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SPLIT COMMON FIXED POINT PROBLEM FOR STRICT PSEUDO-CONTRACTION TYPE MAPPINGS IN BANACH SPACES

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Abstract. A class of nonlinear mappings in p-uniformly convex and uniformly smooth Banach spaces is proposed. We call each mapping in this class a strict pseudo-contraction type mapping. This class contains the classes of strict pseudo-contraction mappings in Hilbert spaces and metric resolvents of maximal monotone operators in Banach spaces. Then, we study the split common fixed point problem for a finite family of strict pseudo-contraction-type mappings in p-uniformly convex and uniformly smooth Banach spaces. We propose an inertial Halpern-type algorithm with the step size independent on the prior estimate of the norm of the bounded linear operator and prove strong convergence theorem.
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1. Introduction

Let E be a Banach space. Then a mapping $T : E \to E$ is said to be firmly nonexpansive type if

$$\langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle \geq 0, \quad \forall x, y \in E,$$

where J is the normalized duality mapping on E. These mappings were introduced and investigated by Kohsaka and Takahashi [20](see also [3]). The class of firmly nonexpansive type mappings is an important class since it includes firmly nonexpansive mappings in Hilbert spaces, the metric projections onto a closed convex set and resolvents of maximal monotone operators in Banach spaces.

Let $\mathcal{H}$ be a Hilbert space. We recall that a mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be k-strict pseudo-contraction (in the sense of Browder-Petryshyn [5]) if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2. \tag{1.1}$$

These mappings are extensions of nonexpansive mappings which satisfy the inequality (1.1) with $k = 0$. Strict pseudo-contractions mappings have more powerful applications than firmly nonexpansive mappings and nonexpansive mappings do in solving inverse problems (see Scherzer [25]).

Let E and F be Banach spaces and $A : E \to F$ be a bounded linear operator. Let $\{C_i\}_{i=1}^p$ and $\{Q_j\}_{j=1}^r$ be two finite families of nonempty closed and convex subsets of E and F, respectively. The multiple-set split feasibility problem (MSSFP) [10] requires to seek an element $x^\star \in E$ satisfying:

$$x^\star \in \bigcap_{i=1}^{p} C_i \quad and \quad Ax^\star \in \bigcap_{j=1}^{r} Q_j.$$

This problem applied for modeling inverse problems often arise in many real-world application problems such as signal and image processing, medical image reconstruction, etc (see [6, 8, 9, 10] for details). Various algorithms and some interesting results have been studied in order to solve it, (see, for example [26, 27, 31] and the references therein).

A generalization of the MSSFP is the split common fixed point problem (SCFPP) [12]. Let $S_i : E \to E$, $(i = 1, ..., p)$ and $T_j : F \to F$, $(j = 1, ..., r)$ be nonlinear mappings. The SCFPP is formulated as:

$$x^\star \in \bigcap_{i=1}^{p} Fix(S_i) \quad and \quad Ax^\star \in \bigcap_{j=1}^{r} Fix(T_j). \tag{1.2}$$

Here $Fix(S_i) := \{x \in E : S_i(x) = x\}$ is the set of fixed points of S_i and $Fix(T_j)$ is the set of fixed points of T_j. Recently, some authors have studied the SCFPP for a pair of mappings of different classes in Banach spaces (see, for example [15, 29, 30] and the references therein).

On the other hand, the inertial technique has become of great interest to many researchers mainly due to its nice convergence characteristics as well as improving the performance of algorithms. The main idea of these methods is to make use of two previous iterates to update the next iterate, which results in speeding up the algorithm's convergence. Recently, authors have shown considerable interest in studying inertial type algorithms, see for example [2, 16, 17, 23] and the references therein.

The remainder of this paper is organized as follows. In Section 2, we collect some preliminary knowledge and some related lemmas. Section 3 is devoted to the study of the properties of strict pseudo-contraction type mappings in a p-uniformly convex and uniformly smooth Banach space. In Section 4, we propose an inertial algorithm for solving the split common fixed point problem for a finite family of strict pseudo-contraction type mappings, and prove its strong convergence of its variant under mild conditions. Two applications of our main theorem to solving the multiple-set split feasibility problem and the split common null point problem in p-uniformly convex and uniformly smooth Banach spaces are presented in Section 5.

2. Preliminaries

In this section, we recall some definitions and results that will be used later. Throughout this paper, we consider $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let E be a real Banach space with norm $\|.\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. Let $S(E) = \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. The modulus of convexity and smoothness are defined respectively by

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : x, y \in S(E), \|x - y\| \geq \epsilon\}, \quad \epsilon \in (0, 2].$$

and

$$\rho_E(\tau) = \sup\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S(E)\}, \qquad \tau > 0.$$

E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$, and p-uniformly convex if there exists a $c_p > 0$ so that $\delta_E(\epsilon) \geq c_p \epsilon^p$ for any $\epsilon \in (0, 2]$. The space E is called uniformly smooth if $\lim_{\tau \to 0} \dfrac{\rho_E(\tau)}{\tau} = 0$, and q-uniformly smooth if there exists a $C_q > 0$ so that $\rho_E(\tau) \leq C_q \tau^q$ for any $\tau > 0$. It is observe that every p-uniformly convex (q-uniformly smooth) space is uniformly convex (uniformly smooth) space. It is known that E is p-uniformly convex (q-uniformly smooth) if and only if its dual E^* is q-uniformly smooth (p-uniformly convex), see [21]. Furthermore, L_r (or ℓ_r) spaces and the Sobolev spaces $W_{k,r}$ are $\min\{r, 2\}$-uniformly smooth for every $r > 1$, see [32].

We shall denote by J_E^p, the duality mapping from E to 2^{E^*} given by

$$J_E^p(x) = \{f \in E^*, \langle x, f \rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\},$$

for every $x \in E$. In particular, $J_E = J_E^2$ is called the normalized duality mapping. In this case, we assume that E is a p-uniformly convex and uniformly smooth which implies that its dual space E^* is q-uniformly smooth and uniformly convex. In this situation, it is known that the duality mapping J_E^p is one-to-one, single-valued and satisfies $J_E^p = (J_{E^*}^q)^{-1}$ where $J_{E^*}^q$ is the duality mapping of E^*. Moreover, if E is uniformly smooth then the duality mapping J_E^p is norm-to-norm uniformly continuous on bounded subsets of E (see [14, 28] for more details).

Lemma 2.1 ([32]). *If E is a q-uniformly smooth Banach space, then there is a constant $c_q > 0$ such that:*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_E^q(x) \rangle + c_q \|y\|^q, \tag{2.1}$$

for all $x, y \in E$, where c_q is called the q-uniform smoothness coefficient of E.

It is known that, the q-uniform smoothness coefficient of L_p (or ℓ_p) for $p > 2$, is $c_q = (p - 1)$ (see e.g. [32]).

Lemma 2.2 ([13]). *If E is a p-uniformly convex Banach space ($p \geq 2$), then there exists a constant $d_p > 0$ such that for all $x, y \in E$,*

$$\|x - y\| \leq (\frac{p}{d_p^2})^{\frac{1}{p-1}} \quad \|J_E^p(x) - J_E^p(y)\|^{\frac{1}{p-1}}. \tag{2.2}$$

Let C be a nonempty closed and convex subset of E. The metric projection of $x \in E$ onto C is the unique element $P_C x \in C$ that minimizes the norm distance to x, i.e. $\|x - P_C x\| = \min_{y \in C} \|x - y\|$. It has been employed successfully in optimization, approximation theory, and fixed point theory. The metric projection can also be characterized by a variational inequality:

$$\langle y - P_C x, J_E^p(x - P_C x)\rangle \leq 0, \quad \forall y \in C. \tag{2.3}$$

Let $f : E \to \mathbb{R}$, be a Gâteaux differentiable convex function. The Bregman distance with respect to f is defined by

$$\Delta_f(x, y) := f(x) - f(y) - \langle x - y, \nabla f(y)\rangle, \quad x, y \in E.$$

In the particular case if $f_p = \frac{1}{p}\|.\|^p$ $(1 < p < \infty)$, then the gradient ∇f of f is coincident with the generalized duality mapping J_E^p. So we have

$$\begin{aligned}\Delta_p(x, y) =& \frac{1}{p}(\|x\|^p - \|y\|^p) + \langle y - x, J_E^p(y)\rangle \\ =& \frac{1}{p}\|x\|^p + \frac{1}{q}\|y\|^p - \langle x, J_E^p(y)\rangle.\end{aligned}$$

For a p-uniformly convex space, the Bregman distance has the following property important properties:

(i) Three point identity: for each $x, y, z \in E$,

$$\Delta_p(x, y) + \Delta_p(y, z) - \Delta_p(x, z) = \langle x - y, J_E^p(z) - J_E^p(y)\rangle.$$

(ii) Two point identity: for each $x, y \in E$,

$$\Delta_f(x, y) + \Delta_f(y, x) = \langle x - y, J_E^p(x) - J_E^p(y)\rangle. \tag{2.4}$$

(iii) For each $x, y \in E$,

$$\tau\|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_E^p(x) - J_E^p(y)\rangle, \tag{2.5}$$

where $\tau > 0$ is some fixed number [26].

