Fixed Point Theory, 26(2025), No. 2, 539-552
DOI: 10.24193/fpt-ro.2025.2.12
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

EXISTENCE OF A UNIQUE MILD SOLUTION TO A FRACTIONAL THERMOSTAT MODEL VIA A RUS'S FIXED POINT THEOREM

J. HARJANI*, B. LÓPEZ** AND K. SADARANGANI***

*Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria,
Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain
E-mail: jackie.harjani@ulpgc.es

**Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria,
Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain
E-mail: belen.lopez@ulpgc.es

***Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria,
Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain
E-mail: kishin.sadarangani@ulpgc.es

Abstract. The purpose of this paper is to study the existence of a unique solution of a nonlinear fractional boundary value problem which can be considered as the fractional version of the thermostat model. Our analysis is based on a Rus's fixed point theorem involving two metrics.
Key Words and Phrases: Fractional boundary value problem, Green's function, Rus's fixed point theorem.
2010 Mathematics Subject Classification: 34A08, 47H10, 49L20, 47H09, 47H10.

1. Introduction

In this paper, we are concerned with the existence of a unique solution of the following fractional boundary value problem

$$\begin{cases} {}^cD^{\alpha}_{a^+}u(t) + f(t,u(t)) = 0, & a < t < b, \\ u'(a) = 0, \quad \beta D^{\alpha-1}_{a^+}u(b) + u(\eta) = 0, \end{cases} \tag{1.1}$$

where ${}^cD^{\alpha}_{a^+}$ denotes the Caputo fractional derivative of order α, $1 < \alpha \leq 2$, $\beta > 0$ and $a < \eta < b$.

Problem (1.1) has been treated in [9] in the particular case $a = 0$ and $b = 1$, where the authors used as main tool Guo-Krasnoselskii's fixed point theorem on cones.

On the other hand, Problem (1.1) can be considered as the fractional analogue to the following problem

$$\begin{cases} u''(t) + f(t,u(t)) = 0, & 0 < t < 1, \\ u'(0) = 0, \quad \beta u'(1) + u(\eta) = 0, \end{cases} \tag{1.2}$$

which was studied in [6]. Moreover, Problem (1.2) models a thermostat insulated at $t = 0$ with the controller at $t = 1$ adding or discharging heat depending on the temperature detected by the sensor at $t = \eta$.

In the present work, the main tool used in the proof of our result is a Rus's fixed point theorem involving two metrics.

2. Background

We start this section recalling some definitions and results about fractional calculus theory. This material appears in [10, 8].

Definition 2.1 Suppose $\alpha \geq 0$ and $f : [a, b] \to \mathbb{R}$. The Riemann-Liouville fractional integral of order α is defined as

$$(I_{a^+}^{\alpha} f)(t) = \begin{cases} f(t) & \text{if } \alpha = 0, \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f(s) ds, & \text{if } \alpha > 0. \end{cases}$$

Definition 2.2 Let $\alpha \geq 0$ and $f : [a, b] \to \mathbb{R}$. The Caputo fractional derivative of order α is defined by

$$({}^c D_{a^+}^{\alpha} f)(t) = \begin{cases} f(t) & \text{if } \alpha = 0, \\ I_{a^+}^{n-\alpha}(D^n f)(t), & \text{if } \alpha > 0 \text{ and } n = [\alpha] + 1 \end{cases}$$

(Here $[\alpha]$ denotes the integer part of α).

Lemma 2.3 Suppose that $f \in \mathcal{C}(a, b) \cap L^1(a, b)$ with a fractional derivative of order $\alpha > 0$ belonging to $\mathcal{C}(a, b) \cap L^1(a, b)$. Then

$$I_{a^+}^{\alpha}({}^c D_{a^+}^{\alpha} f)(t) = f(t) + c_0 + c_1(t-a) + \cdots + c_{n-1}(t-a)^{n-1},$$

for any $t \in [a, b]$, where $c_i \in \mathbb{R}$, $i = 0, 1, \cdots, n-1$ and $n = [\alpha] + 1$.

