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EXISTENCE OF UNIQUE SOLUTION FOR HILFER FRACTIONAL DIFFERENTIAL EQUATION WITH MULTI-POINT BOUNDARY VALUE CONDITIONS

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Abstract. In this paper, two fixed point theorems about decreasing operators are given, and the existence of the unique solution for the higher order Hilfer fractional differential equation with multi-point boundary conditions is studied by using them. In addition, an iterative sequence is established to approximate the unique solution. As an application, two concrete examples are given to illustrate our results.
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1. Introduction

With the development of social science, and the increasing demand of complex engineering applications, fractional calculus theory and its applications have been widely concerned, especially the fractional differential equation abstracted from practical problems has become a research hotspot of many mathematical workers. Most scholars investigated Riemann-Liouville and Caputo boundary value problems, see [12], [9], [6], [7], [5], [2], [3]. The Hilfer fractional derivative, which is related to both Riemann-Liouville derivative and Caputo derivative, provides new ideas for researchers, see [8], [1]. However, at present, there are few articles devoted to boundary value problems involving the Hilfer fractional derivatives.

Yong et al. [8] have studied the following multi-point boundary value problems of the Hilfer fractional differential equations at resonance:

$$\begin{cases} D_{0+}^{\alpha,\beta}x(t) = f\left(t, x(t)\right), \quad t \in (0,T] =: J \setminus \{0\}, \\ I_{0+}^{1-\gamma}x(0) = \sum\limits_{i=1}^{m} c_i x(\tau_i), \quad \tau_i \in J, \quad \Gamma(\gamma) = \sum\limits_{i=1}^{m} c_i(\tau_i)^{\gamma-1}, \end{cases}$$

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where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$, $f : (0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function, $D_{0+}^{\alpha,\beta}$ is Hilfer fractional derivative of order α and type β, $c_i (i = 1, 2, \cdots, m)$ are positive real numbers, $\tau_i (i = 1, 2, \cdots, m)$ satisfy $0 < \tau_1 < \tau_2 < ... < \tau_m \leq T$.

In [1], Athasit et al. used the standard fixed point theorem to consider the existence of solutions for the following boundary value problems:

$$\begin{cases} {}^H D^{\alpha,\beta} x(t) = F\left(t, x(t)\right), \quad t \in J, \\ x(a) = 0, \quad x(b) = \sum\limits_{i=1}^{m} \delta_i I^{\varphi_i} x(\tilde{\xi_i}), \quad \varphi_i > 0, \quad \delta_i \in \mathbb{R}, \quad \tilde{\xi_i} \in [a, b], \end{cases}$$

where ${}^H D^{\alpha,\beta}$ is a Hilfer fractional derivative of order α and type β, $1 < \alpha < 2$, $0 \leq \beta \leq 1$, $F : J \times \mathbb{R} \longrightarrow p(\mathbb{R})$ is a multivalued map, $p(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$. But in [1], the authors did not provide iterative methods to approach the unique solution obtained.

In this paper, we will study the boundary value problems of nonlinear fractional differential equation with the higher Hilfer derivative:

$$\begin{cases} D_{0+}^{\alpha,\beta} x(t) + f\left(t, x(t)\right) = a, \quad t \in [0, 1], \\ x(0) = x'(0) = \cdots = x^{(n-3)}(0) = D_{0+}^{\gamma-1} x(t)|_{t=0} = 0, \\ x'(1) = \sum\limits_{i=1}^{m} b_i x(\xi_i), \end{cases} \tag{1}$$

where $D_{0+}^{\alpha,\beta}$ is a Hilfer fractional derivative of order α and type β, $n - 1 < \alpha < n$, $0 \leq \beta \leq 1$, $D_{0+}^{\gamma-1}$ is Riemann-Liouville fractional derivative, $\gamma = \alpha + n\beta - \alpha\beta$, $m, n \in N$, $n \geq 3$, $a \in \mathbb{R}$, $b_i, \xi_i \in \mathbb{R}$, for all $i = 1, 2, \cdots, m$, $b_i \geq 0$, $0 < \xi_1 < \cdots < \xi_m < 1$, $f \in C([0, 1] \times \mathbb{R})$ is a nonlinear function.

The structure of this paper is as follows. In Section 2, we give the definition of new convex operators and two fixed point theories about decreasing operators. In Section 3, We introduce some definitions and lemmas which are the basis of our theorems. In Section 4, in two cases, i.e. $a > 0$ and $a \leq 0$, we obtain that the problem (1) has the unique solution in $P_{h,\delta}$ or P_h, correspondingly, and an iterative sequence can be obtained to approximate the unique solution. In Section 5, We give two examples to illustrate our main results.

2. Fixed point theorems of decreasing operators

First, we mainly give two fixed point theorems of decreasing operators. Let $(E, \|\cdot\|)$ be a real Banach space with a zero element θ. A cone P of E means a convex cone, in the sense: $\alpha P + \beta P \subseteq P$, for all $\alpha, \beta \geq 0$. Further, the relation $x \leq y$ if $y - x \in P$ is a partial order when the cone P is pointed: $P \bigcap (-P) = \{0\}$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. P is called normal cone if there exists $M > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq M\|y\|$, in this case, M is called the normality constant of P. If $x \leq y$ implies $Ax \geq Ay$, then the operator $A : E \to E$ is called a decreasing operator. Given $h > \theta$, $\delta \in P$ with $\theta \leq \delta \leq h$.

Letting h, δ as before, define $x \sim h$ if there exist $\nu_1, \nu_2 > 0$ such that $\nu_1 h \leq x \leq \nu_2 h$ (hence, $\nu_1 \leq \nu_2$). Then, define $P_h = \{x \in E; x \sim h\}$, $P_{h,\delta} = \{x \in E; x + \delta \sim h\}$.

Definition 2.1 Let $T : P_{h,\delta} \to E$ be an operator, and $\psi : (0,1) \to (0,1)$ be such that $\mu < \psi(\mu) < 1$, for all $\mu \in (0,1)$. We say that T is a $\psi - (h,\delta)-$ convex operator provided

$$T(\mu x + (\mu - 1)\delta) \leq \frac{1}{\psi(\mu)} Tx + (\frac{1}{\psi(\mu)} - 1)\delta, for\ all\ x \in P_{h,\delta}, \mu \in (0,1).$$

Lemma 2.2 *Suppose that T is a decreasing $\psi - (h,\delta)-$ convex operator with $Th \in P_{h,\delta}$, and P is a normal cone, then*

(i) the operator equation $Tx = x$ has a unique solution $x^* \in P_{h,\delta}$;

(ii) for any $x_0 \in P_{h,\delta}$, making the sequence $x_n = Tx_{n-1}, n = 1, 2, \cdots$, we can obtain $\|x_n - x^*\| \to 0$ as $n \to \infty$.

Proof. (i) We prove the existence of solutions. By $h \in P_{h,\delta}$, $Th \in P_{h,\delta}$, we have $h + \delta \sim h$, $Th + \delta \sim h$. According to transitivity of " $\sim$ ", we get $h + \delta \sim Th + \delta$. So there are $\lambda_2 \geq \lambda_1 > 0$, such that

$$\lambda_1(h + \delta) \leq Th + \delta \leq \lambda_2(h + \delta).$$

Taking $m_0 = min\{\lambda_1, \frac{1}{\lambda_2}\}$, then there exists $m_0 \in (0,1)$, such that

$$m_0 h + (m_0 - 1)\delta \leq Th \leq \frac{1}{m_0} h + (\frac{1}{m_0} - 1)\delta. \tag{2}$$

Therefore, for all $n \in N_+$, we get

$$m_0^n h + (m_0^n - 1)\delta \leq Th \leq \frac{1}{m_0^n} h + (\frac{1}{m_0^n} - 1)\delta.$$

Since T is a $\psi - (h,\delta)-$ convex operator, we have $m_0 < \psi(m_0) < 1$. Then there exists $r \in N_+$, such that

$$(\frac{\psi(m_0)}{m_0})^r \geq \frac{1}{m_0}. \tag{3}$$

Let $a_n = m_0^n h + (m_0^n - 1)\delta$, $b_n = \frac{1}{m_0^n} h + (\frac{1}{m_0^n} - 1)\delta$, $n = 1, 2, \cdots$. Then,

$$a_n = m_0 a_{n-1} + (m_0 - 1)\delta, \ n = 1, 2, \cdots, \tag{4}$$

$$b_n = \frac{1}{m_0} b_{n-1} + (\frac{1}{m_0} - 1)\delta, \ n = 1, 2, \cdots. \tag{5}$$