Let C be a nonempty, closed and convex subset of E. The Bregman projection is defined by ([4])

$$\Pi_C x = \arg min_{y \in C} \Delta_p(y, x), \quad x \in E.$$

The following characterization of the Bregman projection may be found in [28].

$$\langle y - \Pi_C x, J_E^p(x) - J_E^p(\Pi_C x)\rangle \leq 0, \quad \forall y \in C. \tag{2.6}$$

Following [1, 11], we make use of the function $V_p : E \times E^* \to [0, +\infty)$ associated with f_p which is defined by:

$$V_p(x, x^*) = \frac{1}{p}\|x\|^p - \langle x, x^*\rangle + \frac{1}{q}\|x^*\|^q, \;\; \forall x \in E, x^* \in E^*.$$

So $V_p(x, x^*) = \Delta_p(x, J^q_{E^*}(x^*))$ for all $x \in E$ and $x^* \in E^*$. Moreover, by the subdifferential inequality, we have

$$V_p(x, x^*) + \langle J^q_{E^*}(x^*) - x, y^* \rangle \leq V_p(x, x^* + y^*), \tag{2.7}$$

for all $x \in E$ and $x^*, y^* \in E^*$, (see [19]). Furthermore V_p is convex in the second variable, (see [22]). Then for all $z \in E$, we have

$$\Delta_p\left(z, J^q_{E^*}\left(\sum_{i=1}^{m} t_i J^p_E(x_i)\right)\right) \leq \sum_{i=1}^{m} t_i \Delta_p(z, x_i), \tag{2.8}$$

where $\{x_i\}_{i=1}^m \subset E$ and $\{t_i\}_{i=1}^m \subset (0,1)$ with $\sum_{i=1}^m t_i = 1$.

Lemma 2.3 ([24]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences of E. If $\{y_n\}$ is bounded and* $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$, *then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Let E be a p-uniformly convex and uniformly smooth Banach space. Let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by $D(A)$, that is, $D(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be monotone if $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in D(A)$, $u \in Ax$ and $v \in Ay$. A monotone mapping A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone mapping on E. The set of null points of A is denoted by $A^{-1}0 = \{x \in E : 0 \in Ax\}$. For each $x \in E$ and $\mu > 0$, we define the metric resolvent of maximal monotone operator A by

$$Q^A_\mu(x) = (I + \mu(J^p_E)^{-1}A)^{-1}(x), \quad \forall x \in E. \tag{2.9}$$

It is known that $A^{-1}0 = Fix(Q^A_\mu)$ and

$$\langle Q^A_\mu(x) - Q^A_\mu(y), J^p_E(x - Q^A_\mu(x)) - J^p_E(y - Q^A_\mu(y)) \rangle \geq 0, \tag{2.10}$$

for all $x, y \in E$, (see [3] for details).

The following lemma plays a crucial role in proof of our main result.

Lemma 2.4 ([18]). *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$\begin{cases} s_{n+1} \leq (1-\eta_n)s_n + \eta_n\delta_n, & n \geq 0, \\ s_{n+1} \leq s_n - \varrho_n + \zeta_n, & n \geq 0, \end{cases}$$

where $\{\eta_n\}$ is a sequence in $(0,1)$, $\{\varrho_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\zeta_n\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=1}^\infty \eta_n = \infty$,
(ii) $\lim_{n\to\infty} \zeta_n = 0$,
(iii) $\lim_{k\to\infty} \varrho_{n_k} = 0$, *implies* $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence* $\{n_k\} \subset \{n\}$,

Then $\lim_{n\to\infty} s_n = 0$.

3. Strict pseudo-contraction type mappings

In this section, we introduce and discuss strict pseudo-contraction type mappings in a p-uniformly convex and uniformly smooth Banach space.

Definition 3.1. Let E be a p-uniformly convex which is also uniformly smooth. We call a mapping $T : E \to E$ is strict pseudo-contraction type if there exists a constant $0 \le k < 1$ such that for any $x, y \in E$

$$\langle Tx - Ty, J_E^p(x - Tx) - J_E^p(y - Ty)\rangle \ge -k\|J_E^p(x - Tx) - J_E^p(y - Ty)\|^q. \quad (3.1)$$

In this case, we call the mapping T is k-strict pseudo-contraction type.

Example 3.2. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a k-strict pseudo-contraction mapping. Then T is $\frac{1+k}{2}$-strict pseudo-contraction type. Indeed, we know that every Hilbert space is 2-uniformly convex and uniformly smooth Banach space and $J_{\mathcal{H}}^p = I$. Also, we know the following identity in Hilbert space:

$$2\langle u - v, u - w\rangle = \|u - w\|^2 + \|v - u\|^2 - \|v - w\|^2, \quad \forall u, v, w \in \mathcal{H}.$$

Setting $u = Ty - Tx$, $v = 0$ and $w = y - x$ in above identity we get

$$\begin{aligned}\langle Ty - Tx, (Ty - Tx) - (y - x)\rangle &= \frac{1}{2}\|(Ty - Tx) - (y - x)\|^2 \\ &+ \frac{1}{2}\|(Ty - Tx)\|^2 - \frac{1}{2}\|y - x\|^2 \\ &\le \frac{1}{2}\|(Ty - Tx) - (y - x)\|^2 + \frac{1}{2}\|y - x\|^2 \\ &+ \frac{k}{2}\|(Ty - Tx) - (y - x)\|^2 - \frac{1}{2}\|y - x\|^2 \\ &= (\frac{1 + k}{2})\|(Ty - Tx) - (y - x)\|^2.\end{aligned}$$

This implies that

$$\langle Tx - Ty, (x - Tx) - (y - Ty)\rangle \ge -(\frac{1 + k}{2})\|(x - Tx) - (y - Ty)\|^2.$$

Remark 3.3. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping. Then from Example 3.2, T is $\frac{1}{2}$-strict pseudo-contraction type mapping.

Example 3.4. Let $\mathcal{H}$ be a Hilbert space. A mapping $T : \mathcal{H} \to \mathcal{H}$ is called α-averaged with $\alpha \in (0, 1)$ if $T = (1 - \alpha)I + \alpha S$, where $S : \mathcal{H} \to \mathcal{H}$ is nonexpansive. In this case, from Remark 3.3, we have

$$\langle Sx - Sy, (x - Sx) - (y - Sy)\rangle \ge -(\frac{1}{2})\|(x - Sx) - (y - Sy)\|^2, \quad (3.2)$$

and hence

$$\langle x - y, (x - Sx) - (y - Sy)\rangle \ge \frac{1}{2}\|(x - Sx) - (y - Sy)\|^2. \quad (3.3)$$

Multiplying both parts of inequality (3.2) by α^2 we get

$$\langle \alpha(Sx - Sy), \alpha((x - Sx) - (y - Sy))\rangle \ge -(\frac{1}{2})\|\alpha((x - Sx) - (y - Sy))\|^2. \quad (3.4)$$

Multiplying both sides of inequality (3.3) by $\alpha(1-\alpha)$ we obtain

$$\langle (1-\alpha)(x-y), \alpha((x-Sx)-(y-Sy))\rangle \geq (\frac{1-\alpha}{2\alpha})\|\alpha((x-Sx)-(y-Sy))\|^2. \quad (3.5)$$

Note that $I-T=\alpha(I-S)$. Now inequalities (3.4) and (3.5) yield

$$\langle Tx-Ty, (x-Tx)-(y-Ty)\rangle \geq -(\frac{2\alpha-1}{2\alpha})\|(x-Tx)-(y-Ty)\|^2.$$

Thus every α-averaged mapping is $(\frac{2\alpha-1}{2\alpha})$-strict pseudo-contraction type.