Lemma 2.4 For $f \in L^1(a, b)$ and $\alpha, \beta > 0$, we have:

(a) ${}^c D_{a^+}^{\alpha} I_{a^+}^{\alpha} f(t) = f(t)$,

(b) $I_{a^+}^{\alpha}(I_{a^+}^{\beta} f)(t) = (I_{a^+}^{\alpha+\beta} f)(t)$.

Next, we recall some results in connection with our Problem (1.1) which will be used later, and they appear in [2].

Lemma 2.5 Let $g \in \mathcal{C}[a, b]$. Then $u \in \mathcal{C}[a, b]$ is a solution to the fractional boundary value problem

$$\begin{cases} {}^c D_{a^+}^{\alpha} u(t) + g(t) = 0, & a < t < b, \\ u'(a) = 0, \quad \beta D_{a^+}^{\alpha-1} u(b) + u(\eta) = 0, \end{cases} \tag{2.1}$$

where $1 < \alpha \leq 2$, $\beta > 0$ and $a < \eta < b$, if

$$u(t) = \int_a^b G(t, s) g(s) ds,$$

being $G(t,s)$ the Green's function given by

$$G(t,s) = \beta + \mathcal{H}_\eta(s) - \mathcal{H}_t(s),$$

where, for $r \in [a,b]$, the function $\mathcal{H}_r : [a,b] \to \mathbb{R}$ is defined by

$$H_r(\tau) = \begin{cases} \dfrac{(r-\tau)^{\alpha-1}}{\Gamma(\alpha)}, & a \le \tau \le r \le b, \\ 0, & a \le r < \tau \le b \end{cases}$$

Remark 2.6 In (Proposition 1 of [2]), it is proved the following facts:

(a) $\max\{G(t,s) : t,s \in [a,b]\} = \beta + \dfrac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}$.

(b) $\min\{G(t,s) : t,s \in [a,b]\} = \beta - \dfrac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}$.

Notice that, taking into account Remark 2.6 (b), when $\beta\Gamma(\alpha) \ge (b-\eta)^{\alpha-1}$, we have $G(t,s) \ge 0$, and, thus, by Remark 2.6 (a),

$$0 \le G(t,s) \le \beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}.$$

Next, we state the fixed point theorem due to Rus which is the main tool used in the proof of our result. This result appears in [11].

Theorem 2.7 Let X be a nonempty set and d and ρ two metrics on X such that (X,d) is a complete metric space. Suppose that $T : X \to X$ is a continuous mapping with respect to the metric d on X and

(i) there exists $c > 0$ such that

$$d(Tx,Ty) \le c\rho(x,y) \quad \text{for any} \quad x,y \in X,$$

(ii) there exists $\alpha \in (0,1)$ satisfying

$$\rho(Tx,Ty) \le \alpha\rho(x,y) \quad \text{for any} \quad x,y \in X,$$

then T has a unique fixed point.

3. Main result

In what follows, we suppose that $\beta\Gamma(\alpha) \ge (b-\eta)^{\alpha-1}$ and, therefore, $G(t,s) \ge 0$.

Our point of starting in this section is to prove that the Green's function $G(t,s)$ appearing in Lemma 2.5 is of Holder-type respect to the first variable.

Previously, we recall the following well known fact.

Suppose that $f : [0,1] \to \mathbb{R}$ is a concave function with $f(0) = 0$ then $|f(x) - f(y)| \le f(|x-y|)$.

Particularly, we have that, when $0 < \beta \le 1$

$$|x^\beta - y^\beta| \le |x-y|^\beta, \quad \text{for any} \quad x,y \in [0,1],$$

since the function $f(x) = x^\beta$ is concave and $f(0) = 0$.

Now, we need the following result.