Take $y_0 = a_r$, $z_0 = b_r$, then $y_0, z_0 \in P_{h,\delta}$ and $y_0 \le z_0$.
On the one hand, from (2)-(4), monotonicity and convexity of T , we have

$$\begin{aligned}
Ty_0 &= Ta_r = T(m_0 a_{r-1} + (m_0 - 1)\delta) \\
&\le \frac{1}{\psi(m_0)} Ta_{r-1} + (\frac{1}{\psi(m_0)} - 1)\delta \\
&= \frac{1}{\psi(m_0)} T(m_0 a_{r-2} + (m_0 - 1)\delta) + (\frac{1}{\psi(m_0)} - 1)\delta \\
&\le \frac{1}{\psi(m_0)} [\frac{1}{\psi(m_0)} Ta_{r-2} + (\frac{1}{\psi(m_0)} - 1)\delta] + (\frac{1}{\psi(m_0)} - 1)\delta \\
&= \frac{1}{(\psi(m_0))^2} Ta_{r-2} + (\frac{1}{(\psi(m_0))^2} - 1)\delta \\
&\le \cdots \le \frac{1}{(\psi(m_0))^r} Th + (\frac{1}{(\psi(m_0))^r} - 1)\delta \\
&\le \frac{1}{(\psi(m_0))^r} [\frac{1}{m_0} h + (\frac{1}{m_0} - 1)\delta] + (\frac{1}{(\psi(m_0))^r} - 1)\delta \\
&\le \frac{1}{m_0^r} h + (\frac{1}{m_0^r} - 1)\delta = z_0.
\end{aligned}$$

On the other hand, from (2), (3), (5), convexity and monotonicity of T , we get

$$\begin{aligned}
Tz_0 &= Tb_r = T(\frac{1}{m_0} b_{r-1} + (\frac{1}{m_0} - 1)\delta) \\
&\ge \psi(m_0)\{T(m_0[\frac{1}{m_0} b_{r-1} + (\frac{1}{m_0} - 1)\delta] + (m_0 - 1)\delta) - (\frac{1}{\psi(m_0)} - 1)\delta\} \\
&= \psi(m_0) Tb_{r-1} + (\psi(m_0) - 1)\delta \\
&= \psi(m_0) T(\frac{1}{m_0} b_{r-2} + (\frac{1}{m_0} - 1)\delta) + (\psi(m_0) - 1)\delta \\
&\ge \psi(m_0)[\psi(m_0) Tb_{r-2} + (\psi(m_0) - 1)\delta] + (\psi(m_0) - 1)\delta \\
&= (\psi(m_0))^2 Tb_{r-2} + ((\psi(m_0))^2 - 1)\delta \\
&\ge \cdots \ge (\psi(m_0))^r Th + ((\psi(m_0))^r - 1)\delta \\
&\ge (\psi(m_0))^r [m_0 h + (m_0 - 1)\delta] + ((\psi(m_0))^r - 1)\delta \\
&\ge m_0^r h + (m_0^r - 1)\delta = y_0.
\end{aligned}$$

Thus,

$$y_0 \le z_0, \; Ty_0 \le z_0, \; Tz_0 \ge y_0. \tag{6}$$

Suppose

$$y_n = Tz_{n-1}, \; z_n = Ty_{n-1}, \; n = 1, 2, \cdots . \tag{7}$$

Using (6), (7) and monotonicity of operator T, we have

$$y_0 \le y_1 \le \cdots \le y_n \le \cdots \le z_n \le \cdots \le z_1 \le z_0, \tag{8}$$

and

$$y_n, \; z_n \in P_{h,\delta}, \; n = 1, 2, \cdots .$$

Then, $\{\eta \in (0,1); y_n \geq \eta z_n + (\eta - 1)\delta\}$ is nonempty, for each $n \in N$.
Let

$$\eta_n = sup\{\eta > 0 : y_n \geq \eta z_n + (\eta - 1)\delta\}.$$

The following we will prove that the sequence $\{\eta_n\}$ is increasing.
Since $y_n \leq z_n$ and P is a closed set , then $\eta_n \in (0,1)$ and $y_n \geq \eta_n z_n + (\eta_n - 1)\delta$.
Therefore,

$$y_{n+1} \geq y_n \geq \eta_n z_n + (\eta - 1)\delta \geq \eta_n z_{n+1} + (\eta - 1)\delta.$$

This shows that $\eta_{n+1} \geq \eta_n$, i.e. $\{\eta_n\}$ is an increasing sequence.
Assume $\lim_{n\to\infty} \eta_n = \eta^*$. Now we prove $\eta^* = 1$. If $0 < \eta^* < 1$, we consider the following two cases:

(1) There exists $k \in N_+$, such that $\eta_k = \eta^*$.
Since for any $n \geq k$, $\eta_n = \eta^*$, then when $n \geq k$,

$$\begin{aligned}
y_{n+1} &= Tz_n \geq T[\frac{1}{\eta_n} y_n + (\frac{1}{\eta_n} - 1)\delta] \\
&= T[\frac{1}{\eta^*} y_n + (\frac{1}{\eta^*} - 1)\delta] \\
&\geq \psi(\eta^*)\{T(\eta^*[\frac{1}{\eta^*} y_n + (\frac{1}{\eta^*} - 1)\delta] + (\eta^* - 1)\delta) - (\frac{1}{\psi(\eta^*)} - 1)\delta\} \\
&= \psi(\eta^*)Ty_n + (\psi(\eta^*) - 1)\delta \\
&= \psi(\eta^*)z_{n+1} + (\psi(\eta^*) - 1)\delta.
\end{aligned}$$

So $\eta^* = \eta_{n+1} \geq \psi(\eta^*) > \eta^*$.This is a contradiction.

(2) For any $n \in N_+$, $\eta_n < \eta^*$, then $0 < \frac{\eta_n}{\eta^*} < 1$, and

$$\begin{aligned}
y_{n+1} &= Tz_n \geq T[\frac{1}{\eta_n} y_n + (\frac{1}{\eta_n} - 1)\delta] \\
&\geq \psi(\frac{\eta_n}{\eta^*})\{T(\frac{\eta_n}{\eta^*}[\frac{1}{\eta_n} y_n + (\frac{1}{\eta_n} - 1)\delta] + (\frac{\eta_n}{\eta^*} - 1)\delta) - (\frac{1}{\psi(\frac{\eta_n}{\eta^*})} - 1)\delta\} \\
&= \psi(\frac{\eta_n}{\eta^*})T[\frac{1}{\eta^*} y_n + (\frac{1}{\eta^*} - 1)\delta] + [\psi(\frac{\eta_n}{\eta^*}) - 1]\delta \\
&\geq \psi(\frac{\eta_n}{\eta^*})[\psi(\eta^*)Ty_n + (\psi(\eta^*) - 1)\delta] + [\psi(\frac{\eta_n}{\eta^*}) - 1]\delta \\
&= \psi(\frac{\eta_n}{\eta^*})\psi(\eta^*)Ty_n + [\psi(\frac{\eta_n}{\eta^*})\psi(\eta^*) - 1]\delta \\
&= \psi(\frac{\eta_n}{\eta^*})\psi(\eta^*)z_{n+1} + [\psi(\frac{\eta_n}{\eta^*})\psi(\eta^*) - 1]\delta.
\end{aligned}$$

Therefore, $\eta_{n+1} \geq \psi(\frac{\eta_n}{\eta^*})\psi(\eta^*) > \frac{\eta_n}{\eta^*}\psi(\eta^*)$. When $n \to \infty$, we get $\eta^* \geq \psi(\eta^*) > \eta^*$. This is also a contradiction. Then according to $\lim_{n\to\infty} \eta_n = 1$, P is a normal cone and the completeness of E, we can prove that y_n and z_n have the same limit value. Let this limit value be x^*. Paying attention to the reducibility of T and combinate (7), thereby, we get that the operator T has a fixed point x^* in $P_{h,\delta}$. This part of the proof is similar to the reference [8], we omit it here.