Example 3.5. Let C be a nonempty closed and convex subset of p-uniformly convex and uniformly smooth Banach space E. It is known that (see [3] for details)

$$\langle P_C x - P_C y, J_E^p(x-P_C x) - J_E^p(y-P_C y)\rangle \geq 0.$$

Therefore the metric projection P_C is 0-strict pseudo-contraction type mapping.

Example 3.6. Let E be a p-uniformly convex and uniformly smooth Banach space and $B: E \to 2^{E^*}$ be a maximal monotone operator. Then from inequality (2.10), we have the metric resolvent operator Q_μ^B is 0-strict pseudo-contraction type.

Now we will establish the demiclosedness principle for strict pseudo-contraction type mappings. Demiclosedness principles play an important role in convergence analysis of fixed point algorithms.

Lemma 3.7. *Let E be a p-uniformly convex which is also uniformly smooth. Let $T: E \to E$ be a k-strict pseudo-contraction type mapping. Assume that $\{x_n\}$ is a sequence in E such that $x_n \rightharpoonup x^*$ and $\|x_n - Tx_n\| \to 0$, then $x^* = Tx^*$.*

Proof. The assumptions yield $Tx_n \rightharpoonup x^*$ and $\|J_E^p(x_n - Tx_n)\| = \|x_n - Tx_n\|^{p-1} \to 0$. Since T is k-strict pseudo-contraction type, we have that

$$\langle Tx_n - Tx^*, J_E^p(x_n-Tx_n) - J_E^p(x^*-Tx^*)\rangle \geq -k\|J_E^p(x_n-Tx_n) - J_E^p(x^*-Tx^*)\|^q.$$

Therefore

$$-\|x^*-Tx^*\|^p = \langle x^*-Tx^*, -J_E^p(x^*-Tx^*)\rangle \geq -k\|J_E^p(x^*-Tx^*)\|^q.$$

Note that $\|J_E^p(x^*-Tx^*)\|^q = \|(x^*-Tx^*)\|^{q(p-1)} = \|(x^*-Tx^*)\|^p$. Hence

$$(k-1)\|x^*-Tx^*\|^p \geq 0.$$

Since $0 \leq k < 1$ we get $x^* = Tx^*$.

Lemma 3.8. *Let E be a p-uniformly convex which is also uniformly smooth. Let $T: E \to E$ be a k-strict pseudo-contraction type mapping. Let $x^* \in Fix(T)$ and $x \in E$, then we have*

$$\langle x-x^*, J_E^p(x-Tx)\rangle \geq (1-k)\|(x-Tx)\|^p.$$

Proof. Take $x^* \in Fix(T)$ and $x \in E$. Since T is a k-strict pseudo-contraction type mapping we have

$$\begin{aligned}-k\|J_E^p(x-Tx)\|^q &\le \langle Tx - x^*, J_E^p(x-Tx)\rangle \\ &= \langle x - x^*, J_E^p(x-Tx)\rangle - \langle x - Tx, J_E^p(x-Tx)\rangle \\ &= \langle x - x^*, J_E^p(x-Tx)\rangle - \|x-Tx\|^p.\end{aligned}$$

This implies that

$$\langle x - x^*, J_E^p(x-Tx)\rangle \ge (1-k)\|(x-Tx)\|^p.$$

Lemma 3.9. *Let E be a p-uniformly convex which is also uniformly smooth. Let $T: E \to E$ be a k-strict pseudo-contraction type mapping. Then $Fix(T)$ is closed and convex.*

Proof. From Lemma 3.8 we can easily observe that $Fix(T)$ is closed. We show that $Fix(T)$ is convex. Take $x, y \in Fix(T)$ and $\alpha \in [0,1]$. Put $z = \alpha x + (1-\alpha)y$. Utilizing Lemma 3.8 we obtain

$$\begin{aligned}0 &= \langle z - \alpha x - (1-\alpha)y, J_E^p(z-Tz)\rangle \\ &= \alpha\langle z - x, J_E^p(z-Tz)\rangle + (1-\alpha)\langle z-y, J_E^p(z-Tz)\rangle \\ &\ge \alpha(1-k)\|z-Tz\|^p + (1-\alpha)(1-k)\|z-Tz\|^p \\ &= (1-k)\|z-Tz\|^p.\end{aligned}$$

This implies that $z = Tz$. Hence $Fix(T)$ is convex.

4. Split common fixed points problem

In this section, we present our method for solving the split common fixed points problem for strict pseudo-contraction type mappings in p-uniformly convex and uniformly smooth Banach spaces. Subsequently, we analyze and establish strong convergence of the proposed algorithm. We begin with the following assumptions under which our strong convergence is obtained.

Assumption 4.1. E and F are p-uniformly convex and uniformly smooth Banach spaces. c_q and c_q' are the q-uniform smoothness coefficients of F^* and E^*, respectively. Furthermore, we assume that the following hold:

(i) For each $i \in \{1,2,...,m\}$, $S_i : F \to F$ is a k_i-strict pseudo-contraction type mapping.

(ii) For each $i \in \{1,2,...,m\}$, $T_i : E \to E$ is a l_i-strict pseudo-contraction type mapping.

(iii) $A : E \to F$ is a bounded linear operator and $A^* : F^* \to E^*$ is the adjoint of A.

(iv) $\Omega = \{x^* \in \bigcap_{i=1}^m Fix(T_i) : \quad Ax^* \in \bigcap_{i=1}^m Fix(S_i)\} \neq \emptyset$.

Next, we state the conditions under which our parameters are chosen.

Assumption 4.2. Suppose that $\{\gamma_{n,i}\}$, $\{t_{n,i}\}$,$\{\beta_n\}$ and $\{\varepsilon_n\}$ are positive sequences satisfying the following conditions:

(i) $\gamma_{n,i} \in (0,1], \sum_{i=1}^m \gamma_{n,i} = 1$ and $\liminf_{n\to\infty} \gamma_{n,i} > 0$,

(ii) $\{t_{n,i}\} \subset (0, (\frac{q(1-l_i)}{c_q'})^{\frac{1}{q-1}})$ and $\liminf_{n\to\infty} t_{n,i}((1-l_i) - \frac{c_q' t_{n,i}^{q-1}}{q}) > 0$,

(iii) $\{\varepsilon_n\}$ is a positive sequence such that $\lim_{n\to\infty}\frac{\varepsilon_n}{\beta_n}=0$ where $\{\beta_n\}\subset(0,1)$ satisfies $\lim_{n\to\infty}\beta_n=0$ and $\sum_{n=0}^{\infty}\beta_n=\infty$.

We now present the proposed method of this paper.

Algorithm 1

Initialization Take $\nu, x_1, x_0 \in E$ arbitrarily. Choose sequences $\{\gamma_{n,i}\}$, $\{t_{n,i}\}$, $\{\beta_n\}$ and $\{\varepsilon_n\}$ such that the Assumption 4.2 hold.

Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$. Calculate x_{n+1} as follows:

Step 1: Compute $w_n = J^q_{E^*}\left(J^p_E(x_n) + \theta_n(J^p_E(x_{n-1}) - J^p_E(x_n))\right)$, where $0 \leq \theta_n \leq \overline{\theta}_n$ and $\theta^* \in (0,1)$ such that

$$\overline{\theta}_n = \begin{cases} \min\left\{\frac{\varepsilon_n}{\|J^p_E(x_n)-J^p_E(x_{n-1})\|}, \theta^*\right\}, & x_n \neq x_{n-1} \\ \theta^*, & \text{otherwise.} \end{cases} \tag{4.1}$$

Step 2: For each $i \in \{1,2,...,m\}$, compute

$$u_{n,i} = J^q_{E^*}(J^p_E(w_n) - r_{n,i}A^*J^p_F(Aw_n - S_i(Aw_n)))$$

here the stepsizes are chosen in such a way that for small enough $\epsilon > 0$,

$$r_{n,i} \in \left(\epsilon, \left(\frac{(\frac{q(1-k_i)}{C_q})\|Aw_n - S_i(Aw_n)\|^p}{\|A^*J^p_F(Aw_n - S_i(Aw_n))\|^q} - \epsilon\right)^{\frac{1}{q-1}}\right), \tag{4.2}$$

if $n \in \Lambda = \{k : Aw_n - S_i(Aw_n) \neq 0\}$, otherwise $r_{n,i} = r_i$ is any nonnegative real number.

Step 3: For each $i \in \{1,2,...,m\}$, compute

$$z_{n,i} = J^q_{E^*}(J^p_E(u_{n,i}) - t_{n,i}J^p_E(u_{n,i} - T_i(u_{n,i}))).$$

Step 4: Compute

$$y_n = J^q_{E^*}(\sum_{i=1}^{m}\gamma_{n,i}J^p_E(z_{n,i})).$$

Step 5: Compute

$$x_{n+1} = J^q_{E^*}(\beta_n J^p_E\nu + (1-\beta_n)J^p_E y_n).$$

Set $n := n+1$ and go to step 1.

Remark 4.3. If $A^*J^p_F(Aw_n - S_i(Aw_n)) = 0$. Given $x^* \in \Omega$. By Lemma 3.8, we get

$$\begin{aligned} 0 =& \langle w_n - x^*, A^*J^p_F(Aw_n - S_i(Aw_n))\rangle \\ =& \langle Aw_n - Ax^*, J^p_F(Aw_n - S_i(Aw_n))\rangle \\ \geq& (1-k_i)\|Aw_n - S_i(Aw_n)\|^p. \end{aligned}$$

This implies that $Aw_n - S_i(Aw_n) = 0$.

Remark 4.4. From Lemma 3.9 and since A is linear operator, we have Ω is closed and convex. Therefore, the Bregman projection Π_Ω from E onto Ω is well-defined.

Now, we are ready to analyze the strong convergence of Algorithm 1.

Theorem 4.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 1 under Assumption 4.1 and Assumption 4.2. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = \Pi_\Omega \nu$.*

Proof. Suppose $x^* = \Pi_\Omega \nu$. First, we show that $\{x_n\}$ is bounded. Utilizing Lemma 2.1 and Lemma 3.8 for each $i \in \{1, 2, ..., m\}$ we have

$$\begin{aligned}\Delta_p(x^*, u_{n,i}) =& V_p(x^*, J_E^p(w_n) - r_{n,i} A^* J_F^p(Aw_n - S_i(Aw_n))\\ =& \frac{1}{p}\|x^*\|^p - \langle x^*, J_E^p(w_n)\rangle + r_{n,i}\langle Ax^*, J_F^p(Aw_n - S_i(Aw_n))\rangle\\ &+\frac{1}{q}\|J_E^p(w_n) - r_{n,i}A^* J_F^p(Aw_n - S_i(Aw_n))\|^q\\ \leq& \frac{1}{p}\|x^*\|^p - \langle x^*, J_E^p(w_n)\rangle + r_{n,i}\langle Ax^*, J_F^p(Aw_n - S_i(Aw_n))\rangle\\ &+\frac{1}{q}\|J_E^p(w_n)\|^q - r_{n,i}\langle Aw_n, J_F^p(Aw_n - S_i(Aw_n))\rangle\\ &+\frac{c_q(r_{n,i})^q}{q}\|A^* J_F^p(Aw_n - S_i(Aw_n))\|^q\\ =& \frac{1}{p}\|x^*\|^p - \langle x^*, J_E^p(w_n)\rangle + \frac{1}{q}\|w_n\|^p\\ &-r_{n,i}\langle Aw_n - Ax^*, J_F^p(Aw_n - S_i(Aw_n))\rangle\\ &+\frac{c_q(r_{n,i})^q}{q}\|A^* J_F^p(Aw_n - S_i(Aw_n))\|^q\\ \leq& \Delta_p(x^*, w_n) - r_{n,i}(1-k_i)\|Aw_n - S_i(Aw_n)\|^p \\ &+\frac{c_q(r_{n,i})^q}{q}\|A^* J_F^p(Aw_n - S_i(Aw_n))\|^q.\end{aligned} \tag{4.3}$$

For $n \in \Lambda$, from the definition of $r_{n,i}$ follows

$$\|A^* J_F^p(Aw_n - S_i(Aw_n))\|^q(\epsilon + (r_{n,i})^{q-1}) \leq (\frac{q(1-k_i)}{c_q})\|Aw_n - S_i(Aw_n)\|^p.$$

This implies that

$$\begin{aligned}(\tfrac{\epsilon c_q}{q})r_{n,i}\|A^* J_F^p(Aw_n - S_i(Aw_n))\|^q \leq& r_{n,i}(1-k_i)\|Aw_n - S_i(Aw_n)\|^p\\ &- \tfrac{c_q(r_{n,i})^q}{q}\|A^* J_F^p(Aw_n - S_i(Aw_n))\|^q.\end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), we get

$$\Delta_p(x^*, u_{n,i}) \leq \Delta_p(x^*, w_n) - (\frac{\epsilon c_q}{q})r_{n,i}\|A^* J_F^p(Aw_n - S_i(Aw_n))\|^q. \tag{4.5}$$

Applying Lemma 2.1 and Lemma 3.8 again, we obtain

$$
\begin{aligned}
\Delta_p(x^*, z_{n,i}) =& V_p(x^*, J_E^p(u_{n,i}) - t_{n,i}J_E^p(u_{n,i} - T_i(u_{n,i}))\\
=& \frac{1}{p}\|x^*\|^p - \langle x^*, J_E^p(u_{n,i})\rangle + t_{n,i}\langle x^*, J_E^p(u_{n,i} - T_i(u_{n,i}))\rangle\\
&+ \frac{1}{q}\|J_E^p(u_{n,i}) - t_{n,i}J_E^p(u_{n,i} - T_i(u_{n,i}))\|^q\\
\leq& \frac{1}{p}\|x^*\|^p - \langle x^*, J_E^p(u_{n,i})\rangle + t_{n,i}\langle x^*, J_E^p(u_{n,i} - T_i(u_{n,i}))\rangle\\
&+ \frac{1}{q}\|J_E^p(u_{n,i})\|^q - t_{n,i}\langle u_{n,i}, J_E^p(u_{n,i} - T_i(u_{n,i}))\rangle\\
&+ \frac{{c_q}'(t_{n,i})^q}{q}\|J_E^p(u_{n,i} - T_i(u_{n,i}))\|^q\\
=& \frac{1}{p}\|x^*\|^p - \langle x^*, J_E^p(u_{n,i})\rangle + \frac{1}{q}\|u_{n,i}\|^p\\
&- t_{n,i}\langle u_{n,i} - x^*, J_F^p(u_{n,i} - T_i(u_{n,i}))\rangle + \frac{{c_q}'(t_{n,i})^q}{q}\|J_F^p(u_{n,i} - T_i(u_{n,i}))\|^q\\
\leq& \Delta_p(x^*, u_{n,i}) - t_{n,i}(1 - l_i)\|u_{n,i} - T_i(u_{n,i})\|^p\\
&+ \frac{{c_q}'(t_{n,i})^q}{q}\|u_{n,i} - T_i(u_{n,i})\|^p\\
=& \Delta_p(x^*, u_{n,i}) - t_{n,i}((1 - l_i) - \frac{{c_q}'(t_{n,i})^{q-1}}{q})\|u_{n,i} - T_i(u_{n,i})\|^p. \qquad (4.6)
\end{aligned}
$$