Lemma 3.1 Suppose that $0 < \beta \leq 1$ and $t_1, t_2, s \in [a, b]$ with $s \leq t_1, t_2$. Then

$$|(t_1 - s)^\beta - (t_2 - s)^\beta| \leq |t_1 - t_2|^\beta.$$

Proof. Since $t_1, t_2, s \in [a, b]$ and $s \leq t_1, t_2$, we have

$$0 \leq \frac{t_1 - s}{b - a} \leq 1 \quad \text{and} \quad 0 \leq \frac{t_2 - s}{b - a} \leq 1,$$

and, taking into account the above mentioned fact, it follows

$$\begin{aligned} |(t_1 - s)^\beta - (t_2 - s)^\beta| &= (b - a)^\beta \left| \left(\frac{t_1 - s}{b - a}\right)^\beta - \left(\frac{t_2 - s}{b - a}\right)^\beta \right| \\ \leq (b - a)^\beta \left| \frac{t_1 - s}{b - a} - \frac{t_2 - s}{b - a} \right|^\beta &= |t_1 - t_2|^\beta. \end{aligned}$$

Proposition 3.2 The function $G(-, s)$ is $(\alpha - 1)$-Holder on $[a, b]$, this is

$$|G(t_1, s) - G(t_2, s)| \leq \frac{1}{\Gamma(\alpha)} |t_1 - t_2|^{\alpha - 1},$$

for any $t_1, t_2, s \in [a, b]$.

Proof. Recall that the function $G(t, s)$ has the following expression

$$G(t, s) = \beta + \mathcal{H}_\eta(s) - \mathcal{H}_t(s),$$

where, for $r \in [a, b]$, the function $\mathcal{H}_r : [a, b] \to \mathbb{R}$ is defined as

$$H_r(\tau) = \begin{cases} \dfrac{(r - \tau)^{\alpha - 1}}{\Gamma(\alpha)}, & a \leq \tau \leq r \leq b, \\ 0, & a \leq r < \tau \leq b \end{cases}$$

Moreover, in our case $1 < \alpha \leq 2$, this is, $0 < \alpha - 1 \leq 1$.

Fix $s \in [a, b]$ and $t_1, t_2 \in [a, b]$. We can consider the following cases.

Case 1. $t_1, t_2 \leq s$.

In this case, we have

$$|G(t_2, s) - G(t_1, s)| = |\mathcal{H}_{t_2}(s) - \mathcal{H}_{t_1}(s)| = 0.$$

Case 2. $t_1, t_2 \geq s$.

Under this assumption, it follows:

$$\begin{aligned} |G(t_2, s) - G(t_1, s)| &= |\mathcal{H}_{t_2}(s) - \mathcal{H}_{t_1}(s)| \\ &= \left| \frac{(t_2 - s)^{\alpha - 1}}{\Gamma(\alpha)} - \frac{(t_1 - s)^{\alpha - 1}}{\Gamma(\alpha)} \right| \\ &= \frac{1}{\Gamma(\alpha)} |(t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1}| \\ &\leq \frac{1}{\Gamma(\alpha)} |t_2 - t_1|^{\alpha - 1}, \end{aligned}$$

where we have used Lemma 3.1.

Case 3. $t_1 \leq s \leq t_2$.

For this case, we infer

$$\begin{aligned}|G(t_2,s)-G(t_1,s)| &= |\mathcal{H}_{t_2}(s)-\mathcal{H}_{t_1}(s)|\\ &= |\mathcal{H}_{t_2}(s)| = \left|\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\right|\\ &= \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \le \frac{(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}.\end{aligned}$$

Summarizing, we have

$$|G(t_1,s)-G(t_2,s)| \le \frac{1}{\Gamma(\alpha)}|t_1-t_2|^{\alpha-1},$$

for any $s, t_1, t_2 \in [a,b]$.

This completes the proof.

Now, we put $N = \beta + \dfrac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}$.

(Recall that under our assumption $\beta\Gamma(\alpha) \ge (b-\eta)^{\alpha-1}$, we have $0 \le G(t,s) \le \beta + \dfrac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)} = N$).

Next, we present the main result of the paper, but previously we need the following definition.