We prove the uniqueness of solutions. Let y^* be also a fixed point of T in $P_{h,\delta}$. Since $x^*, y^* \in P_{h,\delta}$, we have $x^* + \delta \sim y^* + \delta$, namely, there exists $\lambda_4 \geq \lambda_3 > 0$, such that

$$\lambda_3(y^* + \delta) \leq x^* + \delta \leq \lambda_4(y^* + \delta).$$

Taking $d_0 = min\{\lambda_3, \frac{1}{\lambda_4}\}$, then there exists $d_0 \in (0, 1]$, such that

$$d_0(y^* + \delta) \leq x^* + \delta \leq \frac{1}{d_0}(y^* + \delta),$$

which implies that

$$d_0 y^* + (d_0 - 1)\delta \leq x^* \leq \frac{1}{d_0} y^* + (\frac{1}{d_0} - 1)\delta.$$

When $d_0 = 1$, it's obvious that $x^* = y^*$.
When $d_0 \in (0, 1)$, let

$$\hat{d} = sup\{0 < d < 1 : dy^* + (d-1)\delta \leq x^* \leq \frac{1}{d} y^* + (\frac{1}{d} - 1)\delta\},$$

obviously, $0 < \hat{d} \leq 1$, and since P is a closed set, we have

$$\hat{d} y^* + (\hat{d} - 1)\delta \leq x^* \leq \frac{1}{\hat{d}} y^* + (\frac{1}{\hat{d}} - 1)\delta.$$

We will prove $\hat{d} = 1$. If $0 < \hat{d} < 1$, then

$$\begin{aligned} x^* = Tx^* &\geq T[\frac{1}{\hat{d}} y^* + (\frac{1}{\hat{d}} - 1)\delta] \\ &\geq \psi(\hat{d})\{T(\hat{d}[\frac{1}{\hat{d}} y^* + (\frac{1}{\hat{d}} - 1)\delta] + (\hat{d} - 1)\delta) - (\frac{1}{\psi(\hat{d})} - 1)\delta\} \\ &= \psi(\hat{d}) T y^* + (\psi(\hat{d}) - 1)\delta \\ &= \psi(\hat{d}) y^* + (\psi(\hat{d}) - 1)\delta, \end{aligned}$$

and

$$\begin{aligned} x^* &\leq T[\hat{d} y^* + (\hat{d} - 1)\delta] \\ &\leq \frac{1}{\psi(\hat{d})} T y^* + (\frac{1}{\psi(\hat{d})} - 1)\delta \\ &= \frac{1}{\psi(\hat{d})} y^* + (\frac{1}{\psi(\hat{d})} - 1)\delta, \end{aligned}$$

hence,

$$\psi(\hat{d}) y^* + (\psi(\hat{d}) - 1)\delta \leq x^* \leq \frac{1}{\psi(\hat{d})} y^* + (\frac{1}{\psi(\hat{d})} - 1)\delta.$$

According to the definition of $\hat{d}$, we have $\psi(\hat{d}) \leq \hat{d}$, it and $\psi(\hat{d}) > \hat{d}$ contradict each other. So $\hat{d} = 1$, then, $x^* = y^*$. In summary, the operator equation $Tx = x$ has a unique solution.

(ii) We prove that for any $x_0 \in P_{h,\delta}$, let $x_n = Tx_{n-1}$, $n = 1, 2, \cdots$, we have $\|x_n - x^*\| \to 0$ as $n \to \infty$. Since $x^*, x_0 \in P_{h,\delta}$, so there exists $\zeta \in (0, 1]$, such that

$$\zeta x^* + (\zeta - 1)\delta \le x_0 \le \frac{1}{\zeta}x^* + (\frac{1}{\zeta} - 1)\delta.$$

When $\zeta = 1$, we get $x_0 = x^*$. By $Tx^* = x^*$, then $\|x_n - x^*\| = \|x^* - x^*\| = 0$, i.e. the conclusion has been proved.
When $\zeta \in (0, 1)$, let

$$y'_0 = \zeta x^* + (\zeta - 1)\delta,\ z'_0 = \frac{1}{\zeta}x^* + (\frac{1}{\zeta} - 1)\delta,$$

then

$$y'_0 \le x_0 \le z'_0,\ y'_0 \le x^* \le z'_0. \tag{9}$$

We construct two iterative sequences:

$$y'_n = Tz'_{n-1},\ z'_n = Ty'_{n-1},\ n = 1, 2, \cdots. \tag{10}$$

Combining (9), (10) and the reducibility of operator T, we get

$$y'_n \le x_n \le z'_n,\ n = 1, 2, \cdots, \tag{11}$$

and

$$y'_n \le x^* \le z'_n,\ n = 1, 2, \cdots. \tag{12}$$

Moreover, using (10) and convexity and reducibility of T, we have

$$y'_1 = Tz'_0 = T(\frac{1}{\zeta}x^* + (\frac{1}{\zeta} - 1)\delta) \ge \psi(\zeta)Tx^* + (\psi(\zeta) - 1)\delta \ge \zeta x^* + (\zeta - 1)\delta = y'_0,$$

and

$$z'_1 = Ty'_0 = T(\zeta x^* + (\zeta - 1)\delta) \le \frac{1}{\psi(\zeta)}Tx^* + (\frac{1}{\psi(\zeta)} - 1)\delta \le \frac{1}{\zeta}x^* + (\frac{1}{\zeta} - 1)\delta = z'_0,$$

consequently, $y'_2 = Tz'_1 \ge Tz'_0 = y'_1$ and $z'_2 = Ty'_1 \le Ty'_0 = z'_1$, and so on, we can get

$$y'_n \ge y'_{n-1},\ z'_n \le z'_{n-1},\ n = 1, 2, \cdots, \tag{13}$$

further more, if $m \ge n$ and $m, n \in \mathbb{N}$, we have

$$y'_m = Tz'_{m-1} \le Tz'_m \le Ty'_m = z'_{m+1} \le z'_n. \tag{14}$$

According to (13) and (14), we have

$$y'_0 \le y'_1 \le \cdots \le y'_n \le \cdots \le z'_n \le \cdots \le z'_1 \le z'_0.$$

Similar to the above proof of (i), namely see the reference [8], we can get that $\{y'_n\}$ and $\{z'_n\}$ have the same limit value. Taking the limit of (12), we obtain

$$\lim_{n\to\infty} y'_n = \lim_{n\to\infty} z'_n = x^*.$$

Using (11), we get that

$$\lim_{n\to\infty} x_n = x^*.$$

The proof of the lemma is completed.

Lemma 2.3 *Suppose E is a real Banach space, $P \subset E$ is normal , $h > \theta$, and suppose $T : P \to P$ is a decreasing operator , if operator T satisfies the following two conditions:*

(L_1) there is $h_0 \in P_h$ such that $Th_0 \in P_h$;

(L_2) for any $x \in P$ and $\mu \in (0, 1)$, there exists $\mu < \psi(\mu) < 1$ such that $T(\mu x) \leq \frac{1}{\psi(\mu)}Tx$.

then the operator T has a unique fixed point $x^* \in P_h$. In addition, take any $g_0 \in P_h$, constructing the sequence $g_n = Tg_{n-1}, n = 1, 2, \cdots$, we can get $\|g_n - x^*\| \to 0$ as $n \to \infty$.

Proof. (1) We prove that the operator T has a fixed point.

1) The following we show that $T : P_h \to P_h$. For any $x \in P_h$, there exists $\eta_1 \in (0, 1)$, such that

$$\eta_1 h \leq x \leq \frac{1}{\eta_1}h.$$

Using (L_2), we have

$$Tx = T(\eta \cdot \frac{1}{\eta}x) \leq \frac{1}{\psi(\eta)}T(\frac{1}{\eta}x),\ \eta \in (0, 1),$$

then

$$T(\frac{1}{\eta}x) \geq \psi(\eta)Tx,\ \forall x \in P,\ \eta \in (0, 1). \tag{15}$$

From (L_2), formula (15) and monotonicity of T, we obtain

$$Tx \leq T(\eta_1 h) \leq \frac{1}{\psi(\eta_1)}Th,$$

and

$$Tx \geq T(\frac{1}{\eta_1}h) \geq \psi(\eta_1)Th.$$

Consequently, we get $Tx \sim Th$.

Since $Th_0 \in P_h$, then there exists $\lambda_6 \geq \lambda_5 > 0$, such that $\lambda_5 h \leq Th_0 \leq \lambda_6 h$. According to $h_0 \in P_h$, then there exists $\eta_0 \in (0, 1)$, such that $\eta_0 h_0 \leq h \leq \frac{1}{\eta_0}h_0$. So we get

$$Th \leq T(\eta_0 h_0) \leq \frac{1}{\psi(\eta_0)}Th_0 \leq \frac{\lambda_6}{\psi(\eta_0)}h,$$

and

$$Th \geq T(\frac{1}{\eta_0}h_0) \geq \psi(\eta_0)Th_0 \geq \psi(\eta_0)\lambda_5 h.$$

Hence, $Th \in P_h$, i.e. $Th \sim h$. By using transitivity of equivalence relation " $\sim$ ", we have $Tx \sim h$, namely, $Tx \in P_h$, so the operator T is $P_h \to P_h$.