By Assumption 4.2 (ii) for all $n \in \mathbb{N}$, we have

$$
\Delta_p(x^*, z_{n,i}) \leq \Delta_p(x^*, u_{n,i}) \leq \Delta_p(x^*, w_n), \quad (i = 1, 2, .., m). \qquad (4.7)
$$

Applying inequality (2.8), we get

$$
\Delta_p(x^*, y_n) = \Delta_p(x^*, J_{E^*}^q(\sum_{i=1}^m \gamma_{n,i}J_E^p(z_{n,i}))) \leq \sum_{i=1}^m \gamma_{n,i}\Delta_p(x^*, z_{n,i}). \qquad (4.8)
$$

From the definition of w_n follows

$$
\begin{aligned}
\Delta_p(x^*, w_n) =& \Delta_p(x^*, J_{E^*}^q(J_E^p(x_n) + \theta_n(J_E^p(x_{n-1}) - J_E^p(x_n))) \qquad (4.9)\\
\leq& (1 - \theta_n)\Delta_p(x^*, x_n) + \theta_n\Delta_p(x^*, x_{n-1}).
\end{aligned}
$$

From inequalities (4.7),(4.8) and (4.9) we get

$$\begin{aligned}\Delta_p(x^*,x_{n+1}) \leq&\beta_n\Delta_p(x^*,\nu)+(1-\beta_n)\Delta_p(x^*,y_n)\\ \leq&\beta_n\Delta_p(x^*,\nu)+(1-\beta_n)\Delta_p(x^*,w_n)\\ \leq&\beta_n\Delta_p(x^*,\nu)+(1-\beta_n)[(1-\theta_n)\Delta_p(x^*,x_n)+\theta_n\Delta_p(x^*,x_{n-1})] \qquad (4.10)\\ \leq&\max\{\Delta_p(x^*,\nu),\max\{\Delta_p(x^*,x_n),\Delta_p(x^*,x_{n-1})\}\}\\ &\vdots\\ \leq&\max\{\Delta_p(x^*,\nu),\max\{\Delta_p(x^*,x_1),\Delta_p(x^*,x_0)\}\}.\end{aligned}$$

Therefore, $\Delta_p(x^*,x_n)$ is bounded and by (2.5), the sequence $\{x_n\}$ is also bounded. Consequently, $\{w_n\}$, $\{y_n\}$, $\{u_{n,i}\}$ are all bounded. We have $\theta_n\|J_E^p(x_n)-J_E^p(x_{n-1})\|\leq\varepsilon_n$ for all n, which together with $\lim_{n\to\infty}\frac{\varepsilon_n}{\beta_n}=0$ implies that

$$\lim_{n\to\infty}\frac{\theta_n}{\beta_n}\|J_E^p(x_n)-J_E^p(x_{n-1})\|=0. \qquad (4.11)$$

Utilizing Lemma 2.2 we get that

$$\lim_{n\to\infty}\frac{\theta_n}{\beta_n}\|x_n-x_{n-1}\|=0. \qquad (4.12)$$

Since the sequence $\{x_n\}$ is bounded, there exists a constant $M>0$ such that

$$\begin{aligned}\Delta_p(x^*,x_{n-1})-\Delta_p(x^*,x_n)=&\frac{1}{q}\|x_{n-1}\|^p-\langle x^*,J_E^p(x_{n-1})\rangle+\frac{1}{p}\|x^*\|^p\\ &-(\frac{1}{q}\|x_n\|^p-\langle x^*,J_E^p(x_n)\rangle+\frac{1}{p}\|x^*\|^p)\\ =&\frac{1}{q}(\|x_{n-1}\|^p-\|x_n\|^p)+\langle x^*,J_E^p(x_n)-J_E^p(x_{n-1})\rangle\\ \leq&\frac{1}{q}M\|x_{n-1}-x_n\|+\|J_E^p(x_n)-J_E^p(x_{n-1})\|\|x^*\|.\end{aligned}$$

By virtue of (4.12) and (4.11) we have

$$\lim_{n\to\infty}\frac{\theta_n}{\beta_n}(|\Delta_p(x^*,x_{n-1})-\Delta_p(x^*,x_n)|)=0. \qquad (4.13)$$

From (4.9) we get

$$\begin{aligned}\beta_n\Delta_p(x^*,\nu)+&(1-\beta_n)\Delta_p(x^*,w_n)\\ \leq&\beta_n\Delta_p(x^*,\nu)+(1-\beta_n)[(1-\theta_n)\Delta_p(x^*,x_n)+\theta_n\Delta_p(x^*,x_{n-1})]\\ =&\Delta_p(x^*,x_n)+\beta_n\left[\Delta_p(x^*,\nu)-\Delta_p(x^*,x_n)\right]\\ &+\beta_n\left[(1-\beta_n)\frac{\theta_n}{\beta_n}[\Delta_p(x^*,x_{n-1})-\Delta_p(x^*,x_n)]\right]. \qquad (4.14)\\ \leq&\Delta_p(x^*,x_n)+\beta_n(K_1+K_2),\end{aligned}$$

where $K_1=\sup_{n\in\mathbb{N}}\{|\Delta_p(x^*,\nu)-\Delta_p(x^*,x_n)|\}$ and
$K_2=\sup_{n\in\mathbb{N}}\{|\frac{\theta_n}{\beta_n}[\Delta_p(x^*,x_{n-1})-\Delta_p(x^*,x_n)]|\}$.

Utilizing (4.5), (4.6), (4.8) and (4.14) we have

$$
\begin{aligned}
\Delta_p(x^*, x_{n+1}) \leq &(1-\beta_n)\Delta_p(x^*, y_n) + \beta_n \Delta_p(x^*, \nu) \\
\leq &(1-\beta_n)\sum_{i=1}^{m}\gamma_{n,i}\Delta_p(x^*, z_{n,i}) + \beta_n \Delta_p(x^*, \nu)\rangle \\
\leq &\beta_n \Delta_p(x^*, \nu) + (1-\beta_n)\sum_{i=1}^{m}\gamma_{n,i}\Delta_p(x^*, u_{n,i}) \\
&-(1-\beta_n)\sum_{i=1}^{m}\gamma_{n,i}t_{n,i}((1-l_i) - \frac{{c_q}'(t_{n,i})^{q-1}}{q})\|u_{n,i} - T_i(u_{n,i})\|^p \\
\leq &\beta_n \Delta_p(x^*, \nu) + (1-\beta_n)\Delta_p(x^*, w_n) \\
&-(1-\beta_n)\sum_{i=1}^{m}\gamma_{n,i}t_{n,i}((1-l_i) - \frac{{c_q}'(t_{n,i})^{q-1}}{q})\|u_{n,i} - T_i(u_{n,i})\|^p \\
&-(1-\beta_n)\sum_{i=1}^{m}\gamma_{n,i}(\frac{\epsilon c_q}{q})r_{n,i}\|A^* J_F^p(Aw_n - S_i(Aw_n))\|^q \\
\leq &\Delta_p(x^*, x_n) + \beta_n(K_1 + K_2) \\
&-(1-\beta_n)\sum_{i=1}^{m}\gamma_{n,i}(\frac{\epsilon c_q}{q})r_{n,i}\|A^* J_F^p(Aw_n - S_i(Aw_n))\|^q \\
&-(1-\beta_n)\sum_{i=1}^{m}\gamma_{n,i}t_{n,i}((1-l_i) - \frac{{c_q}'(t_{n,i})^{q-1}}{q})\|u_{n,i} - T_i(u_{n,i})\|^p.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\Delta_p(x^*, x_{n+1}) \leq &\Delta_p(x^*, x_n) + \beta_n(K_1 + K_2) \\
&-(1-\beta_n)\sum_{i=1}^{m}\gamma_{n,i}(\frac{\epsilon\, c_q}{q})r_{n,i}\|A^* J_F^p(Aw_n - S_i(Aw_n))\|^q \\
&-(1-\beta_n)\sum_{i=1}^{m}\gamma_{n,i}t_{n,i}((1-l_i) - \frac{{c_q}'(t_{n,i})^{q-1}}{q})\|u_{n,i} - T_i(u_{n,i})\|^p. \quad (4.15)
\end{aligned}
$$