Definition 3.3 A function $x \in \mathcal{C}[a,b]$ is said to be a mild solution to Problem (1.1) if it is a fixed point of the operator T defined as

$$(Tx)(t) = \int_a^b G(t,s)f(s,x(s))ds, \quad \text{for} \quad t \in [a,b].$$

Theorem 3.4 Let $f : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that there exists a constant $M > 0$ satisfying that

$$|f(t,x)-f(t,y)| \le M|x-y|,$$

for any $t \in [a,b]$ and $x, y \in \mathbb{R}_+$ and let $t_0 \in [a,b]$ be such that $f(t_0,0) > 0$.

If $M \cdot N(b-a)^{1/2} < 1$, where N is the above mentioned constant, then Problem (1.1) has a unique nontrivial nonnegative mild solution.

Proof. Let P be the cone given by

$$P = \{x \in \mathcal{C}[a,b] : x(t) \ge 0, \quad \text{for} \quad t \in [a,b]\}.$$

It is well known that P is a closed subset of the complete metric space $(\mathcal{C}[a,b], d_\infty)$, where d_∞ denotes the classical supremum distance, this is, for $x, y \in \mathcal{C}[a,b]$,

$$d_\infty(x,y) = \sup\{|x(t)-y(t)| : t \in [a,b]\}.$$

Therefore, (P, d_∞) is a complete metric space.

On the other hand, in P we can also consider the distance ρ given by

$$\rho(x,y) = \left(\int_a^b |x(s)-y(s)|^2 ds\right)^{1/2}.$$

Now, for any $x \in P$, we consider the operator defined on P as

$$(Tx)(t) = \int_a^b G(t,s)f(s,x(s))ds, \quad \text{for} \quad t \in [a,b].$$

Taking into account that $G(t,s) \geq 0$, for $t,s \in [a,b]$ and, the fact that f applies $[a,b] \times \mathbb{R}_+$ into $\mathbb{R}_+$, it follows that, for $x \in P$, $(Tx)(t) \geq 0$, for any $t \in [a,b]$.

In the sequel, we will prove that if $x \in P$, then $Tx \in \mathcal{C}[a,b]$.

To do this, we take a sequence $(t_n) \subset [a,b]$ and $t_0 \in [a,b]$ such that $t_n \to t_0$ and we will prove that $(Tx)(t_n) \to (Tx)(t_0)$.

In fact,

$$\begin{aligned} |(Tx)(t_n) - (Tx)(t_0)| &= \left| \int_a^b G(t_n,s)f(s,x(s))ds - \int_a^b G(t_0,s)f(s,x(s))ds \right| \\ &= \left| \int_a^b (G(t_n,s) - G(t_0,s))f(s,x(s))ds \right| \\ &\leq \int_a^b |G(t_n,s) - G(t_0,s)||f(s,x(s))|ds. \end{aligned}$$

Since $f : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, particularly,
$f : [a,b] \times [0, \|x\|_\infty] \to \mathbb{R}_+$ is bounded because $[a,b] \times [0, \|x\|_\infty]$ is a compact subset of $[a,b] \times \mathbb{R}_+$ (here, $\|x\|_\infty$ denotes de supremum norm, this is,
$\|x\|_\infty = \sup\{|x(t)| : t \in [a,b]\}$.

Put $R = \sup\{f(s,y) : (x,y) \in [a,b] \times [0, \|x\|_\infty]\}$.

Taking into account Proposition 3.2, from the last inequality we infer

$$\begin{aligned} |(Tx)(t_n) - (Tx)(t_0)| &\leq \int_a^b |G(t_n,s) - G(t_0,s)|f(s,x(s))ds \\ &\leq \frac{(t_n - t_0)^{\alpha-1}}{\Gamma(\alpha)} R(b-a), \end{aligned}$$

and, consequently, letting $n \to \infty$, we get that $(Tx)(t_n) \to (Tx)(t_0)$.

Therefore, T applies P into itself.

Next, we will prove that assumptions of Theorem 2.7 are satisfied.