2) Now we prove that there exists $\hat{y}_0, \hat{z}_0 \in P_h$, $b \in (0, 1)$, such that

$$b\hat{z}_0 \leq \hat{y}_0 < \hat{z}_0, \tag{16}$$

and

$$\hat{y}_0 \leq T\hat{z}_0 \leq T\hat{y}_0 \leq \hat{z}_0. \tag{17}$$

From the above the proof of 1), we have known that $Th \in P_h$, then there exists $\eta_2 \in (0,1)$, such that

$$\eta_2 h \leq Th \leq \frac{1}{\eta_2} h. \tag{18}$$

According to the condition (L_2), we have $\eta_2 < \psi(\eta_2) < 1$. So we take $r \in N_+$, such that

$$(\frac{\psi(\eta_2)}{\eta_2})^r \geq \frac{1}{\eta_2}, \tag{19}$$

then,

$$\eta_2^r h \leq Th \leq \frac{1}{\eta_2^r} h.$$

Let

$$\hat{y}_0 = \eta_2^r h,\ \hat{z}_0 = \frac{1}{\eta_2^r} h, \tag{20}$$

then $\hat{y}_0, \hat{z}_0 \in P_h$ and $\hat{y}_0 = \eta_2^r h = \eta_2^{2r} \cdot \frac{1}{\eta_2^r} h = \eta_2^{2r} \cdot \hat{z}_0 < \hat{z}_0$. Taking any $b \in (0, \eta_2^{2r})$, then $b \in (0,1)$ and $\hat{y}_0 \geq b\hat{z}_0$, therefore, $b\hat{z}_0 \leq \hat{y}_0 < \hat{z}_0$. On the other hand, since T is a decreasing operator, then $T\hat{y}_0 \geq T\hat{z}_0$, and we combine formula (15) and (18)-(20), we have

$$\begin{aligned}
T\hat{y}_0 &= T(\eta_2^r h) = T(\eta_2 \cdot \eta_2^{r-1} h) \\
&\leq \frac{1}{\psi(\eta_2)} T(\eta_2^{r-1} h) = \frac{1}{\psi(\eta_2)} T(\eta_2 \cdot \eta_2^{r-2} h) \\
&\leq \frac{1}{(\psi(\eta_2))^2} T(\eta_2^{r-2} h) \\
&\leq \cdots \leq \frac{1}{(\psi(\eta_2))^r} Th \\
&\leq \frac{1}{\eta_2^{r-1}} \cdot \frac{1}{\eta_2} h = \frac{1}{\eta_2^r} h = \hat{z}_0,
\end{aligned}$$

and

$$\begin{aligned}
T\hat{z}_0 &= T(\frac{1}{\eta_2^r} h) = T(\frac{1}{\eta_2} \cdot \frac{1}{\eta_2^{r-1}} h) \\
&\geq \psi(\eta_2) T(\frac{1}{\eta_2^{r-1}} h) = \psi(\eta_2) T(\frac{1}{\eta_2} \cdot \frac{1}{\eta_2^{r-2}} h) \\
&\geq (\psi(\eta_2))^2 T(\frac{1}{\eta_2^{r-2}} h) \\
&\geq \cdots \geq (\psi(\eta_2))^r Th \\
&\geq \eta_2^{r-1} \cdot \eta_2 h = \eta_2^r h = \hat{y}_0,
\end{aligned}$$

therefore, $\hat{y}_0 \leq T\hat{z}_0 \leq T\hat{y}_0 \leq \hat{z}_0$.

3) The following by constructing sequences $\hat{y}_n$ and $\hat{z}_n$, we prove that these two sequences have the same limit value, thereby, the limit value is a fixed point of the

operator T.
Suppose

$$\hat{y}_n = T\hat{z}_{n-1},\ \hat{z}_n = T\hat{y}_{n-1},\ n = 1, 2, \cdots . \tag{21}$$

Using (17), (21) and monotonicity of operator T, we get,

$$\hat{y}_0 \le \hat{y}_1 \le \cdots \le \hat{y}_n \le \cdots \le \hat{z}_n \le \cdots \le \hat{z}_1 \le \hat{z}_0. \tag{22}$$

Using formula (16), there exists $b \in (0, 1)$, such that

$$\hat{y}_n \ge \hat{y}_0 \ge b\hat{z}_0 \ge b\hat{z}_n,\ n = 1, 2, \cdots .$$

We let

$$\hat{\eta}_n = sup\{\hat{\eta} > 0 : \hat{y}_n \ge \hat{\eta}\hat{z}_n\}.$$

Now we prove that the sequence $\{\hat{\eta}_n\}$ is increasing.
Since $\hat{y}_n \le \hat{z}_n$ and P is a closed set , then $\hat{\eta_n} \in (0, 1)$ and $\hat{y}_n \ge \hat{\eta}_n\hat{z}_n$,
so

$$\hat{y}_{n+1} \ge \hat{y}_n \ge \hat{\eta}_n\hat{z}_n \ge \hat{\eta}_n\hat{z}_{n+1},$$

it indicates that $\hat{\eta}_{n+1} \ge \hat{\eta}_n$, namely, $\{\hat{\eta}_n\}$ is an increasing sequence.
Supposing $\lim_{n\to\infty} \hat{\eta}_n = \hat{\eta}^*$, then $\hat{\eta}^* = 1$. If $0 < \hat{\eta}^* < 1$, we show that one gets a contradiction. Two cases occur.
case 1 There exists $k \in N_+$, such that $\hat{\eta}_k = \hat{\eta}^*$.
Since for any $n \ge k$, $\hat{\eta}_n = \hat{\eta}^*$, then when $n \ge k$,

$$\begin{aligned}\hat{y}_{n+1} &= T\hat{z}_n \ge T(\frac{1}{\hat{\eta}_n}\hat{y}_n) = T(\frac{1}{\hat{\eta}^*}\hat{y}_n)\\ &\ge \psi(\hat{\eta}^*)T\hat{y}_n = \psi(\hat{\eta}^*)\hat{z}_{n+1}.\end{aligned}$$

So $\hat{\eta}^* = \hat{\eta}_{n+1} \ge \psi(\hat{\eta}^*) > \hat{\eta}^*$. This is a contradiction.
case 2 For any $n \in N_+$, $\hat{\eta}_n < \hat{\eta}^*$. Then $0 < \frac{\hat{\eta}_n}{\hat{\eta}^*} < 1$, and

$$\begin{aligned}\hat{y}_{n+1} &= T\hat{z}_n \ge T(\frac{1}{\hat{\eta}_n}\hat{y}_n)\\ &\ge \psi(\frac{\hat{\eta}_n}{\hat{\eta}^*})T(\frac{1}{\hat{\eta}^*}\hat{y}_n)\\ &\ge \psi(\frac{\hat{\eta}_n}{\hat{\eta}^*})\psi(\hat{\eta}^*)T\hat{y}_n\\ &= \psi(\frac{\hat{\eta}_n}{\hat{\eta}^*})\psi(\hat{\eta}^*)\hat{z}_{n+1}.\end{aligned}$$

Therefore, $\hat{\eta}_{n+1} \ge \psi(\frac{\hat{\eta}_n}{\hat{\eta}^*})\psi(\hat{\eta}^*) > \frac{\hat{\eta}_n}{\hat{\eta}^*}\psi(\hat{\eta}^*)$. If $n \to \infty$, then we get $\hat{\eta}^* \ge \psi(\hat{\eta}^*) > \hat{\eta}^*$. This is also a contradiction. Then according to $\lim_{n\to\infty} \hat{\eta}_n = 1$, P is a normal and E is a complete space, we can prove that $\hat{y}_n$ and $\hat{z}_n$ have the same limit value. Assuming the limit value is x^*. Then from formula (21) and T is decreasing, we can obtain that the operator T has a fixed point x^* in P_h. This part of the proof is similar to the reference [11], we also omit here.

The following we prove that the fixed point of T is unique. Let y^* be also a fixed point of T in P_h. Since $x^*, y^* \in P_h$, so we get $x^* \sim y^*$, namely, there exists $\lambda_8 \geq \lambda_7 > 0$, such that

$$\lambda_7 y^* \leq x^* \leq \lambda_8 y^*.$$

Taking $\eta_3 = min\{\lambda_7, \frac{1}{\lambda_8}\}$, then there exists $\eta_3 \in (0,1]$, such that

$$\eta_3 y^* \leq x^* \leq \frac{1}{\eta_3} y^*.$$

When $\eta_3 = 1$, obviously, $x^* = y^*$.
When $\eta_3 \in (0,1)$, let

$$\tilde{\eta} = sup\{0 < \eta < 1 : \eta y^* \leq x^* \leq \frac{1}{\eta} y^*\},$$

it's obvious that $0 < \tilde{\eta} \leq 1$, since P is closed , then we have

$$\tilde{\eta} y^* \leq x^* \leq \frac{1}{\tilde{\eta}} y^*,$$

and $\tilde{\eta} = 1$. If $0 < \tilde{\eta} < 1$, then

$$x^* = Tx^* \geq T(\frac{1}{\tilde{\eta}} y^*) \geq \psi(\tilde{\eta}) Ty^* = \psi(\tilde{\eta}) y^*,$$

and

$$x^* \leq T(\tilde{\eta} y^*) \leq \frac{1}{\psi(\tilde{\eta})} Ty^* = \frac{1}{\psi(\tilde{\eta})} y^*.$$

Therefore,

$$\psi(\tilde{\eta}) y^* \leq x^* \leq \frac{1}{\psi(\tilde{\eta})} y^*.$$

From the definition of $\tilde{\eta}$, we obtain $\psi(\tilde{\eta}) \leq \tilde{\eta}$. But $\psi(\tilde{\eta}) \in (\tilde{\eta}, 1)$, this is a contradiction. Consequently, $\tilde{\eta} = 1$, then $x^* = y^*$. Above all, the operator T has a unique fixed point.

