTS FOR WEAKLY PICARD NON-SELF OPERATORS ON $\mathbb{R}^m_+$-METRIC SPACES

VERONICA ILEA*, ADELA NOVAC** AND DIANA OTROCOL**,***

*Babeş-Bolyai University, Department of Mathematics,
1 M. Kogălniceanu Street, 400084 Cluj-Napoca, Romania
E-mail: veronica.ilea@ubbcluj.ro

**Technical University of Cluj-Napoca, Department of Mathematics,
28 Memorandumului Street, 400114 Cluj-Napoca
E-mail: adela.novac@math.utcluj.ro, diana.otrocol@math.utcluj.ro

***Tiberiu Popoviciu Institute of Numerical Analysis, Romanian Academy,
P.O.Box. 68-1, 400110 Cluj-Napoca, Romania

Abstract. The aim of this paper is to extend the graphic contraction principle, well-known for self operators, to non-self operators. Data dependence of fixed points results for non-self operators are also discussed. The results complement and extend results given in the paper: V. Ilea, A. Novac, D. Otrocol, Fixed point results for non-self operators on $\mathbb{R}^m_+$-metric spaces, Fixed Point Theory, 26(2025), no. 1, 177-188.
Key Words and Phrases: $\mathbb{R}^m_+$-metric spaces, fixed point, weakly Picard operator, non-self operator, data dependence of fixed point.
2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

In is well known that one of the most important tool in the theory of metric spaces is the contraction principle known as Banach fixed-point theorem; also known as Banach-Caccioppoli theorem, the contraction principle or contraction mapping theorem. This theorem guarantees the existence and uniqueness of a fixed point for self operators in complete metric spaces. Also it provides a constructive method to find the mentioned fixed point (Picard's method of successive approximations). The theorem, first given in 1922, is named after the great mathematician, Stefan Banach (1892-1945). Initially the theorem was stated in Banach normed spaces. In 1920 Stefan Banach presented his doctoral dissertation. A year later, he published the results of his doctorate in Fundamenta Mathematicae.

In 1930 similar results were obtained independently by Renato Caccioppoli (1904-1959), who rediscovered and generalized Banach's theorem for complete metric space. Due to this fact, for many mathematicians, the theorem is known under the name Banach-Caccioppoli theorem.

Since then, a lot of papers were dedicated to improve that result. Several extensions of this result tried to relax the metric structure of the space, the completeness or the contraction condition itself. Thus, several variants of contraction principle are known for different types of generalized contractions on metric spaces.

The generalization of the Contraction Principle takes place on two directions: on one hand the contraction condition was generalized and on the other hand the metric space was generalized to uniform spaces.

Around the years 1970-1975, there were great mathematicians that reload the theorem for different type of conditions. Here we can mention some of them: M. Edelstein, R. Kannan, L.B. Ciric, M. A. Krasnoselskii, B.E. Rhoades, F.E. Browder, L.F. Guseman, S. Reich, I.A. Rus, S. Bianchini, and others.

On the other hand, many authors discussed the contraction principle in different generalized metric spaces. For example, Branciari introduced the concept of rectangular metric spaces and proved an analogue of the Banach contraction principle in the setting of such a space. Also, we can mention here the importance of the work of Perov, Schroder and Zabrejko [2], [7], [11], [4], [9], [22].

The first variant of contraction principle with the most generous conclusions, that combines and generalizes all previous is the variant given by I.A. Rus in 2016, see [14].

In this paper, we use the following version for saturated principles of graphic contractions in complete $\mathbb{R}^m_+$ metric spaces, see [**?**].

Theorem 1.1. ([**?**]) *(Saturated principle of graphic contractions) Let* (X,d) *be a complete generalized metric space and* $f\colon X \to X$ *be an orbitally continuous* S*-graphic contraction. Then we have that:*

(i) $F_f \neq \emptyset$, *and* $f^n(x) \to x^*(x) =: f^\infty(x) \in F_f,\ \forall x \in X$, *i.e.* f *is a WPO.*
(ii) $F_f = F_{f^n} \neq \emptyset$, *for all* $n \in \mathbb{N}^*$;
(iii) $d(f^n(x), f^\infty(x)) \leq (I-S)^{-1}S^n d(x, f(x)),\ \forall x \in X$ *and* $n \in \mathbb{N}^*$;
(iv) $d(x, f^\infty(x)) \leq (I-S)^{-1} d(x, f(x))$, *for all* $x \in X$, *i.e.,* f *is a* $(I-S)^{-1}$*-WPO;*

In this paper we extend the graphic contraction principle for self operators to non-self operators. Data dependence of fixed points results for non-self operators are also discussed.