Let $x, y \in P$. Then

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &\leq \int_a^b G(t,s)|f(s,x(s)) - f(s,y(s))|ds \\ &\leq M \int_a^b G(t,s)|x(s) - y(s)|ds \\ &\leq M \left(\int_a^b G(t,s)^2 ds \right)^{1/2} \left(\int_a^b |x(s) - y(s)|^2 ds \right)^{1/2} \\ &\leq M \left(\int_a^b G(t,s)^2 ds \right)^{1/2} \cdot \rho(x,y), \end{aligned}$$

where we have used our assumption and Cauchy-Schwartz inequality.

Taking into account that

$$0 \leq G(t,s) \leq \beta + \frac{(\eta - a)^{\alpha-1}}{\Gamma(\alpha)} = N,$$

(see Remark 2.6), from the last inequality, we deduce

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &\leq M \left(\int_a^b N^2 ds \right)^{1/2} \cdot \rho(x,y) \\ &= M \cdot N \cdot (b-a)^{1/2} \cdot \rho(x,y), \end{aligned} \tag{3.1}$$

and this says us that assumption (i) of Theorem 2.7 is satisfied with

$$c = M \cdot N \cdot (b-a)^{1/2}.$$

On the other hand, for $x, y \in P$, we have

$$\begin{aligned} \rho(x,y) &= \left(\int_a^b |x(t) - y(t)|^2 dt \right)^{1/2} \\ &\leq \left(\int_a^b d_\infty(x,y)^2 dt \right)^{1/2} \\ &= (b-a)^{1/2} d_\infty(x,y). \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), it follows

$$\begin{aligned} d_\infty(Tx,Ty) &\leq M \cdot N \cdot (b-a)^{1/2} \rho(x,y) \\ &\leq M \cdot N \cdot (b-a) d_\infty(x,y), \end{aligned}$$

and this proves the continuity of the operator $T : P \to P$ respect to the metric d_∞, which is a assumption appearing in Theorem 2.7.

Finally, for $x, y \in P$, we have

$$\begin{aligned} \rho(Tx,Ty) &= \left(\int_a^b |(Tx)(t) - (Ty)(t)|^2 dt \right)^{1/2} \\ &= \left(\int_a^b \left(\int_a^b (G(t,s)|f(s,x(s)) - f(s,y(s))|)^2 ds \right) dt \right)^{1/2} \\ &\leq \left(\int_a^b \left(M^2 \int_a^b G(t,s)^2 |x(s) - y(s)|^2 ds \right) dt \right)^{1/2} \\ &\leq M \cdot N \left(\int_a^b \left(\int_a^b |x(s) - y(s)|^2 ds \right) dt \right)^{1/2} \\ &= M \cdot N \cdot \rho(x,y)(b-a)^{1/2}. \end{aligned}$$

By our assumption, this is, $M \cdot N \cdot (b-a)^{1/2} < 1$, condition (ii) of Theorem 2.7 is satisfied with $\alpha = M \cdot N \cdot (b-a)^{1/2}$.

Therefore, Theorem 2.7 says us that the operator T has a unique fixed point $x^* \in P$, this is, x^* is a unique mild solution of Problem (1.1).

In the sequel, we prove that the unique mild solution obtained x^* to Problem (1.1) is nontrivial.

In fact, since x^* is a fixed point of the operator T, it follows

$$x^*(t) = \int_a^b G(t,s)f(s,x^*(s))ds.$$

We proceed by contradiction.

Suppose that x^* is trivial, this is, $x^*(t) = 0$ for any $t \in [a,b]$, then

$$\int_a^b G(t,s)f(s,0)ds = 0, \qquad \text{for any} \qquad t \in [a,b]. \tag{3.3}$$

As the integrand $G(t,s)f(s,x^*(x))$ is nonnegative, from (3.3) we deduce that

$$G(t,s)f(s,0) = 0, \quad \text{a.e. } (s).$$

Since $G(t,s) > \beta$ and $\beta > 0$, it follows $f(x,0) = 0$, a.e. (s).

By using our assumption, there exists $t_0 \in [a,b]$ with $f(t_0,0) > 0$ and, consequently, the continuity of f gives us the existence of a neighborhood U of t_0 such that $f(\tau,0) > 0$ for $\tau \in U$ and moreover, $\mu(U) > 0$ where μ is the Lebesgue measure.