(2) We prove that for any $g_0 \in P_h$, let the sequence be $g_n = Tg_{n-1}, n = 1, 2, \cdots$, we get $\|g_n - x^*\| \to 0$ as $n \to \infty$.
The above (1) has proved that $T : P_h \to P_h$, so we have $Tg_0 \in P_h$. We combine $g_0 \in P_h$, then there exists $\zeta_1 \in (0,1)$, such that

$$\zeta_1 g_0 \leq Tg_0 \leq \frac{1}{\zeta_1} g_0. \tag{23}$$

Using (L_2), we have $\psi(\zeta_1) \in (\zeta_1, 1)$. Then there exists $r \in N_+$, such that

$$(\frac{\psi(\zeta_1)}{\zeta_1})^r \geq \frac{1}{\zeta_1}. \tag{24}$$

Let

$$\bar{y}_0 = \zeta_1^r g_0, \ \bar{z}_0 = \frac{1}{\zeta_1^r} g_0, \tag{25}$$

then $\bar{y}_0, \bar{z}_0 \in P_h$, and we obtain

$$\bar{y}_0 \leq g_0 \leq \bar{z}_0. \tag{26}$$

Moreover, $\bar{y}_0 = \zeta_1^r g_0 = \zeta_1^{2r}\bar{z}_0 < \bar{z}_0$. So taking any $b \in (0, \zeta_1^{2r})$, we have $b \in (0,1)$ and $\bar{y}_0 \geq b\bar{z}_0$.
From (26) and monotonicity of T, we obtain

$$T\bar{z}_0 \leq Tg_0 \leq T\bar{y}_0. \tag{27}$$

From (23)-(25), we can also get

$$\begin{aligned}
T\bar{y}_0 &= T(\zeta_1^r g_0) = T(\zeta_1 \cdot \zeta_1^{r-1} g_0)\\
&\leq \frac{1}{\psi(\zeta_1)} T(\zeta_1^{r-1} g_0) = \frac{1}{\psi(\zeta_1)} T(\zeta_1 \cdot \zeta_1^{r-2} g_0)\\
&\leq \frac{1}{(\psi(\zeta_1))^2} T(\zeta_1^{r-2} g_0) \leq \cdots \leq \frac{1}{(\psi(\zeta_1))^r} Tg_0\\
&\leq \frac{1}{\zeta_1^{r-1}} \cdot \frac{1}{\zeta_1} g_0 = \frac{1}{\zeta_1^r} g_0 = \bar{z}_0,
\end{aligned}$$

and

$$\begin{aligned}
T\bar{z}_0 &= T(\frac{1}{\zeta_1^r} g_0) = T(\frac{1}{\zeta_1} \cdot \frac{1}{\zeta_1^{r-1}} g_0)\\
&\geq \psi(\zeta_1) T(\frac{1}{\zeta_1^{r-1}} g_0) = \psi(\zeta_1) T(\frac{1}{\zeta_1} \cdot \frac{1}{\zeta_1^{r-2}} g_0)\\
&\geq (\psi(\zeta_1))^2 T(\frac{1}{\zeta_1^{r-2}} g_0) \geq \cdots \geq (\psi(\zeta_1))^r Tg_0\\
&\geq \zeta_1^{r-1} \cdot \zeta_1 g_0 = \zeta_1^r g_0 = \bar{y}_0,
\end{aligned}$$

that is, we have

$$\bar{y}_0 \leq T\bar{z}_0 \leq T\bar{y}_0 \leq \bar{z}_0. \tag{28}$$

So we construct two iterative sequences:

$$\bar{y}_n = T\bar{z}_{n-1},\ \bar{z}_n = T\bar{y}_{n-1},\ n = 1, 2, \cdots. \tag{29}$$

Combining (27), (29) and the reducibility of operator T, we can obtain

$$\bar{y}_n \leq g_n \leq \bar{z}_n,\ n = 1, 2, \cdots, \tag{30}$$

and

$$\bar{y}_0 \leq \bar{y}_1 \leq \cdots \leq \bar{y}_n \leq \cdots \leq \bar{z}_n \leq \cdots \leq \bar{z}_1 \leq \bar{z}_0.$$

Similar to the above proof of (1), namely, see the reference [11], we can get that $\{\bar{y}_n\}$ and $\{\bar{z}_n\}$ have the same limit value x^*. Taking the limit of (30), we obtain $\lim\limits_{n\to\infty} g_n = x^*$. The proof of the lemma is completed.

3. Preliminaries

Definition 3.1 [6] The Riemann-Liouville fractional integral of $\alpha > 0$ of a function $y : (0, +\infty) \to R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 3.2 [6] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function y is given by

$$D^{\alpha}_{0+}y(t) = \frac{d^n}{dt^n}(I^{n-\alpha}_{0+}y)(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t (t-s)^{n-\alpha-1}y(s)ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$, where $n = [\alpha]+1$, $t > 0$.

Definition 3.3 [3] The generalized Hilfer fractional derivative of order α and type β of a function x is given by

$$D^{\alpha,\beta}_{0+}x(t) = I^{\beta(n-\alpha)}_{0+}(\frac{d}{dt})^n(I^{(1-\beta)(n-\alpha)}_{0+})x(t),$$

where $n-1 < \alpha < n$, $0 \leq \beta \leq 1$.

Remark 3.4 The definition 3.3 also can be expressed as

$$D^{\alpha,\beta}_{0+}x(t) = I^{\beta(n-\alpha)}_{0+}D^{\gamma}_{0+}x(t),$$

where $\gamma = \alpha + n\beta - \alpha\beta$, and it's easy to get $n-1 < \alpha \leq \gamma \leq n$.

Lemma 3.5 [6] *For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha}_{0+}u(t) = 0$ is given by*

$$u(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n},$$

where $c_i \in \mathbb{R}, i = 1,2,\cdots,n, n = [\alpha]+1$.

Lemma 3.6 *Let $y \in C[0,1]$, then $x \in C[0,1]$ is the solution of the following fractional multi-point boundary value problem:*

$$\begin{cases} D^{\alpha,\beta}_{0+}x(t) + y(t) = 0, \quad t \in [0,1], \\ x(0) = x'(0) = \cdots = x^{(n-3)}(0) = D^{\gamma-1}_{0+}x(t)|_{t=0} = 0, \\ x'(1) = \sum\limits_{i=1}^{m} b_ix(\xi_i), \end{cases}$$

where $n-1 < \alpha < n$, $0 \leq \beta \leq 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $m,n \in N$, $n \geq 3$, $b_i, \xi_i \in R$, for all $i = 1,2,\cdots,m$, $0 < \xi_1 < \cdots < \xi_m < 1$, if and only if x satisfies the integral equation

$$x(t) = \int_0^1 G(t,s)y(s)ds, \qquad t \in [0,1],$$

where $G(.,.)$ (the Green function) is given as

$$G(t,s) = g_1(t,s) + \frac{t^{\gamma-2}}{A}\sum_{i=1}^{m} b_ig_1(\xi_i,s) + g_2(t,s), \quad for \quad (t,s) \in [0,1]\times[0,1], \tag{31}$$

with

$$g_1(t,s) = \begin{cases} \frac{t^{\gamma-2}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\gamma-2}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{32}$$

$$g_2(t,s) = \frac{\alpha-\gamma+1}{A\Gamma(\alpha)}(1-s)^{\alpha-2}t^{\gamma-2}, \tag{33}$$

and

$$A=\gamma-2-\sum_{i=1}^{m}b_i\xi_i^{\gamma-2}\neq 0. \tag{34}$$

Proof. From Remark 3.4 we have

$$I_{0+}^{\beta(n-\alpha)}D_{0+}^{\gamma}x(t)=-y(t), \tag{35}$$

Applying the fractional derivative $D_{0+}^{\beta(n-\alpha)}$ on both sides of equation (35), we given

$$D_{0+}^{\gamma}x(t)=-D_{0+}^{\beta(n-\alpha)}y(t).$$

By Lemma 3.5 and $I_{0+}^{\gamma}D_{0+}^{\beta(n-\alpha)}=I_{0+}^{\alpha}$, we have

$$x(t)=-I_{0+}^{\alpha}y(t)+C_1t^{\gamma-1}+C_2t^{\gamma-2}+\cdots+C_nt^{\gamma-n},$$

where $C_i(i=1,2,\cdots,n)\in\mathbb{R}$ is arbitrary constant. Using boundary conditions $x(0)=x'(0)=\cdots=x^{(n-3)}(0)=D_{0+}^{\gamma-1}x(t)|_{t=0}=0$ and $D_{0+}^{\gamma-1}t^{\gamma-1}=\Gamma(\gamma)\neq 0, D_{0+}^{\gamma-1}t^{\gamma-2}=0$, we can obtain in turn that $C_n=C_{n-1}=\cdots=C_3=C_1=0$, then

$$x(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds+C_2t^{\gamma-2}. \tag{36}$$