The results complement and extend results given in the paper: V. Ilea, A. Novac, D. Otrocol, Fixed point results for non-self operators on $\mathbb{R}^m_+$-metric spaces, Fixed Point Theory, 26(2025), no. 1, 177-188.

2. Preliminaries

We begin with some standard notations.

Let (X,d) be a $\mathbb{R}^m_+$-metric space, $Y \subset X$ a nonempty subset of X and $f : Y \to X$ an operator. In the sequel we use the following notations:

$F_f = \{x \in Y : f(x) = x\}$ - the fixed points set of f.

$I(f) = \{Z \subset Y : f(Z) \subset Z, Z \neq \varnothing\}$ - the set of invariant subsets of f.

$(MI)_f = \cup I(f)$- the maximal invariant subset of f.

$P_{cl}(X) = \{Y \subset X |\ Y \text{ closed}\}$.

$(AB)_f(x^*) = \{x \in Y : f^n(x)$ is defined for all $n \in \mathbb{N}$ and $f^n(x) \to x^* \in F_f\}$-the attraction basin of the fixed point x^* with respect to f.

$(BA)_f = \bigcup_{x^* \in F_f} (AB)_f(x^*)$- the attraction basin of f.

$(PH)_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$-the Pompeiu-Hausdorff functional

$$(PH)_d(A,B) = \max\left(\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right).$$

Throughout this paper we consider that $Y \in P_{cl}(X)$.

Following [3] we have:

Definition 2.1. *An operator $f : Y \to X$ is said to be a Picard operator (PO) if:*
(i) $F_f = \{x_f^*\}$;
(ii) $(MI)_f = (BA)_f$.

Definition 2.2. *An operator $f : Y \to X$ is said to be a weakly Picard operator (WPO) if:*
(i) $F_f \neq \emptyset$;
(ii) $(MI)_f = (BA)_f$.

Definition 2.3. *For each WPO $f : Y \to X$ we define the operator $f^\infty : (BA)_f \to (BA)_f$ by $f^\infty(x) = \lim_{n\to\infty} f^n(x)$.*

Remark 2.4. *It is clear that $f^\infty((BA)_f) = F_f$, so f^∞ is a set retraction of $(BA)_f$ to F_f.*

Remark 2.5. *In terms of weakly Picard self operators the above definitions take the following form:*

$$f : Y \to X \text{ is a WPO iff } f|_{(MI)_f} : (MI)_f \to (MI)_f \text{ is a WPO.}$$

For other results on Picard and weakly Picard operators see [1], [2], [12], [13], [18].

3. Saturated contraction principle for graphic S-contraction

Definition 3.1. *An operator $f : Y \to X$ is an S-contraction if $S \in \mathbb{R}_+^{m\times m}$ and the following take place:*

(i) *S is a convergent to zero matrix, i.e. $S^n \to 0$ as $n \to \infty$;*
(ii) *$d(f(x), f(y)) \le Sd(x,y)$, for all $x, y \in Y$.*

Definition 3.2. *An operator $f : Y \to X$ is a graphic S-contraction if $S \in \mathbb{R}_+^{m\times m}$ and:*

(i) *S is a convergent to zero matrix, i.e. $S^n \to 0$ as $n \to \infty$;*
(ii) *$d(f(x), f^2(x)) \le Sd(x, f(x))$, for all $x \in Y$ such that $f(x) \in Y$.*

The first result of the paper is the following:

Theorem 3.3. *(Saturated principle of graphic contractions) Let (X,d) be a complete $\mathbb{R}_+^m$-metric space, $Y \subset X$ a nonempty closed subset and $f : Y \to X$ be an orbitally continuous S-graphic contraction, where $S \in \mathbb{R}_+^{m\times m}$. Then we have that:*

(i) *f is a non-self WPO, i.e.* $F_f \neq \emptyset$, *and* $f^n(x) \to x^*(x) =: f^\infty(x) \in F_f$, $\forall x \in (MI)_f$;
(ii) $F_f = F_{f^n} \neq \emptyset$, *for all* $n \in \mathbb{N}^*$;
(iii) $d(f^n(x), f^\infty(x)) \leq (I-S)^{-1}S^n d(x, f(x))$, *for all* $x \in (MI)_f$;
(iv) $d(x, f^\infty(x)) \leq (I-S)^{-1} d(x, f(x))$, $\forall x \in (MI)_f$ *and* $n \in \mathbb{N}^*$, *i.e., f is a* $(I-S)^{-1} - WPO$.