Taking into account (3.3), we have

$$\begin{aligned} 0 &= \int_a^b G(t,s)f(s,0)ds \geq \int_a^b \beta f(s,0)ds \\ &= \beta \int_a^b f(s,0)ds > \beta \int_U f(s,0)ds > 0, \end{aligned}$$

and this contradicts to (3.3).

Therefore, x^* is nontrivial.

This finishes the proof.

Remark 3.5 Under the assumption of Theorem 3.4, this is $f : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that there exists a constant $M > 0$ satisfying that

$$|f(t,x) - f(t,y)| \leq M|x-y|,$$

for any $t \in [a,b]$ and $x, y \in \mathbb{R}_+$ and $t_0 \in [a,b]$ such that $f(t_0,0) > 0$.

We deduce that, for $x, y \in P$.

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &= \int_a^b G(t,s)|f(s,x(s)) - f(s,y(s))|ds \\ &\leq M \int_a^b G(t,s)|x(s) - y(s)|ds \\ &\leq M \cdot d_\infty(x,y) \int_a^b G(t,s)ds \\ &\leq M \cdot N \cdot (b-a) \cdot d_\infty(x,y). \end{aligned}$$

Therefore, if $M \cdot N \cdot (b-a) < 1$, then the Banach's Contraction Theorem gives us the existence and uniqueness of a mild solution to Problem (1.1).

On the other hand, we have proved that if

$$M \cdot N \cdot (b-a)^{1/2} < 1,$$

then we also get the existence and uniqueness of a mild solution to Problem (1.1).

From this, we can distinguish two cases:

Case 1. If $b - a < 1$, then $(b-a)^{1/2} > (b-a)$ and consequently, the condition

$$M \cdot N \cdot (b-a)^{1/2} < 1$$

implies

$$M \cdot N \cdot (b-a) < M \cdot N \cdot (b-a)^{1/2} < 1$$

and from this, our Theorem 3.4 is better that the use of the Banach's Contraction Theorem.

Case 2. If $b - a \geq 1$, then $(b-a) \geq (b-a)^{1/2}$ and this implies

$$M \cdot N \cdot (b-a)^{1/2} \leq M \cdot N \cdot (b-a) < 1,$$

and thus, in this case, we get a better result by using the Banach's Contraction Theorem that Rus's Theorem.

In the book [7], the authors proved the following result.

Theorem 3.6 (Theorem 7.7 of [7]). Suppose that $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a Lipschitz condition with respect to the second variable, this is, there exists $L > 0$ such that

$$|g(t,x) - g(t,y)| \leq L|x-y|,$$

for any $t \in [a,b]$ and $x, y \in \mathbb{R}$.

If $b - a < \dfrac{2\sqrt{2}}{L}$, then the following boundary value problem

$$\begin{cases} u''(t) = -g(t,u(t)), & a < t < b, \\ u(a) = c, & u(b) = d, \end{cases}$$

where $c, d \in \mathbb{R}$, has a unique mild solution.

Recently, similar results to Theorem 3.6 have appeared in the literature (see [3, 1, 4, 5], for example).

Next, taking into account Theorem 3.4, we obtain a result of the same type to the one given in Theorem 3.6 for our Problem (1.1). More precisely, we have the following corollary.

Corollary 3.7 Suppose that $f : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and such that there exists $M > 0$ satisfying that

$$|f(t,x) - f(t,y)| \leq M|x-y|,$$

for any $t \in [a,b]$ and $x, y \in \mathbb{R}_+$.

If $b - a < \dfrac{1}{M^2 \cdot N^2}$, where $N = \beta + \dfrac{(\eta - a)^{\alpha-1}}{\Gamma(\alpha)}$, then Problem (1.1) has a unique nonnegative mild solution x^*.

Moreover, if there exists $t_0 \in [a,b]$ such that $f(t_0, 0) > 0$, then x^* is nontrivial.