Finding the derivative of equation (36), we have

$$x'(t)=-\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}y(s)ds+C_2(\gamma-2)t^{\gamma-3}.$$

Since $x'(1)=\sum\limits_{i=1}^{m}b_ix(\xi_i)$ and let $A=\gamma-2-\sum\limits_{i=1}^{m}b_i\xi_i^{\gamma-2}\neq 0$, it gives

$$C_2=\frac{1}{A}[\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}y(s)ds-\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}b_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}y(s)ds].$$

By substituting C_2 in (36), we get

$$\begin{aligned}x(t)=&-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds+\frac{t^{\gamma-2}}{A}[\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}y(s)ds\\&-\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}b_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}y(s)ds]\\=&-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds+\frac{t^{\gamma-2}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}y(s)ds-\frac{t^{\gamma-2}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}y(s)ds\\&-\frac{t^{\gamma-2}}{A\Gamma(\alpha)}\sum_{i=1}^{m}b_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}y(s)ds+\frac{t^{\gamma-2}}{A\Gamma(\alpha)}\sum_{i=1}^{m}b_i{\xi_i}^{\gamma-2}\int_0^1(1-s)^{\alpha-2}y(s)ds\\&-\frac{t^{\gamma-2}}{A\Gamma(\alpha)}\sum_{i=1}^{m}b_i{\xi_i}^{\gamma-2}\int_0^1(1-s)^{\alpha-2}y(s)ds+\frac{t^{\gamma-2}}{A\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}y(s)ds\\=&\frac{1}{\Gamma(\alpha)}\int_0^t[t^{\gamma-2}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}]y(s)ds+\frac{1}{\Gamma(\alpha)}\int_t^1t^{\gamma-2}(1-s)^{\alpha-2}y(s)ds\end{aligned}$$

$$+\frac{t^{\gamma-2}}{A}\sum_{i=1}^{m}b_i\frac{1}{\Gamma(\alpha)}\int_0^{\xi_i}[{\xi_i}^{\gamma-2}(1-s)^{\alpha-2}-(\xi_i-s)^{\alpha-1}]y(s)ds$$

$$+\frac{t^{\gamma-2}}{A}\sum_{i=1}^{m}b_i\frac{1}{\Gamma(\alpha)}\int_{\xi_i}^{1}{\xi_i}^{\gamma-2}(1-s)^{\alpha-2}y(s)ds+\int_0^1\frac{(\alpha-\gamma+1)(1-s)^{\alpha-2}t^{\gamma-2}}{A\Gamma(\alpha)}y(s)ds$$

$$=\int_0^1 G(t,s)y(s)ds.$$

The proof is completed.

Hence, x is a solution of the boundary value problem (1) if and only if x satisfies the following integral equation:

$$x(t)=\int_0^1 G(t,s)[f(s,x(s)-a]ds=\int_0^1 G(t,s)f(s,x(s))ds-a\int_0^1 G(t,s)ds.$$

Lemma 3.7 *Supposing $A>0$, then the Green function $G(t,s)$ has the following properties:*

(1) *$G(t,s)$ is continuous and $G(t,s)\geq 0$, for all $(t,s)\in[0,1]$;*

(2) *$lq(s)t^{\gamma-2}\leq G(t,s)\leq lt^{\gamma-2}$, for $t,s\in[0,1]$, where $q(s)=(1-s)^{\alpha-2}-(1-s)^{\alpha-1}$, $l=\frac{1}{A\Gamma(\alpha-1)}$.*

Proof. Obviously, $G(t,s)$ is continuous. Now we prove that $G(t,s)$ is nonnegative. When $0\leq s\leq t\leq 1$,

$$\begin{aligned}g_1(t,s)&=\frac{1}{\Gamma(\alpha)}[t^{\gamma-2}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}]\\&\geq\frac{1}{\Gamma(\alpha)}[t^{\gamma-2}(1-s)^{\alpha-2}-t^{\alpha-1}(1-s)^{\alpha-1}]\\&\geq\frac{1}{\Gamma(\alpha)}[t^{\gamma-2}(1-s)^{\alpha-2}-t^{\gamma-2}(1-s)^{\alpha-1}]\\&=\frac{1}{\Gamma(\alpha)}[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}]t^{\gamma-2}\\&\geq 0.\end{aligned}$$

It is clear that

$$g_1(t,s)=\frac{1}{\Gamma(\alpha)}t^{\gamma-2}(1-s)^{\alpha-2}\geq 0,\ 0\leq t\leq s\leq 1,$$

and

$$g_2(t,s)=\frac{\alpha-\gamma+1}{A\Gamma(\alpha)}(1-s)^{\alpha-2}t^{\gamma-2}\geq 0,\ s,t\in[0,1],$$

so we get $G(t,s)\geq 0$.

Next, we prove that $lq(s)t^{\gamma-2}\leq G(t,s)\leq lt^{\gamma-2}$.

For $0\leq s\leq t\leq 1$, from the above, we have come to the conclusion:

$$g_1(t,s)=\frac{1}{\Gamma(\alpha)}[t^{\gamma-2}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}]\geq\frac{1}{\Gamma(\alpha)}[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}]t^{\gamma-2},$$

on the other hand, we have

$$g_1(t,s)=\frac{1}{\Gamma(\alpha)}[t^{\gamma-2}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}]\le\frac{1}{\Gamma(\alpha)}t^{\gamma-2}.$$

For $0\le t\le s\le 1$,

$$g_1(t,s)=\frac{1}{\Gamma(\alpha)}t^{\gamma-2}(1-s)^{\alpha-2}\ge\frac{1}{\Gamma(\alpha)}[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}]t^{\gamma-2},$$

and

$$g_1(t,s)=\frac{1}{\Gamma(\alpha)}t^{\gamma-2}(1-s)^{\alpha-2}\le\frac{1}{\Gamma(\alpha)}t^{\gamma-2},$$

hence,

$$\frac{1}{\Gamma(\alpha)}[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}]t^{\gamma-2}\le g_1(t,s)\le\frac{1}{\Gamma(\alpha)}t^{\gamma-2},\ \forall t,s\in[0,1].$$

Then we have

$$\begin{aligned}
G(t,s)&=g_1(t,s)+\frac{t^{\gamma-2}}{A}\sum_{i=1}^{m}b_ig_1(\xi_i,s)+g_2(t,s)\\
&\ge\frac{1}{\Gamma(\alpha)}[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}]t^{\gamma-2}\\
&+\frac{t^{\gamma-2}}{A\Gamma(\alpha)}\sum_{i=1}^{m}b_i{\xi_i}^{\gamma-2}[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}]\\
&+\frac{\alpha-\gamma+1}{A\Gamma(\alpha)}[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}]t^{\gamma-2}\\
&=\frac{1}{\Gamma(\alpha)}(1+\frac{\sum_{i=1}^{m}b_i{\xi_i}^{\gamma-2}}{A}+\frac{\alpha-\gamma+1}{A})[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}]t^{\gamma-2}\\
&=\frac{1}{A\Gamma(\alpha-1)}[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}]t^{\gamma-2}\\
&=lq(s)t^{\gamma-2},
\end{aligned}$$

and

$$\begin{aligned}
G(t,s)&\le\frac{1}{\Gamma(\alpha)}t^{\gamma-2}+\frac{t^{\gamma-2}}{A\Gamma(\alpha)}\sum_{i=1}^{m}b_i{\xi_i}^{\gamma-2}+\frac{\alpha-\gamma+1}{A\Gamma(\alpha)}t^{\gamma-2}\\
&=\frac{1}{\Gamma(\alpha)}(1+\frac{\sum_{i=1}^{m}b_i{\xi_i}^{\gamma-2}}{A}+\frac{\alpha-\gamma+1}{A})t^{\gamma-2}\\
&=\frac{1}{A\Gamma(\alpha-1)}t^{\gamma-2}=lt^{\gamma-2}.
\end{aligned}$$

The proof is completed.

4. Main results

Let the space $E = C[0,1]$, and its norm is $\|u\| = max\{|u(t)| : t \in [0,1]\}$. We define a cone

$$P = \{u \in C([0,1]) : u(t) \geq 0, t \in [0,1]\}.$$

Then $P \subset E$ is normal, the normality constant is 1.