Proof. i) It is well-known that an orbitally continuous graphic contraction on a complete metric space has at least one fixed point.

On the other hand

$$
\begin{aligned}
&\sum_{n=0}^{\infty} d(f^n(x), f^{n+1}(x)) = \\
&\quad = d(x, f(x)) + d(f(x), f^2(x)) + d(f^2(x), f^3(x)) + \ldots + d(f^n(x), f^{n+1}(x)) + \ldots \\
&\quad \leq d(x, f(x)) + Sd(x, f(x)) + Sd(f(x), f^2(x)) + \ldots + Sd(f^{n-1}(x), f^n(x)) + \ldots \\
&\quad \leq d(x, f(x)) + Sd(x, f(x)) + S^2 d(x, f(x)) + \ldots + S^n d(x, f(x)) + \ldots \\
&\quad \leq \left(\sum_{n \in \mathbb{N}} S^n\right) d(x, f(x)) < +\infty, \forall x \in (MI)_f.
\end{aligned}
$$

This implies that $(f^n(x))_{n\in\mathbb{N}}$ converges.

Let $x^*(x)$ be the limit of $f^n(x)$. From the continuity of f it follows that $x^* \in F_f$, so $F_f \neq \emptyset$ and $f^n(x) \to x^*(x) =: f^\infty(x) \in F_f$, $\forall x \in (MI)_f$, i.e. f is a WPO.

ii) Let $y \in F_{f^n}, n > 1$. It follows that $f^n(y) = y$. From i) $f^k(y) \to x^*$, if $k \to \infty$. Since $f^{kn}(y) = y, k \in \mathbb{N}^*$ we get that $y = x^*$.

iii) Let $p \in \mathbb{N}$. We have the following estimations:

$$
\begin{aligned}
&d(f^n(x), f^{n+p}(x)) \leq \\
&\leq d(f^n(x), f^{n+1}(x)) + d(f^{n+1}(x), f^{n+p}(x)) \\
&\leq d(f^n(x), f^{n+1}(x)) + d(f^{n+1}(x), f^{n+2}(x)) + \ldots + d(f^{n+p-1}(x), f^{n+p}(x)) \\
&\leq S^n d(x, f(x)) + \ldots + S^{n+p-1} d(x, f(x)) \\
&\leq (S^n + \ldots + S^{n+p-1} + \ldots) d(x, f(x)) \\
&= (I-S)^{-1} S^n d(x, f(x)).
\end{aligned}
$$

Letting $p \to \infty$, we obtain iii), $\forall x \in (MI)_f$ and $n \in \mathbb{N}^*$.

iv) We have:

$$
\begin{aligned}
d(x, f^\infty(x)) &\leq d(x, f(x)) + d(f(x), f^\infty(x)) \\
&\leq d(x, f(x)) + Sd(x, f^\infty(x)).
\end{aligned}
$$

It follows that

$$d(x, f^\infty(x)) \leq (I-S)^{-1} d(x, f(x)),\ \forall x \in (MI)_f. \qquad \square$$

4. Data dependence

In this section, we consider non-self operators in the case of an ordered $\mathbb{R}_+^m$-metric space.

Theorem 4.1. *(Gronwall lemma for graphic S-contractions) Let $(X, d, \leq)$ be an ordered complete $\mathbb{R}_+^m$-metric space, $Y \subset X$ closed, and $f : Y \to X$ be an operator. We suppose that:*

(i) *f is a graphic S-contraction;*
(ii) *f is orbitally continuous.*
(iii) *f is increasing.*

Then:

(a) $x \leq f(x) \Longrightarrow x \leq f^\infty(x),\ \forall x \in (MI)_f$;
(b) $x \geq f(x) \Longrightarrow x \geq f^\infty(x),\ \forall x \in (MI)_f$.