Next, we give an application of our results to the eigenvalues problem and a Lyapunov-type inequality associated to our Problem (1.1).

Corollary 3.8 Suppose the following boundary value problem

$$\begin{cases} {}^cD^{\alpha}_{a^+}u(t) + q(t)u(t) = 0, & a < t < b, \\ u'(a) = 0, & \beta D^{\alpha-1}_{a^+}u(b) + u(\eta) = 0, \end{cases} \tag{3.4}$$

where $1 < \alpha \leq 2$, $\beta > 0$, $a < \eta < b$ and $q : [a, b] \to \mathbb{R}$ is a continuous function.

If Problem (3.4) has a nontrivial mild solution, then the following Lyapunov-type inequality

$$\|q\|_\infty \geq \frac{1}{(b-a)^{1/2}} \cdot \frac{\Gamma(\alpha)}{(\beta\Gamma(\alpha) + (\eta - a)^{\alpha-1})}$$

holds.

Proof. Notice that Problem (3.4) is a particular case of Problem (1.1) with

$$f(t, u) = q(t)u.$$

Therefore, for any $t \in [a, b]$ and $x, y \in \mathbb{R}_+$, we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= |q(t)x - q(t)y| \\ &\leq |q(t)||x-y| \leq \|q\|_\infty |x-y|. \end{aligned}$$

Therefore, the condition appearing in Theorem 3.4 is satisfied with $M = \|q\|_\infty$. Since the trivial solution $u_0 \equiv 0$ is a fixed point of the operator

$$(Tu)(t) = \int_a^b G(t,s)q(s)u(s)ds,$$

if Problem (3.4) has a nontrivial mild solution, this says us that in Theorem 3.4 fails the uniqueness of the mild solution and this happens when

$$M \cdot N \cdot (b-a)^{1/2} = \|q\|_\infty \cdot N \cdot (b-a)^{1/2} \geq 1.$$

Therefore, if Problem (3.4) has a nontrivial mild solution, then

$$\|q\|_\infty \geq \frac{1}{N \cdot (b-a)^{1/2}} = \frac{1}{(b-a)^{1/2}} \cdot \frac{\Gamma(\alpha)}{(\beta\Gamma(\alpha) + (\eta - a)^{\alpha-1})}.$$

This finishes the proof.

Remark 3.9 By Corollary 3.8, we have that if

$$\|q\|_\infty < \frac{1}{(b-a)^{1/2}} \cdot \frac{\Gamma(\alpha)}{(\beta\Gamma(\alpha) + (\eta - a)^{\alpha-1})},$$

then Problem(3.4) has a unique trivial mild solution.

A consequence of Corollary 3.8 is the following result about eigenvalues problem associated to Problem (1.1).

Corollary 3.10 Suppose the following eigenvalues problem

$$\begin{cases} {}^cD^{\alpha}_{a^+}u(t) + \lambda u(t) = 0, & a < t < b, \\ u'(a) = 0, & \beta \cdot D^{\alpha-1}_{a^+}u(b) + u(\eta) = 0, \end{cases} \tag{3.5}$$

where $1 < \alpha \leq 2$, $\beta > 0$, $a < \eta < b$ and $\lambda \geq 0$.

If $\lambda < \dfrac{1}{(b-a)^{1/2}} \cdot \dfrac{\Gamma(\alpha)}{(\beta\Gamma(\alpha) + (\eta - a)^{\alpha-1})}$, then λ is not an eigenvalue.

Proof. Notice that in this case if λ is an eigenvalue, this means that Problem (3.5) has a nontrivial mild solution and, by Corollary 3.8,

$$\lambda \geq \frac{1}{(b-a)^{1/2}} \cdot \frac{\Gamma(\alpha)}{(\beta\Gamma(\alpha) + (\eta - a)^{\alpha-1})}.$$

4. Comparison with other result

In this section we compare our results with the ones given in [9]. Previously, we need to introduce some notation.