Theorem 4.1 *Suppose $a > 0$ and $A > 0$ in Lemma 3.6, $h(t) = alt^{\gamma-2}$, $\delta(t) = a\int_0^1 G(t,s)ds$, any $t \in [0,1]$. Where $G(t,s)$ is the Green function of Lemma 3.7. Define $f : [0,1] \times [-\delta^*, +\infty) \to (-\infty, +\infty)$ is continuous, where $\delta^* = max\{\delta(t), t \in [0,1]\}$. If the following conditions hold:*

(H_1) f is decreasing with respect to the second variable, that is to say: for any $t \in [0,1]$, $-\delta^* \leq x \leq y$, we have $f(t,x(t)) \geq f(t,y(t))$;

(H_2) for any $\mu \in (0,1)$, there exists $\mu < \psi(\mu) < 1$ such that $f(t, \mu x + (\mu-1)y) \leq \frac{1}{\psi(\mu)} f(t,x)$, any $t \in [0,1], x \in [0,+\infty), y \in [0,\delta^*]$;

(H_3) for any $t \in [0,1]$, $f(t,al) \geq 0$ and $f(t,al) \not\equiv 0$.

Then boundary value problem (1) has a unique nontrivial solution $x^* \in P_{h,\delta}$. Moreover, selecting any $x_0 \in P_{h,\delta}$, making the sequence $x_n(t) = \int_0^1 G(t,s)f(s,x_{n-1}(s))ds - a\int_0^1 G(t,s)ds, n = 1,2,\cdots$, we get $\|x_n - x^*\| \to 0$ as $n \to \infty$.

Proof. Firstly, we show under the assumption, $P_{h,\delta}$ is well defined. Since $a > 0$, $G(t,s) \geq 0$, and the property (2) in Lemma 3.7 we obtain

$$\delta(t) = a\int_0^1 G(t,s)ds \geq 0,$$

and

$$\delta(t) \leq al\int_0^1 t^{\gamma-2}ds = alt^{\gamma-2} = h(t).$$

Therefore, $\delta \in P$, $\theta \leq \delta \leq h$, namely, $P_{h,\delta}$ is well defined.

Secondly, for any $t \in [0,1]$, we define an operator $B : P_{h,\delta} \to E$ as follows

$$Bx(t) = \int_0^1 G(t,s)f(s,x(s))ds - a\int_0^1 G(t,s)ds.$$

It can be drawn that $x \in P_{h,\delta}$ is the solution of problem (1) if and only if the operator equation $Bx = x$ has a unique solution in $P_{h,\delta}$. So lastly, we need to illustrate $Bx = x$ has a unique solution x^* in $P_{h,\delta}$. For any $x, y \in P_{h,\delta}$ and $-\delta^* \leq x \leq y$, according to (H_1), we get

$$\begin{aligned} Bx(t) &= \int_0^1 G(t,s)f(s,x(s))ds - a\int_0^1 G(t,s)ds \\ &\geq \int_0^1 G(t,s)f(s,y(s))ds - a\int_0^1 G(t,s)ds = By(t). \end{aligned}$$

So $B : P_{h,\delta} \to E$ is a decreasing operator.

Next, we prove that B is a $\psi-(h,\delta)-$ convex operator. For $x\in P_{h,\delta}$, $\mu\in(0,1)$, from (H_2), there exists $\mu<\psi(\mu)<1$, such that

$$\begin{aligned}B(\mu x(t)+(\mu-1)\delta(t)) &= \int_0^1 G(t,s)f(s,\mu x(s)+(\mu-1)\delta(s))ds - a\int_0^1 G(t,s)ds\\ &\le \frac{1}{\psi(\mu)}\int_0^1 G(t,s)f(s,x(s))ds-\delta(t)\\ &= \frac{1}{\psi(\mu)}[\int_0^1 G(t,s)f(s,x(s))ds-\delta(t)]+(\frac{1}{\psi(\mu)}-1)\delta(t)\\ &= \frac{1}{\psi(\mu)}Bx(t)+(\frac{1}{\psi(\mu)}-1)\delta(t).\end{aligned}$$

Therefore, B is a $\psi-(h,\delta)-$ convex operator. Now let's prove that $Bh\in P_{h,\delta}$, namely, we need to prove that $Bh+\delta\in P_h$. Using the property (2) in Lemma 3.7 and (H_1), we obtain

$$\begin{aligned}Bh(t)+\delta(t) &= \int_0^1 G(t,s)f(s,h(s))ds\\ &\le \int_0^1 lt^{\gamma-2}f(s,als^{\gamma-2})ds\\ &\le t^{\gamma-2}l\int_0^1 f(s,0)ds\\ &= a^{-1}\int_0^1 f(s,0)ds\cdot alt^{\gamma-2}\\ &= a^{-1}\int_0^1 f(s,0)ds\cdot h(t),\end{aligned}$$

and

$$\begin{aligned}Bh(t)+\delta(t) &\ge \int_0^1 lq(s)t^{\gamma-2}f(s,als^{\gamma-2})ds\\ &\ge t^{\gamma-2}l\int_0^1 q(s)f(s,al)ds\\ &= a^{-1}\int_0^1 q(s)f(s,al)ds\cdot alt^{\gamma-2}\\ &= a^{-1}\int_0^1 q(s)f(s,al)ds\cdot h(t).\end{aligned}$$

Since $0\le q(s)\le 1$, $a^{-1}>0$ and from (H_3), we get

$$a^{-1}\int_0^1 f(s,0)ds\ge a^{-1}\int_0^1 q(s)f(s,al)ds>0.$$

So we have $Bh+\delta\in P_h$.

By using Lemma 2.2, the operator equation $Bx = x$ has a unique solution x^* in $P_{h,\delta}$, i.e. boundary value problem (1) has a unique nontrivial solution $x^* \in P_{h,\delta}$. Moreover, selecting any $x_0 \in P_{h,\delta}$, making the sequence

$$x_n(t) = Bx_{n-1}(t) = \int_0^1 G(t,s)f(s,x_{n-1}(s))ds - a\int_0^1 G(t,s)ds,\ n = 1,2,\cdots,$$

we get $\|x_n - x^*\| \to 0$ as $n \to \infty$.

Theorem 4.2 *Let $a \le 0$ and $A > 0$ be as in Lemma 3.6. Further, suppose that the following conditions hold*

(H_4) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and for $t \in [0,1]$, $f(t,1) \not\equiv 0$;

(H_5) for each $t \in [0,1]$, $f(t,x)$ is decreasing about x, i.e. for any $t \in [0,1]$, $0 \le x \le y \Rightarrow f(t,x) \ge f(t,y)$;

(H_6) for any $\mu \in (0,1)$, there exists $\mu < \psi(\mu) < 1$ such that for any $t \in [0,1]$, $x, y \in [0,+\infty)$, $f(t,\mu x) \le \frac{1}{\psi(\mu)} f(t,x)$.

Then boundary value problem (1) has a unique positive solution $x^* \in P_h$, where $h(t) = t^{\gamma-2}$. In addition, for any $x_0 \in P_h$, define a sequence:

$$x_n(t) = \int_0^1 G(t,s)f(s,x_{n-1}(s))ds - a\int_0^1 G(t,s)ds,\ n = 1,2,\cdots,$$

we have $\|x_n - x^*\| \to 0$ as $n \to \infty$.

Proof. We define an operation $B : P \to E$ is

$$Bx(t) = \int_0^1 G(t,s)[f(s,x(s)) - a]ds, \qquad \forall x \in P, t \in [0,1].$$

It is easy to get that $x \in P$ is a positive solution of problem (1) if and only if x is the positive fixed point of B.

On the one hand, since $G(t,s) \ge 0$, $a \le 0$,$f \ge 0$, so for any $x \in P$, we have $Bx \in P$. From (H_5), for any $x, y \in P$ with $0 \le x \le y$, for each $t \in [0,1]$, we have

$$Bx(t) = \int_0^1 G(t,s)[f(s,x(s)) - a]ds \ge \int_0^1 G(t,s)[f(s,y(s)) - a]ds = By(t).$$

Therefore, $B : P \to P$ is a decreasing operation.

On the other hand, take $h_0 = h = t^{\gamma-2}$. Obviously, $h_0 \in P_h$. According to Lemma 3.7 and (H_5), for any $t \in [0,1]$, we get

$$\begin{aligned} Bh_0(t) &= \int_0^1 G(t,s)[f(s,h(s)) - a]ds = \int_0^1 G(t,s)[f(s,s^{\gamma-2}) - a]ds \\ &\le t^{\gamma-2}\{l\int_0^1 [f(s,0) - a]ds\}, \end{aligned}$$

and

$$Bh_0(t) \ge t^{\gamma-2}\{l\int_0^1 q(s)[f(s,1) - a]ds\}.$$

Since $a \le 0$, $f \ge 0$, $f(s,0) \ge f(s,1) \not\equiv 0$, $l > 0$, $0 \le q(s) \le 1$, so

$$l\int_0^1 [f(s,0) - a]ds \ge l\int_0^1 q(s)[f(s,1) - a]ds > 0.$$

This shows that $Bh_0 \in P_h$.