Proof. From Abstract Gronwall Lemma (see [12]) and Theorem 3.3 the conclusion is proved. □

Theorem 4.2. *(Comparison Theorem for graphic S-contractions) Let $(X, d, \leq)$ be an ordered complete $\mathbb{R}_+^m$-metric space, $Y \subset X$ closed, and $f, g, h : Y \to X$ be three operators. We suppose that:*

(i) $f \leq g \leq h$;
(ii) *the operators f, g, h are graphic S-contractions;*
(iii) *f, g, h are orbitally continuous;*
(iv) *the operator g is increasing with respect to $\leq$.*

Then $x \leq y \leq z$ implies that

$$x^* \leq y^* \leq z^* \ \Rightarrow\ f^\infty(x) \leq g^\infty(y) \leq h^\infty(z).$$

Proof. From Abstract Comparison Theorem (see [12]) and Theorem 3.3, the conclusion is proved. □

An important theorem regarding the good WPO is the following.

Theorem 4.3. *Let (X, d) be a complete $\mathbb{R}_+^m$-metric space, $Y \subset X$ a nonempty closed subset and $f : Y \to X$ be an orbitally continuous S-graphic contraction, where $S \in \mathbb{R}_+^{m \times m}$. Then f is a good WPO, i.e.*

$$\sum_{n \in \mathbb{N}} d(f^n(x), f^{n+1}(x)) \leq (I - S)^{-1} d(x, f(x)), \quad \textit{for all } x \in (MI)_f.$$

Proof. For $x \in (MI)_f$, we have:

$$\begin{aligned}
&\sum_{n \in \mathbb{N}} d(f^n(x), f^{n+1}(x)) = \\
&\quad = d(x, f(x)) + d(f(x), f^2(x)) + d(f^2(x), f^3(x)) + \ldots + d(f^n(x), f^{n+1}(x)) + \ldots \\
&\quad \leq \left(\sum_{n \in \mathbb{N}} S^n \right) d(x, f(x)) \\
&\quad = (I - S)^{-1} d(x, f(x)),\ \forall x \in (MI)_f.
\end{aligned}$$

Since $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < \infty$, by definition, f is a good WPO. □

Another result for special WPO is the following.

Theorem 4.4. *Let (X, d) be a complete $\mathbb{R}^m_+$-metric space, $Y \subset X$ a nonempty closed subset and $f : Y \to X$ be an orbitally continuous S-graphic contraction, where $S \in \mathbb{R}^{m\times m}_+$. Then f is a special WPO, i.e.*

$$\sum_{n\in\mathbb{N}} d(f^n(x), f^\infty(x)) \leq (I - B)^{-1} d(x, f^\infty(x)), \text{ for all } x \in (MI)_f,$$

where $B = [I - S(I - S)^{-1}]^{-1} S(I - S)^{-1}$.

Proof. We estimate:

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} d(x, f^\infty(x)) \leq \\
&\leq d(x, f(x)) + d(f(x), f^2(x)) + ... + d(f^n(x), f^{n+1}(x)) + d(f^{n+1}(x), f^\infty(x)) \\
&\leq \left(\sum_{n\in\mathbb{N}} S^n\right) d(x, f(x)) = \\
&= (I - S)^{-1} d(x, f(x)).
\end{aligned}$$

Let $x := f(x)$ in the above estimation.

Follows:

$$\begin{aligned}
d(f(x), f^\infty(x)) &\leq (I - S)^{-1} d(f(x), f^2(x)) \\
&\leq S(I - S)^{-1} d(x, f(x)) \\
&\leq S(I - S)^{-1} [d(x, f^\infty(x)) + d(f^\infty(x), f(x))].
\end{aligned}$$

$$[I - S(I - S)^{-1}] d(f(x), f^\infty(x)) \leq S(I - S)^{-1} d(x, f^\infty(x)).$$

$$d(f(x), f^\infty(x)) \leq [I - S(I - S)^{-1}]^{-1} S(I - S)^{-1} d(x, f^\infty(x)), \ \forall x \in (MI)_f.$$

If we apply the same idea for $x := f^{n-1}(x)$, for all $n \in \mathbb{N}$ we obtain:

$$d(f^{n-1}(x), f^\infty(x)) \leq ((I - A)^{-1} A)^{n-1} d(x, f^\infty(x)),$$

where $A := S(I - S)^{-1}$. Follows:

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} d(f^n(x), f^\infty(x)) = \\
&= d(x, f(x)) + d(f(x), f^2(x)) + ... + d(f^n(x), f^\infty(x)) + ... \\
&\leq (I - B)^{-1} d(x, f^\infty(x)), \text{ with } B := (I - A)^{-1} A.
\end{aligned}$$

□

The following result gives conditions for well posed fixed point problem.