By applying inequality (2.7), we have

$$
\begin{aligned}
\Delta_p(x^*, x_{n+1}) = &\Delta_p(x^*, J_{E^*}^q(\beta_n J_E^p \nu + (1-\beta_n)J_E^p y_n)) \\
= &V_p(x^*, \beta_n J_E^p \nu + (1-\beta_n)J_E^p y_n) \\
\leq &V_p(x^*, \beta_n J_E^p \nu + (1-\beta_n)J_E^p y_n - \beta_n(J_E^p \nu - J_E^p x^*)) \\
&+\beta_n\langle J_E^p \nu - J_E^p x^*, x_{n+1} - x^*\rangle \\
= &V_p(x^*, \beta_n J_E^p x^* + (1-\beta_n)J_E^p y_n) + \beta_n\langle J_E^p \nu - J_E^p x^*, x_{n+1} - x^*\rangle \\
\leq &\beta_n V_p(x^*, J_E^p x^*) + (1-\beta_n)V_p(x^*, J_E^p y_n) + \beta_n\langle J_E^p \nu - J_E^p x^*, x_{n+1} - x^*\rangle \\
= &(1-\beta_n)\Delta_p(x^*, y_n) + \beta_n\langle J_E^p \nu - J_E^p x^*, x_{n+1} - x^*\rangle.
\end{aligned}
$$

Therefore by using inequalities (4.7),(4.8) and (4.9) follows

$$\begin{aligned}\Delta_p(x^*, x_{n+1}) \leq &(1-\beta_n)\Delta_p(x^*, y_n) + \beta_n\langle J_E^p\nu - J_E^p x^*, x_{n+1} - x^*\rangle\\ \leq &(1-\beta_n)[(1-\theta_n)\Delta_p(x^*, x_n) + \theta_n\Delta_p(x^*, x_{n-1})]\\ &+\beta_n\langle J_E^p\nu - J_E^p x^*, x_{n+1} - x^*\rangle. \end{aligned} \tag{4.16}$$

Now we set

$$\chi_n = (1-\beta_n)\frac{\theta_n}{\beta_n}[\Delta_p(x^*, x_{n-1}) - \Delta_p(x^*, x_n)] + \langle J_E^p\nu - J_E^p x^*, x_{n+1} - x^*\rangle,$$

$$\begin{aligned}\varrho_n \ =&(1-\beta_n)\textstyle\sum_{i=1}^m \gamma_{n,i}(\frac{\epsilon c_q}{q})r_{n,i}\|A^*J_F^p(Aw_n - S_i(Aw_n))\|^q\\ &+(1-\beta_n)\textstyle\sum_{i=1}^m \gamma_{n,i}t_{n,i}((1-l_i) - \frac{{c_q}'(t_{n,i})^{q-1}}{q})\|u_{n,i} - T_i(u_{n,i})\|^p,\end{aligned}$$

and

$$\zeta_n = \beta_n(K_1 + K_2), \quad \Gamma_n = \Delta_p(x^*, x_n). \tag{4.17}$$

The inequality (4.15) and inequality (4.16) can be rewritten in the following form:

$$\begin{cases}\Gamma_{n+1} \leq (1-\beta_n)\Gamma_n + \beta_n\chi_n, & n \geq 0,\\ \Gamma_{n+1} \leq \Gamma_n - \varrho_n + \zeta_n, & n \geq 0.\end{cases} \tag{4.18}$$

In order to prove $\Gamma_n \to 0$, by Lemma 2.4, it is sufficient to prove that for any subsequence $\{n_k\} \subset \{n\}$, if $\lim_{k\to\infty} \varrho_{n_k} = 0$, then $\limsup_{k\to\infty} \chi_{n_k} \leq 0$.
We assume that $\lim_{k\to\infty} \varrho_{n_k} = 0$. By Assumption 4.2 and Remark 4.3, we get

$$\lim_{k\to\infty}\|Aw_{n_k} - S_i(Aw_{n_k})\| = 0 = \lim_{k\to\infty}\|u_{n_k,i} - T_i(u_{n_k,i})\|, \qquad i = 1, 2, ..., m. \tag{4.19}$$

Also we have

$$\|J_E^p(w_n) - J_E^p(x_n)\| = \theta_n\|J_E^p(x_{n-1}) - J_E^p(x_n)\| = \beta_n\frac{\theta_n}{\beta_n}\|J_E^p(x_{n-1}) - J_E^p(x_n)\| \to 0.$$

Furthermore for $i = 1, 2, ..., m$, we get that

$$\begin{aligned}\|J_E^p(u_{n_k,i}) - J_E^p(w_{n_k})\| \leq &r_{n_k,i}\|A^*J_F^p(Aw_{n_k} - S_i(Aw_{n_k})\|\\ \leq &r_{n_k,i}\|A^*\|\|Aw_{n_k} - S_i(Aw_{n_k})\|^{p-1} \to 0,\end{aligned}$$

which implies

$$\begin{aligned}\|J_E^p(z_{n_k,i}) - J_E^p(w_{n_k})\| \leq &\|J_E^p(z_{n_k,i}) - J_E^p(u_{n_k,i})\| + \|J_E^p(u_{n_k,i}) - J_E^p(w_{n_k})\|\\ = &t_{n_k,i}\|J_E^p(u_{n_k,i} - T_i(u_{n_k,i}))\| + \|J_E^p(u_{n_k,i}) - J_E^p(w_{n_k})\| \to 0.\end{aligned}$$

In view of Algorithm 1 and above inequality, we conclude that

$$\begin{aligned}\|J_E^p(x_{n+1}) - J_E^p(x_n)\| \leq &\ \|J_E^p(x_{n+1}) - J_E^p(y_n)\| + \|J_E^p(y_n) - J_E^p(x_n)\|\\ = &\ \beta_n\|J_E^p(\nu) - J_E^p(y_n)\| + \|J_E^p(y_n) - J_E^p(x_n)\|\\ \leq &\ \beta_n\|J_E^p(\nu) - J_E^p(y_n)\| + \|J_E^p(w_n) - J_E^p(x_n)\|\\ &+ \textstyle\sum_{i=1}^m \gamma_{n,i}\|J_E^p(z_{n,i}) - J_E^p(w_n)\|\\ \to &\,0.\end{aligned}$$

By uniform continuity of $J_{E^*}^q$ on bounded subset of E^*, we conclude that

$$\lim_{k\to\infty} \|u_{n_k,i} - w_{n_k}\| = \lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = \lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{4.20}$$

Since $\{x_{n_k}\}$ is bounded and E is a reflexive Banach space, there exits a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to z. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup z$. Relation (4.20) implies $u_{n_k,i} \rightharpoonup z, \quad i = 1, 2, .., m$. From (4.19) and Lemma 3.7 we have $z \in \bigcap_{i=1}^m Fix(T_i)$. It follows from (4.20) that $w_{n_k} \rightharpoonup z$. From the continuity of A, we have that $Aw_{n_k} \rightharpoonup Az$. Utilizing (4.19) and Lemma 3.7 we get $Az \in \bigcap_{i=1}^m Fix(S_i)$. This implies that $z \in \Omega$. Next we show that $\limsup_{n\to\infty}\langle x_{n+1} - x^*, J_E^p\nu - J_E^p x^*\rangle \leq 0$. We can choose a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that

$$\limsup_{k\to\infty}\langle x_{n_k} - x^*, J_E^p\nu - J_E^p x^*\rangle = \lim_{j\to\infty}\langle x_{n_{k_j}} - x^*, J_E^p\nu - J_E^p x^*\rangle.$$

Since $x^* = \Pi_\Omega \nu$, utilizing (2.6) we get

$$\lim_{j\to\infty}\langle x_{n_{k_j}} - x^*, J_E^p\nu - J_E^p x^*\rangle = \langle z - x^*, J_E^p\nu - J_E^p x^*\rangle \leq 0. \tag{4.21}$$

From (4.13), (4.20) and (4.21) we arrive at

$$\limsup_{k\to\infty} \chi_{n_k} \leq 0.$$

Hence, all conditions of Lemma 2.4 are satisfied. Therefore, we immediately deduce that $\lim_{n\to\infty}\Gamma_n = \lim_{n\to\infty}\Delta_p(x^*, x_n) = 0$. Therefore it follows from Lemma 2.3 that $\{x_n\}$ converges strongly to x^* as $n \to \infty$. This completes the proof.