Moreover, from (H_6), we get for any $x \in P, \mu \in (0,1)$, there exists $\mu < \psi(\mu) < 1$, such that

$$\begin{aligned} B(\mu x) &= \int_0^1 G(t,s)[f(s,\mu x(s)) - a]ds \leq \int_0^1 G(t,s)[\frac{1}{\psi(\mu)} f(t,x(s)) - \frac{1}{\psi(\mu)} a]ds \\ &= \frac{1}{\psi(\mu)} \int_0^1 G(t,s)[f(t,x(s)) - a]ds = \frac{1}{\psi(\mu)} B(x). \end{aligned}$$

Hence, by Lemma 2.3, boundary value problem (1) has a unique positive solution $x^* \in P_h$, where $h(t) = t^{\gamma-2}$. In addition, for any $x_0 \in P_h$, define a sequence: $x_n(t) = \int_0^1 G(t,s)f(s,x_{n-1}(s))ds - a\int_0^1 G(t,s)ds, n = 1,2,\cdots$, we get $\|x_n - x^*\| \to 0$ as $n \to \infty$. The theorem is proved.

5. Examples

Example 5.1 Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{\frac{11}{3},\frac{1}{3}} x(t) + (\frac{\delta(t)}{\delta^*} x + \delta(t) + 1)^{-\frac{1}{5}} = 3, \quad t \in [0,1], \\ x(0) = x'(0) = D_{0+}^{\frac{25}{9}} x(t)|_{t=0} = 0, \\ x'(1) = \frac{1}{2}x(\frac{1}{4}) + \frac{1}{3}x(\frac{1}{2}) + \frac{1}{4}x(\frac{3}{4}), \end{cases} \tag{37}$$

where $a = 3, n = 4, \alpha = \frac{11}{3}, \beta = \frac{1}{3}, \gamma = \alpha + 4\beta - \alpha\beta = \frac{34}{9}, m = 3, b_1 = \frac{1}{2}, b_2 = \frac{1}{3}, b_3 = \frac{1}{4}, \xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{2}, \xi_3 = \frac{3}{4}, A = \frac{34}{9} - 2 - \frac{1}{2} \times (\frac{1}{4})^{\frac{16}{9}} - \frac{1}{3} \times (\frac{1}{2})^{\frac{16}{9}} - \frac{1}{4} \times (\frac{3}{4})^{\frac{16}{9}} \approx 1.4882 > 0$, and

$$G(t,s) = g_1(t,s) + \frac{t^{\frac{16}{9}}}{A} \sum_{i=1}^{3} b_i g_1(\xi_i, s) + g_2(t,s), \qquad \forall (t,s) \in [0,1] \times [0,1],$$

with

$$g_1(t,s) = \frac{1}{\Gamma(\frac{11}{3})} \begin{cases} t^{\frac{16}{9}}(1-s)^{\frac{5}{3}} - (t-s)^{\frac{8}{3}}, & 0 \leq s \leq t \leq 1, \\ t^{\frac{16}{9}}(1-s)^{\frac{5}{3}}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$g_2(t,s) = \frac{8}{9A\Gamma(\frac{11}{3})}(1-s)^{\frac{5}{3}} t^{\frac{16}{9}}.$$

Take $h(t) = \frac{3}{\Gamma(\frac{11}{3})} t^{\frac{16}{9}}$. Obviously,

$$\begin{aligned} 0 \leq \delta(t) &= 3\int_0^1 G(t,s)ds \\ &= 3\int_0^1 g_1(t,s)ds + \frac{3t^{\frac{16}{9}}}{A} \sum_{i=1}^{3} b_i \int_0^1 g_1(\xi_i,s)ds + 3\int_0^1 g_2(t,s)ds \leq h(t). \end{aligned}$$

Here $\delta^* = max\{\delta(t), t \in [0,1]\}$, $al = \frac{3}{\Gamma(\frac{11}{3})}$, and we can see that

$$f(t,x) = (\frac{\delta(t)}{\delta^*} x + \delta(t) + 1)^{-\frac{1}{5}}.$$

(1) Obviously, $f : [0,1] \times [-\delta^*, +\infty) \to (-\infty, +\infty)$ is continuous and decreasing in x.

(2) Taking $\mu < \psi(\mu) = \mu^{\frac{1}{5}} < 1$, for any $\mu \in (0,1), x \in [0,+\infty), y \in [0,\delta^*]$, we have

$$\begin{aligned}
f(t, \mu x + (\mu - 1)y) &= \{\frac{\delta(t)}{\delta^*}[\mu x + (\mu - 1)y] + \delta(t) + 1\}^{-\frac{1}{5}} \\
&= \mu^{-\frac{1}{5}}\{\frac{\delta(t)}{\delta^*}[x + (1 - \frac{1}{\mu})y] + \frac{1}{\mu}(\delta(t) + 1)\}^{-\frac{1}{5}} \\
&\leq \mu^{-\frac{1}{5}}\{\frac{\delta(t)}{\delta^*}[x + (1 - \frac{1}{\mu})\delta^*] + \frac{1}{\mu}(\delta(t) + 1)\}^{-\frac{1}{5}} \\
&\leq \mu^{-\frac{1}{5}}\{\frac{\delta(t)}{\delta^*}x + \delta(t) + 1\}^{-\frac{1}{5}} \\
&= \frac{1}{\psi(\mu)} f(t,x).
\end{aligned}$$

(3) It is obvious that $f(t, al) = (\frac{3\delta(t)}{\Gamma(\frac{11}{3})\delta^*} + \delta(t) + 1)^{-\frac{1}{5}} \geq 0$ and $f(t, al) \not\equiv 0$.

Thus, from Theorem 4.1, we can obtain that the problem (37) has a unique non-trivial solution $x^* \in P_{h,\delta}$, where $\delta(t) = 3\int_0^1 G(t,s)ds$, $h(t) = \frac{3}{\Gamma(\frac{11}{3})} t^{\frac{16}{9}}$.

Example 5.2 Consider the following boundary value problem:

$$\begin{cases}
D_{0+}^{\frac{15}{4},\frac{1}{2}} x(t) + t^3 (x(t) + c)^{-\frac{1}{9}} = -1, \quad t \in [0,1], \\
x(0) = x'(0) = x''(0) = D_{0+}^{\frac{27}{8}} x(t)|_{t=0} = 0, \\
x'(1) = \frac{1}{3}x(\frac{1}{5}) + \frac{1}{4}x(\frac{1}{6}) + \frac{1}{5}x(\frac{1}{7}) + \frac{1}{6}x(\frac{1}{8}),
\end{cases} \tag{38}$$

where $c > 0, a = -1, n = 5, \alpha = \frac{19}{4}, \beta = \frac{1}{2}, \gamma = \alpha + 5\beta - \alpha\beta = \frac{39}{8}, m = 4, b_1 = \frac{1}{3}, b_2 = \frac{1}{4}, b_3 = \frac{1}{5}$, $b_4 = \frac{1}{6}$, $\xi_1 = \frac{1}{5}, \xi_2 = \frac{1}{6}, \xi_3 = \frac{1}{7}, \xi_4 = \frac{1}{8}$, $A = \frac{39}{8} - 2 - \frac{1}{3} \times (\frac{1}{5})^{\frac{23}{8}} - \frac{1}{4} \times (\frac{1}{6})^{\frac{23}{8}} - \frac{1}{5} \times (\frac{1}{7})^{\frac{23}{8}} - \frac{1}{6} \times (\frac{1}{8})^{\frac{23}{8}} \approx 2.8691 > 0$. Here $f(t,x) = t^3(x(t) + c)^{-\frac{1}{9}}$.

(1) It is obvious that $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and for $t \in [0,1]$, $f(t,1) = t^3(1+c)^{-\frac{1}{9}} \not\equiv 0$;

(2) for each $t \in [0,1]$, $f(t,x)$ is decreasing about x;

(3) for any $\mu \in (0,1)$, taking $\psi(\mu) = \mu^{\frac{1}{9}}$, then $\psi(\mu) \in (\mu, 1)$, we have

$$f(t, \mu x) = t^3(\mu x + c)^{-\frac{1}{9}} = t^3 \mu^{-\frac{1}{9}}(x + \frac{c}{\mu})^{-\frac{1}{9}} \leq t^3 \mu^{-\frac{1}{9}}(x + c)^{-\frac{1}{9}} = \frac{1}{\psi(\mu)} f(t,x).$$

Therefore, by Theorem 4.2, we get the problem (38) has a unique positive solution $x^* \in P_h$, where $h(t) = t^{\frac{23}{8}}$, $t \in [0,1]$.

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