Theorem 4.5. *Let (X, d) be a complete $\mathbb{R}^m_+$-metric space, $Y \subset X$ a nonempty closed subset and $f : Y \to X$ be an orbitally continuous S-graphic contraction, where $S \in \mathbb{R}^{m\times m}_+$. Then the fixed point problem for $f|_{(AB)_f(x^*)} : (AB)_f(x^*) \to (AB)_f(x^*)$ is well posed, $\forall x^* \in F_f$.*

Proof. Let $x_n \in (AB)_f(x^*)$, we have:

$$\begin{aligned} d(x_n, x^*) &\le d(x_n, f(x_n)) + d(f(x_n), x^*) \\ &\le d(x_n, f(x_n)) + d(f(x_n), f^2(x_n) + d(f^2(x_n), x^*) \\ &\le d(x_n, f(x_n)) + d(f(x_n), f^2(x_n) + ... + d(f^{n-1}(x_n), f^n(x_n)) + d(f^n(x_n), x^*) \\ &\le d(x_n, f(x_n)) + Sd(x_n, f(x_n)) + ... + S^{n-1}d(x_n, f(x_n)) + d(f^n(x_n), x^*) \\ &= (I + S + S^2 + \ldots + S^{n-1} + ...)d(x_n, f(x_n)) + d(f^n(x_n), x^*) \\ &\le (I - S)^{-1}d(x_n, f(x_n)). \end{aligned}$$

Letting $n \to \infty$ and since $x_n \in (AB)_f(x^*)$, we obtain the conclusion. □

Next we have established the conditions for the limit shadowing property.

Theorem 4.6. *Let (X, d) be a complete $\mathbb{R}^m_+$-metric space, $Y \subset X$ a nonempty closed subset and $f : Y \to X$ be an orbitally continuous S-graphic contraction, where $S \in \mathbb{R}^{m\times m}_+$. Then, $\forall x^* \in F_f$, the operator $f|_{(AB)_f(x^*)} : (AB)_f(x^*) \to (AB)_f(x^*)$ has the limit shadowing property.*

Proof. Let $x_n \in (AB)_f(x^*)$.

$$d(x_{n+1}, f^\infty(x)) \le d(x_{n+1}, f(x_n)) + d(f(x_n), f^\infty(x)).$$

By applying Theorem 3.3 (iii), follows:

$$d(x_{n+1}, f^\infty(x)) \le [I - S(I - S)^{-1}]^{-1}S(I - S)^{-1}d(x_n, f^\infty(x)).$$ □

In what follow, we give two results for Pompeiu-Hausdorff functional.

Theorem 4.7. *Let (X, d) be a complete $\mathbb{R}^m_+$-metric space, $Y \subset X$ a nonempty closed subset and $f : Y \to X$ be an orbitally continuous S-graphic contraction, where $S \in \mathbb{R}^{m\times m}_+$. Let $g : Y \to X$ be such that:*

(1) *g is $(I - S)^{-1}$-WPO on $(MI)_f$;*
(2) *there exists $\eta \in \mathbb{R}^m_+$: $d(f(x), g(x)) \le \eta$, $\forall x \in Y$.*

Then $H_d(F_f, F_g) \le (I - S)^{-1}\eta$. Here $H_d = (H_{d_1}, \ldots, H_{d_m})$ stands for Pompeiu-Housdorff functional.

Proof. Let $x \in F_g$. From Theorem 3.3(iv) it follows that

$$\begin{aligned} d(x, f^\infty(x)) &\le (I - S)^{-1}d(x, f(x)) = (I - S)^{-1}d(g(x), f(x)) \\ &= (I - S)^{-1}\eta. \end{aligned}$$

Similarly, if $y \in F_f$, then

$$\begin{aligned} d(y, g^\infty(y)) &\le (I - S)^{-1}d(y, g(y)) = (I - S)^{-1}d(f(y), g(y)) \\ &= (I - S)^{-1}\eta. \end{aligned}$$

Now the conclusion follows from [11]. □

Theorem 4.8. *Let (X, d) be a complete $\mathbb{R}^m_+$-metric space, $Y \subset X$ a nonempty closed subset and $f : Y \to X$ be an orbitally continuous S-graphic contraction, where $S \in \mathbb{R}^{m\times m}_+$. Let $f_n : Y \to X$, $n \in \mathbb{N}^*$, be such that:*

(1) $f_n,\ n \in \mathbb{N}$ *are* $(I-S)^{-1}$*-WPOs on* $(MI)_f$*;*

(2) $f_n \stackrel{unif}{\to} f$*, as* $n \to \infty$.