We consequently obtain the following results in Hilbert spaces.

Corollary 4.6. *Let E and F be Hilbert spaces and let $A : E \to F$ be a bounded linear operator. Let for $i = 1, 2, ..., m$, $S_i : F \to F$ be a finite family of k_i-strict pseudo-contraction mappings and let $T_i : E \to E$ be a finite family of l_i-strict pseudo-contraction mappings. Suppose that $\Omega = \{x^* \in \bigcap_{i=1}^m Fix(T_i) : \quad Ax^* \in \bigcap_{i=1}^m Fix(S_i)\} \neq \emptyset$. Let $\theta^* \in (0, 1)$ and $\{\beta_n\}$ be a sequence in $(0, 1)$. For $\nu, x_0, x_1 \in E$, let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} w_n = x_n + \theta_n(x_{n-1} - x_n), \\ u_{n,i} = w_n - r_{n,i}A^*(Aw_n - S_i(Aw_n)), \\ z_{n,i} = u_{n,i} - t_{n,i}(u_{n,i} - T_i(u_{n,i})), i = 1, 2, ..., m \\ y_n = \sum_{i=1}^m \gamma_{n,i}\, z_{n,i} \\ x_{n+1} = \beta_n\nu + (1 - \beta_n)y_n \qquad \forall n \geq 1, \end{cases}$$

where $0 \leq \theta_n \leq \overline{\theta}_n$ such that

$$\overline{\theta}_n = \begin{cases} min\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \theta^*\}, & x_n \neq x_{n-1} \\ \theta^*, & otherwise. \end{cases} \tag{4.22}$$

Suppose the stepsizes are chosen in such a way that for small enough $\epsilon > 0$,

$$r_{n,i} \in (\epsilon, \frac{(1 - k_i)\|Aw_n - S_i(Aw_n)\|^2}{\|A^*(Aw_n - S_i(Aw_n))\|^2} - \epsilon) \quad if\, n \in \Lambda = \{k : Aw_n - S_i(Aw_n) \neq 0\}, \tag{4.23}$$

otherwise $r_{n,i} = r_i$ *is any nonnegative real number. Suppose that the following conditions are satisfied:*
(i) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=1}^{\infty}\beta_n = \infty$,
(ii) $\{t_{n,i}\} \subset (0,(1-l_i))$ *and* $\liminf_{n\to\infty} t_{n,i}((1-l_i)-t_{n,i}) > 0$,
(iii) $\gamma_{n,i} \in (0,1], \sum_{i=1}^{m}\gamma_{n,i} = 1$ *and* $\liminf_{n\to\infty}\gamma_{n,i} > 0$,
(iv) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty}\dfrac{\varepsilon_n}{\beta_n} = 0$.
Then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega\nu$.
Proof. A Hilbert space $\mathcal{H}$ is 2- uniformly smooth Banach space which has the best smoothness number $1 > 0$. We know that for each $i \in \{1,2,...,m\}$, S_i is $\frac{1+k_i}{2}$-strict pseudo-contraction type and T_i is $\frac{1+l_i}{2}$-strict pseudo-contraction type mapping. Thus we obtain the desired result by Theorem 4.5.

The following result is a strong convergence theorem for solving the split common fixed point problem for averaged mappings in Hilbert spaces.

Corollary 4.7. *Let* E *and* F *be Hilbert spaces and let* $A : E \to F$ *be a bounded linear operator. Let for* $i = 1,2,...,m$, $S_i : F \to F$ *be a finite family of* α_i*-averaged mappings and let* $T_i : E \to E$ *be a finite family of* ζ_i*-averaged mappings. Suppose that* $\Omega = \{x^* \in \bigcap_{i=1}^m Fix(T_i) : \quad Ax^* \in \bigcap_{i=1}^m Fix(S_i)\} \neq \emptyset$. *For* $\nu, x_0, x_1 \in E$, *let* $\{x_n\}$ *be a sequence defined by:*

$$\begin{cases} w_n = x_n + \theta_n(x_{n-1} - x_n), \\ y_n = \sum_{i=1}^m \gamma_{n,i} T_i(w_n - r_{n,i}A^*(Aw_n - S_i(Aw_n))) \\ x_{n+1} = \beta_n\nu + (1-\beta_n)y_n \qquad \forall n \geq 1, \end{cases}$$

where $0 \leq \theta_n \leq \overline{\theta}_n$ *and* $\theta^* \in (0,1)$ *such that*

$$\overline{\theta}_n = \begin{cases} min\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \theta^*\}, & x_n \neq x_{n-1} \\ \theta^*, & otherwise. \end{cases} \tag{4.24}$$

Suppose the stepsizes are chosen in such a way that for small enough $\epsilon > 0$,

$$(r_{n,i}) \in (\epsilon, \frac{\frac{1}{\alpha_i}\|Aw_n - S_i(Aw_n)\|^2}{\|A^*(Aw_n - S_i(Aw_n))\|^2} - \epsilon) \quad if\, n \in \Lambda = \{k : Aw_n - S_i(Aw_n) \neq 0\}, \tag{4.25}$$

otherwise $r_{n,i} = r_i$ *is any nonnegative real number. Let the sequences* $\{\gamma_{n,i}\}$, $\{\beta_n\}$ *and* $\{\varepsilon_n\}$ *satisfy the Assumption 4.2. Then the sequence* $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega\nu$.
Proof. Since every ζ_i-averaged mapping is $(\frac{2\zeta_i - 1}{2\zeta_i})$-strict pseudo-contraction type, we have $2(1 - \frac{2\zeta_i-1}{2\zeta_i}) = \frac{1}{\zeta_i} > 1$. Now taking $t_{n,i} = 1$ in Theorem 4.5 we obtain the desired result.

5. Applications

In this section, utilizing Theorem 4.5, we get new strong convergence theorems which are connected with the multiple-set split feasibility problem and the split common null point problem in p-uniformly convex and uniformly smooth Banach spaces.

5.1. **Multiple-set split feasibility problem.**

The following convergence result for solving the multiple-set split feasibility problem in Banach spaces follows from Theorem 4.5.

Corollary 5.1. *Let E and F be p-uniformly convex and uniformly smooth Banach spaces. Let $\{C_i\}_{i=1}^m$ and $\{Q_i\}_{i=1}^m$ be two finite families of nonempty closed and convex subsets of E and F respectively. Let $A : E \to F$ be a bounded linear operator and $A^* : F^* \to E^*$ be the adjoint of A. Suppose that $\Omega = \{x^* \in \bigcap_{i=1}^m C_i : \quad Ax^* \in \bigcap_{i=1}^m Q_i\} \neq \emptyset$. For $\nu, x_0, x_1 \in E$, let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} w_n = J_{E^*}^q \left(J_E^p(x_n) + \theta_n(J_E^p(x_{n-1}) - J_E^p(x_n))\right), \\ u_{n,i} = J_{E^*}^q(J_E^p(w_n) - r_{n,i}A^*J_F^p(Aw_n - P_{Q_i}(Aw_n))), \\ z_{n,i} = J_{E^*}^q(J_E^p(u_{n,i}) - t_{n,i}J_E^p(u_{n,i} - P_{C_i}(u_{n,i}))), i = 1, 2, ..., m \\ y_n = J_{E^*}^q(\sum_{i=1}^m \gamma_{n,i}J_E^p(z_{n,i})) \\ x_{n+1} = J_{E^*}^q(\beta_n J_E^p \nu + (1 - \beta_n)J_E^p y_n) \qquad \forall n \geq 1, \end{cases}$$

where $0 \leq \theta_n \leq \overline{\theta}_n$ and $\theta^* \in (0, 1)$ such that

$$\overline{\theta}_n = \begin{cases} min\{\frac{\varepsilon_n}{\|J_E^p(x_n) - J_E^p(x_{n-1})\|}, \theta^*\}, & x_n \neq x_{n-1} \\ \theta^*, & otherwise. \end{cases} \tag{5.1}$$