Associated to Problem (1.1), we put

$$f_0 = \lim_{u \to 0^+} \min_{t \in [0,1]} \frac{f(t,u)}{u},$$

$$f_0^s = \lim_{u \to 0^+} \max_{t \in [0,1]} \frac{f(t,u)}{u},$$

$$f_\infty = \lim_{u \to \infty} \max_{t \in [0,1]} \frac{f(t,u)}{u}$$

and

$$f_\infty^s = \lim_{u \to \infty} \min_{t \in [0,1]} \frac{f(t,u)}{u}.$$

The main result in [9] is the following.

Theorem 4.1 Suppose that $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and assume that one of the following conditions is satisfied:

(i) $f_0 = \infty$ and $f_\infty = 0$. (Sublinear case)

(ii) $f_0^s = 0$ and $f_\infty^s = \infty$. (Superlinear case)

Under assumption $\beta\Gamma(\alpha) > (1-\eta)^{\alpha-1}$, the following problem

$$\begin{cases} {}^cD_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u'(0) = 0, & \beta D_{0^+}^{\alpha-1} u(1) + u(\eta) = 0, \end{cases} \tag{4.1}$$

has at least one positive mild solution.

In the sequel, we present an example where we can use our results while Theorem 4.1 do not work.

Example 4.2 Consider the following fractional boundary value problem

$$\begin{cases} {}^cD_{0^+}^{3/2} u(t) + \frac{1}{2}(t + \arctan(t + u(t)) = 0, & 0 < t < 1, \\ u'(0) = 0, & \frac{1}{5} D_{0^+}^{1/2} u(1) + u(1/2) = 0. \end{cases} \tag{4.2}$$

Notice that Problem (4.2) is a particular case of Problem (1.1) with $\alpha = \frac{3}{2}$, $a = 0$, $b = 1$, $f(t,u) = \frac{1}{2}(t + \arctan(t+u))$ and $\eta = \frac{1}{2}$.

Moreover, for $t \in [0,1]$ and $u, v \in \mathbb{R}_+$, we have

$$\begin{aligned} |f(t,u) - f(t,v)| &= \tfrac{1}{2}|\arctan(t+u) - \arctan(t+v)| \\ &\leq \tfrac{1}{2}\arctan(|t+u-(t+v)|) \\ &= \tfrac{1}{2}\arctan(|u-v|) \leq \tfrac{1}{2}|u-v|, \end{aligned}$$

where we have used the fact that

$$|\arctan x - \arctan y| \leq \arctan(|x-y|).$$

Hence, the condition appearing in Theorem 3.4 is satisfied with $M = \frac{1}{2}$.

Morerover, since

$$\begin{aligned} N &= \beta + \frac{(\eta - a)^{\alpha-1}}{\Gamma(\alpha)} = \frac{1}{5} + \frac{(\frac{1}{2})^{1/2}}{\Gamma(3/2)} \\ &\approx 0'2 + \frac{0'7071}{0'8863} \approx 0'2 + 0'7979 = 0'9979 < 1 \end{aligned}$$

Theorem 3.4 says us that Problem (4.2) has a unique nontrivial nonnegative mild solution, because $f(1/2,0) = \frac{1}{2}(\frac{1}{2} + \arctan(1/2)) > 0$.

On the other hand, as

$$\begin{aligned} f_0 &= \lim_{u\to 0^+} \min_{t\in[0,1]} \frac{1/2(t + \arctan(t+u))}{u} \\ &= \lim_{u\to 0^+} \frac{1/2\arctan(u)}{u} = \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} f_0^s &= \lim_{u\to 0^+} \max_{t\in[0,1]} \frac{1/2(t + \arctan(t+u))}{u} \\ &= \lim_{u\to 0^+} \frac{1/2(1 + \arctan(1+u))}{u} = \infty \end{aligned}$$

and, therefore, Problem (4.2) is not in sublinear nor in superlinear case. This says us that Problem (4.2) cannot be treated by Theorem 4.1.

Acknowledgment. The authors of the paper have been partially supported by grant PID2019-106093GB-I00 from the Spanish Government.

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Received: June 16, 2023; Accepted: January 13, 2024.