Then $H_d(F_{f_n}, F_f) \to 0$*, as* $n \to \infty$.

Proof. Let $x \in F_{f_n}$. From Theorem 3.3(iv) it follows that

$$d(x, f^\infty(x)) \leq (I-S)^{-1} d(x, f(x)) = (I-S)^{-1} d(f_n(x), f(x)).$$

Similarly, if $y \in F_f$, then

$$d(y, f_n^\infty(y)) \leq (I-S)^{-1} d(y, f_n(y)) = (I-S)^{-1} d(f(y), f_n(y)).$$

Now the conclusion follows from the condition (2). □

References

[1] V. Berinde, S. Măruşter, I.A. Rus, *Saturated contraction principles for non-self operators, generalizations and applications*, Filomat, **31-11**(2017), 3391-3406.

[2] V. Berinde, A. Petruşel, I.A. Rus, *Remarks on the terminology of the mappings in fixed point iterative methods in metric space*, Fixed Point Theory, **24**(2023), no. 2, 525-540.

[3] A. Chiş-Novac, R. Precup, I.A. Rus, *Data dependence of fixed points for non-self generalized contractions*, Fixed Point Theory, **10**(2009), no. 1, 73-87.

[4] L.B. Ciric, *Generalized contraction and fixed point theorems*, Publ. Inst. Math., **12**(1971), 19-26.

[5] V. Ilea, A. Novac, D. Otrocol, *Fixed point results for non-self operators on* $\mathbb{R}^m$*-metric spaces*, Fixed Point Theory, 26(2025), no. 1, 177-188.

[6] V. Ilea, D. Otrocol, I.A. Rus, M.A. Şerban, *Applications of fibre contraction principle to some classes of functional integral equations*, Fixed Point Theory, **23**(2022), no. 1, 279-292.

[7] J.M. Ortega, W.C. Reinboldt, *On a class of approximative iterative processes*, Arch. Rat. Mech. Anal., **23**(1967), 352-365.

[8] A. Petruşel, I.A. Rus, *Graphic contractions principle and applications*, 395-416. In: Th. M. Rassias (ed.), Mathematical Analysis and Applications, Springer, 2019.

[9] B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., **226**(1977), 257-290.

[10] I.A. Rus, *On the method of successive approximations*, Revue Roum. Math. Pures et Appl., **17** (1972), 1433-1437.

[11] I.A. Rus, *Generalized Contractions and Applications,* Cluj University Press, Cluj-Napoca, 2001.

[12] I.A. Rus, *Picard operators and applications*, Scientiae Mathematicae Japonicae, **58**(2003), no. 1, 191-219.

[13] I.A. Rus, *Metric space with fixed point property with respect to contractions*, Stud. Univ. Babeş-Bolyai Math., **51**(2006), no. 3, 115-121.

[14] I.A. Rus, *The theory of a metrical fixed point theorem: theoretical and applicative relevances*, Fixed Point Theory, **9**(2008), no. 2, 541-559.

[15] I.A. Rus, *Some variants of contraction principle, generalizations and applications*, Stud. Univ. Babeş-Bolyai Math., **61**(2016), no. 3, 343-358.

[16] I.A. Rus, *Relevant classes of weakly Picard operators*, Anal. Univ. de Vest Timişoara Math. Inform., **54**(2016), 2-19.

[17] I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj University Press, Cluj-Napoca 2008.

[18] I.A. Rus, A. Petruşel, M.A. Şerban, *Weakly Picard Operators: Equivalent definitions, applications and open problems*, Fixed Point Theory, **7**(2006), no. 1, 3-22.

[19] I.A. Rus, M.A. Şerban, *Some generalizations of a Cauchy lemma and applications*, Topics in Mathematics, Computer Science and Philosophy, Presa Universitară Clujeană, 2008, 173-181.

[20] M.A. Şerban, *Fibre contraction theorem in generalized metric spaces*, Automat. Comput. Appl. Math., **16**(2007), no. 1, 139-144.

[21] M.A. Şerban, *Saturated fibre contraction principle*, Fixed Point Theory, **18**(2017), no. 2, 729-740.
[22] P.P. Zabrejko, *K-metric and K-normed spaces: Survey*, Collect. Math., **48**(1997), 825-859.

Received: February 2, 2024; Accepted: May 25, 2024.