Suppose the stepsizes are chosen in such a way that for small enough $\epsilon > 0$,

$$r_{n,i} \in \left(\epsilon, \left(\frac{(\frac{q}{c_q})\|Aw_n - P_{Q_i}(Aw_n)\|^p}{\|A^*J_F^p(Aw_n - P_{Q_i}(Aw_n))\|^q} - \epsilon\right)^{\frac{1}{q-1}}\right),$$

if $n \in \Lambda = \{k : Aw_n - P_{Q_i}(Aw_n) \neq 0\}$, otherwise $r_{n,i} = r_i$ is any nonnegative real number. Suppose that the following conditions are satisfied:

(i) $\beta_n \in (0, 1), \lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^\infty \beta_n = \infty$,

(ii) $\{t_{n,i}\} \subset (0, (\frac{q}{{c_q}'})^{\frac{1}{q-1}})$ *and* $\liminf_{n\to\infty} t_{n,i}(1 - \frac{{c_q}' t_{n,i}^{q-1}}{q}) > 0$,

(iii) $\gamma_{n,i} \in (0, 1], \sum_{i=1}^m \gamma_{n,i} = 1$ *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$,

(iv) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \dfrac{\varepsilon_n}{\beta_n} = 0$.

Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = \Pi_\Omega \nu$.

Proof. We know that for $i \in \{1, 2, ..., m\}$, P_{C_i} and P_{Q_i} are 0-strict pseudo-contraction type mappings. Therefore, we have the desired result from Theorem 4.5.

Remark 5.2. In the existing algorithms ([15, 26, 27, 29, 30]) to solve multiple-set split feasibility problem, we need to calculate Π_{C_i} and P_{Q_i}. But in our algorithm we need to calculate P_{C_i} and P_{Q_i}. Therefore, our algorithm is different from the existing algorithms.

5.2. **Split common null point problem.** The following result is a strong convergence theorem for solving the split common null point problem in Banach spaces.

Corollary 5.3 *Let E and F be p-uniformly convex and uniformly smooth Banach spaces. Let for $i = 1, 2, ..., m$, $B_i : E \to 2^{E^*}$ and $G_i : F \to 2^{F^*}$ be maximal monotone operators. Let for $i = 1, 2, ..., m$, $Q^{B_i}_{s_i}$, be metric resolvent operators of B_i for $s_i > 0$ and $Q^{G_i}_{\mu_i}$, be metric resolvent operators of G_i for $\mu_i > 0$. Let $A : E \to F$ be a bounded linear operator and $A^* : F^* \to E^*$ be the adjoint of A. Suppose that $\Omega = (\bigcap_{i=1}^m B_i^{-1}0) \cap (\bigcap_{i=1}^m A^{-1}(G_i{}^{-1}0) \neq \emptyset$. For $\nu, x_0, x_1 \in E$, let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} w_n = J^q_{E^*}\left(J^p_E(x_n) + \theta_n(J^p_E(x_{n-1}) - J^p_E(x_n))\right), \\ u_{n,i} = J^q_{E^*}(J^p_E(w_n) - r_{n,i}A^*J^p_F(Aw_n - Q^{G_i}_{\mu_i}(Aw_n))), \\ z_{n,i} = J^q_{E^*}(J^p_E(u_{n,i}) - t_{n,i}J^p_E(u_{n,i} - Q^{B_i}_{s_i}(u_{n,i}))), i = 1, 2, ..., m \\ y_n = J^q_{E^*}(\sum_{i=1}^m \gamma_{n,i}J^p_E(z_{n,i})) \\ x_{n+1} = J^q_{E^*}(\beta_n J^p_E\nu + (1-\beta_n)J^p_E y_n) \qquad \forall n \geq 1, \end{cases}$$

where $0 \leq \theta_n \leq \overline{\theta}_n$ and $\theta^* \in (0,1)$ such that

$$\overline{\theta}_n = \begin{cases} min\{\frac{\varepsilon_n}{\|J^p_E(x_n) - J^p_E(x_{n-1})\|}, \theta^*\}, & x_n \neq x_{n-1} \\ \theta^*, & otherwise. \end{cases} \tag{5.2}$$

Suppose the stepsizes are chosen in such a way that for small enough $\epsilon > 0$,

$$r_{n,i} \in \left(\epsilon, \left(\frac{(\frac{q}{c_q})\|Aw_n - Q^{G_i}_{\mu_i}(Aw_n)\|^p}{\|A^*J^p_F(Aw_n - Q^{G_i}_{\mu_i}(Aw_n))\|^q} - \epsilon\right)^{\frac{1}{q-1}}\right),$$

if $n \in \Lambda = \{k : Aw_n - Q^{G_i}_{\mu_i}(Aw_n) \neq 0\}$, otherwise $r_{n,i} = r_i$ is any nonnegative real number. Suppose that the following conditions are satisfied:
(i) $\beta_n \in (0,1), \lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=1}^\infty \beta_n = \infty$,
(ii) $\{t_{n,i}\} \subset (0, (\frac{q}{c_q{}'})^{\frac{1}{q-1}})$ *and* $\liminf_{n\to\infty} t_{n,i}(1 - \frac{c_q{}' t^{q-1}_{n,i}}{q}) > 0$,
(iii) $\gamma_{n,i} \in (0,1], \sum_{i=1}^m \gamma_{n,i} = 1$ *and* $\liminf_{n\to\infty}\gamma_{n,i} > 0$,
(iv) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty}\dfrac{\varepsilon_n}{\beta_n} = 0$.
Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = \Pi_\Omega\nu$.
Proof. From inequality (2.10), for $i \in \{1, 2, ..., m\}$, we have $Q^{B_i}_{s_i}$ and $Q^{G_i}_{\mu_i}$ are 0-strict pseudo-contraction type mappings. Now applying Theorem 4.5 with $T_i = Q^{B_i}_{s_i}$ and $S_i = Q^{G_i}_{\mu_i}$, the proof follows.

Remark 5.4. Corollary 5.3, generalizes the result of Byrne et al. [7] from solving the split common null point problem in a Hilbert space to p-uniformly smooth and uniformly convex Banach space. Also in our proposed algorithm, the step size $r_{n,i}$ are independent of the norm of A.

6. Some open problems

In [3], Aoyama et al. discussed some properties of mappings of type (P), (Q) and (R) in Banach spaces. These are all generalizations of firmly nonexpansive mappings in Hilbert spaces. Let E be a smooth Banach space. Then a mapping $T : E \to E$ is said to be of type (P) if $\langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle \geq 0, \quad \forall x, y \in E$.
The mapping $T : E \to E$ is said to be of type (Q) if

$$\langle Tx - Ty, (Jx - JTx) - (Jy - JTy)\rangle \geq 0, \quad \forall x, y \in E.$$

The mapping $T : E \to E$ is said to be of type (R) if

$$\langle JTx - JTy, (x - Tx) - (y - Ty)\rangle \geq 0, \quad \forall x, y \in E.$$

It is known that the classes of mappings of type (P), (Q) and (R) correspond to three types of resolvents of monotone operators in Banach spaces.

In this paper we propose the class of strict pseudo-contraction type mappings in Banach spaces which contains the classes of strict pseudo-contraction mappings in Hilbert spaces and the mappings of type (P) in Banach spaces. The above considerations give rise to the following problems.

Problem 6.1. Find a new class of mappings with demiclosedness principle which contains the classes of strict pseudo-contraction mappings in Hilbert spaces and the mappings of type (Q) (or the mappings of type (R)) in Banach spaces. Find an efficient algorithm for approximating the solution of the split common fixed point problem for the new class of mappings.

Problem 6.2. Find an efficient algorithm for approximating the solution of the split common fixed point problem for strict pseudo-contraction type mappings in a more general space, such as reflexive Banach space.

Problem 6.3. Find applications of the proposed algorithm to some practical optimization problems.

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